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Generalized LCM matrices

By ANTAL BEGE (Tîrgu Mureş)

This paper is dedicated to Kálmán Győry, András Sárközy on the occsasion of their 70th birthday, Attila Pethő, János Pintz on the occasion of their 60th birthday

Abstract. Let f be an arithmetical function. The matrix $[f[i,j]]_{n\times n}$ given by the value of f in least common multiple of $[i,j]$, $f\big([i,j]\big)$ as its i, j entry is called the least common multiple (LCM) matrix. We consider the generalization of this matrix where the elements are in the form $f\big(n,[i,j]\big)$ and $f\big(n,i,j,[i,j]\big)$.

1. Introduction

The classical Smith determinant was introduced in 1875 by H. J. S. Smith [12] who also proved that

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \dots & (1,n) \\ (2,1) & (2,2) & \dots & (2,n) \\ \dots & \dots & \dots & \dots \\ (n,1) & (n,2) & \dots & (n,n) \end{vmatrix} = \varphi(1)\varphi(2)\dots\varphi(n), \tag{1}$$

where (i,j) represents the greatest common divisor of i and j, and $\varphi(n)$ denotes the Euler totient function.

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The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \dots & f((1,n)) \\ f((2,1)) & f((2,2)) & \dots & f((2,n)) \\ \dots & \dots & \dots & \dots \\ f((n,1)) & f((n,2)) & \dots & f((n,n)) \end{bmatrix}.$$

There are quite a few generalized forms of GCD matrices, which can be found in several references [1], [3], [7], [8], [11].

H. J. S. Smith [12] also evaluated the determinant of

$$\big[[i,j]\big]_{n\times n} = \begin{bmatrix} [1,1] & [1,2] & \dots & [1,n] \\ [2,1] & [2,2] & \dots & [2,n] \\ \dots & \dots & \dots & \dots \\ [n,1] & [n,2] & \dots & [n,n] \end{bmatrix},$$

and proved that

$$\det \big[[i,j]\big]_{n\times n} = (n!)^2 g(1)g(2)\dots g(n) = \prod_{k=1}^{N} \varphi(k) \prod_{p|k} (-p).$$

where $g(n) = \frac{1}{n}\sum_{d|n} d\mu(d)$, $\mu(n)$ being the classical Möbius function.
The structure of an LCM matrix $\big[[i,j]\big]_{n\times n}$ is the following (I. Korkee, P. Haukkanen [10])

$$\big[[i,j]\big]_{n\times n} = AA^T$$

where $A = [a_{ij}]_{n\times n}$,

$$a_{ij} = \begin{cases} \sqrt{g(j)}, & \text{if } \ j \mid i \\ 0, & \text{if } \ j \nmid i \end{cases}.$$

The LCM matrix with respect to f is

$$[f[i,j]]_{n\times n} = \begin{bmatrix} f([1,1]) & f([1,2]) & \dots & f([1,n]) \\ f([2,1]) & f([2,2]) & \dots & f([2,n]) \\ \dots & \dots & \dots & \dots \\ f([n,1]) & f([n,2]) & \dots & f([n,n]) \end{bmatrix}.$$

Results concerning LCM matrices appear in papers S. Beslin [2], K. Bourque, S. Ligh [4], W. Feng, S. Hong, J. Zhao [6] P. Haukkanen, J. Wang and J. Sillanpää [7].

In this paper we study matrices which have as variables the least common multiple and the indices

$$\big[f(n,[i,j])\big]_{n\times n}=\begin{bmatrix} f(n,[1,1]) & f(n,[1,2]) & \dots & f(n,[1,n]) \\ f(n,[2,1]) & f(n,[2,2]) & \dots & f(n,[2,n]) \\ \dots & \dots & \dots & \dots \\ f(n,[n,1]) & f(n,[n,2]) & \dots & f(n,[n,n]) \end{bmatrix}$$

and the more general form matrices

$$\big[f(n,i,j,[i,j])\big]_{n\times n}=\begin{bmatrix} f(n,1,1,[1,1]) & f(n,1,2,[1,2]) & \dots & f(n,1,n,[1,n]) \\ f(n,2,1,[2,1]) & f(n,2,2,[2,2]) & \dots & f(n,2,n,[2,n]) \\ \dots & \dots & \dots & \dots \\ f(n,n,1,[n,1]) & f(n,n,2,[n,2]) & \dots & f(n,n,n,[n,n]) \end{bmatrix}$$

2. Generalized LCM matrices

Theorem 2.1. *For a given totally multiplicative arithmetical function $g(n)$ let*

$$f(n,[i,j])=g([i,j])\sum_{k\le\frac{n}{[i,j]}}g(k).$$

Then

$$\big[f(n,[i,j])\big]_{n\times n}=C_n^T\operatorname{diag}\big(g(1),\ g(2),\dots,g(n)\big)C_n, \tag{2}$$

where $C_n=[c_{ij}]_{n\times n}$

$$c_{ij}=\begin{cases}1, & \text{if } j\mid i\\ 0, & \text{if } j\nmid i\end{cases}.$$

For a determinant we have

$$\det\big[f(n,[i,j])\big]_{n\times n}=g(1)g(2)\dots g(n). \tag{3}$$

Proof. After multiplication, the general element of $A=(a_{ij})_{n\times n}$,

$$A=C_n^T\operatorname{diag}\big(g(1),\ g(2),\dots,g(n)\big)C_n$$

is

$$a_{ij}=\sum_{k=1}^{n}c_{ki}g(k)c_{kj}=\sum_{\substack{i\mid k\\ j\mid k\\ k\le n}}g(k)=\sum_{\substack{[i,j]\mid k\\ k\le n}}g(k)=\sum_{l\le\frac{n}{[i,j]}}g\big([i,j]l\big).$$

Because $g(n)$ is totally multiplicative

$$a_{ij} = g\big([i,j]\big) \sum_{\ell \le \frac{n}{[i,j]}} g(\ell) = f\big([i,j]\big).$$

If we calculate the determinant of both parts of (2) we have (3). □

Particular cases

Example 1. If $g(n) = 1$, then

$$f\big(n,[i,j]\big) = \left\lfloor \frac{n}{[i,j]} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x.
From Theorem 2.1 we have

$$\left[\left\lfloor \frac{n}{[i,j]} \right\rfloor\right]_{n\times n} = C_n^T \operatorname{diag}\big(1,1,\dots,1\big) C_n, \qquad \det\left[\left\lfloor \frac{n}{[i,j]} \right\rfloor\right]_{n\times n} = 1.$$

Example 2. If $g(n) = n$, then

$$f\big(n,[i,j]\big) = \frac{\left\lfloor \frac{n}{[i,j]} \right\rfloor \left\lfloor \frac{n}{[i,j]} + 1 \right\rfloor}{2}.$$

The decomposition of generalized LCM matrix is

$$\big[f(n,[i,j])\big]_{n\times n} = C_n^T \operatorname{diag}\big(1,\ 2,\dots,n\big) C_n,$$

and the determinant

$$\det\big[f(n,[i,j])\big]_{n\times n} = n!.$$

Example 3. If $g(n) = (-1)^{\Omega(n)}$ is a Liouville function, then

$$f\big(n,[i,j]\big) = (-1)^{\Omega\left([i,j]\right)} \sum_{k \le \frac{n}{[i,j]}} (-1)^{\Omega(k)}$$

and

$$\big[f(n,[i,j])\big]_{n\times n} = C_n^T \operatorname{diag}\big(1,\ -1,\dots,(-1)^{\Omega(n)}\big) C_n,$$
$$\det\big[f(n,[i,j])\big]_{n\times n} = (-1)^{\sum_{k=1}^{n} \Omega(k)}.$$

We remark that matrices related to the greatest integer function appeared in [9], [5].

Theorem 2.2. *For a given totally multiplicative function g let*

$$f(n,i,j,[i,j]) = \sum_{k\le n} g(k) - g(i)\sum_{l\le \frac{n}{i}} g(l) - g(j)\sum_{l\le\frac{n}{j}} g(l) + g([i,j])\sum_{k\le\frac{n}{[i,j]}} g(k).$$

Then

$$\big[f(n,i,j,[i,j])\big]_{n\times n} = D_n^T \operatorname{diag}[g(1),g(2),\dots,g(n)]D_n,$$

where $D_n = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & \text{if } \ j\nmid i \\ 0, & \text{if } \ j \mid i \end{cases}.$$

Proof. After multiplication the general element of the matrix

$$A = [a_{ij}]_{n\times n} = D_n^T \operatorname{diag}[g(1),g(2),\dots,g(n)]D_n$$

is

$$\begin{aligned} a_{ij} &= \sum_{\substack{i\nmid k\\ j\nmid k\\ k\le n}} g(k) = \sum_{k\le n} g(k) - \sum_{i|k} g(k) - \sum_{j|k} g(k) + \sum_{\substack{i|k\\ j|k\\ k\le n}} g(k)\\ &= \sum_{k\le n} g(k) - \sum_{il\le n} g(il) - \sum_{jl\le n} g(jl) + \sum_{\substack{[i,j]|k\\ k\le n}} g(k) \end{aligned}$$

The total multiplicativity of g implies,

$$\begin{aligned} a_{ij} &= \sum_{k\le n} g(k) - g(i)\sum_{l\le\frac{n}{i}} g(l) - g(j)\sum_{l\le\frac{n}{j}} g(l) + g([i,j])\sum_{k\le\frac{n}{[i,j]}} g(k)\\ &= f(n,i,j,[i,j]). \end{aligned}$$ □

Particular cases

Example 4. If $g(n)=1$, then

$$f\big(n,i,j,[i,j]\big) = \tau(n) - \tau\left(\left\lfloor\frac{n}{i}\right\rfloor\right) - \tau\left(\left\lfloor\frac{n}{j}\right\rfloor\right) + \left\lfloor\frac{n}{[i,j]}\right\rfloor,$$

where $\tau(n) = \sum_{d|n} 1$. By Theorem 2.2

$$\left[f\left(n,i,j,\left\lfloor\frac{n}{[i,j]}\right\rfloor\right)\right]_{n\times n} = D_n^T \operatorname{diag}\big(1,\ 1,\dots,1\big)D_n.$$

Example 5. If $g(n) = n$, then

$$f\big(n,i,j,[i,j]\big) = \sigma(n) - \sigma\left(\left\lfloor \frac{n}{i} \right\rfloor\right) - \sigma\left(\left\lfloor \frac{n}{j} \right\rfloor\right) + \frac{\left\lfloor \frac{n}{[i,j]} \right\rfloor \left\lfloor \frac{n}{[i,j]} + 1 \right\rfloor}{2},$$

where $\sigma(n) = \sum_{d|n} d$.

The general form of a generalized LCM matrix is

$$\big[f(n,i,j,[i,j])\big]_{n\times n} = D_n^T \operatorname{diag}\big(1,2,\dots,n\big) D_n.$$

Example 6. If $g(n) = (-1)^{\Omega(n)}$ is the Liouville function then

$$f\big(n,i,j,[i,j]\big) = \sum_{k\le n}(-1)^{\Omega(k)} - (-1)^{\Omega(i)} \sum_{l\le \frac{n}{i}}(-1)^{\Omega(l)} - (-1)^{\Omega(j)} \sum_{l\le \frac{n}{j}}(-1)^{\Omega(l)} g$$
$$+ (-1)^{\Omega\big([i,j]\big)} \sum_{k\le \frac{n}{[i,j]}} (-1)^{\Omega(k)}$$

and

$$\big[f(n,i,j,[i,j]\big]_{n\times n} = D_n^T \operatorname{diag}\big(1,-1,\dots,(-1)^{\Omega(n)}\big) D_n.$$

Remark 2.1. Due to the fact that the first line of the matrix $[f(n,i,j,[i,j])]_{n\times n}$ contains only 0-s, the determinant of the matrix will always be 0.

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ANTAL BEGE
DEPARTMENT OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF SAPIENTIA
P.O. 9, P. O. BOX 4
RO-540485 TÎRGU MUREŞ
ROMANIA

E-mail: abege@ms.sapientia.ro

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