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# $q$-series: dimension estimates, linear independence

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*To Kálmán Győry, Attila Pethő, János Pintz, András Sárközy:*
*Ad multos annos!*

**Abstract.** Entire transcendental solutions $f$ of functional equations $f(q^m z) = R_0(z)f(z) + R_1(z)$ with polynomial coefficients $R_0$, $R_1$ are arithmetically studied. The purpose of this note is to report on recent progress on lower bounds for the dimension of the $K$-vector space generated by 1 and the values of these $f$ and their derivatives at $m$ successive powers of $q$, where $K$ is $\mathbb{Q}$ or an imaginary quadratic number field. In favorable circumstances, linear independence can be obtained, even in a quantitative form.

## 1. Introduction and main result

The aim of this note is to report on generalizations of earlier linear independence results on the values of entire functions $f$ satisfying a functional equation of the form

$$f(q^m z) = R_0(z)f(z) + R_1(z), \tag{1}$$

where $m \in \mathbb{N} := \{1, 2, \dots\}$, $R_0, R_1 \in K[z]$ and $q \in K$, $K$ *denoting always either* $\mathbb{Q}$ *or an imaginary quadratic number field.* It was shown in [9] that $f$ can be linearly expressed by 1 and certain basic hypergeometric series. In [7] and [9], estimates for the dimension of the $K$-vector space spanned by 1 and the $f(\alpha q^{-\mu}), 0 \le \mu < m$, with $\alpha \in K^\times$ were proved and, in [17] the corresponding dimension estimate in

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the case $m = 1$ was obtained for 1 and the $f^{(\kappa)}(\alpha), 0 \le \kappa < k$. Note also that a $p$-adic analogue of [9] is established in [10].

The main result of the present note is the following statement.

**Theorem 1.** *Assume that $q \in K$ with $|q| > 1$ is the quotient $u/v$ of non-zero $u, v \in O_K$, the ring of integers of $K$, and let $\eta := (\log|v|)/(\log|u|)$. Suppose that $f$ is an entire transcendental solution of the functional equation* (1) *with $R_0, R_1 \in K[z]$, $\deg R_0 =: \ell \in \mathbb{N}$, with $f(0) = 1$ if $R_1(0) = 0$ and $R_0(0) \in q^{-\mathbb{N}}$ if $R_1(0) \neq 0$. Let $\alpha \in K^\times$ satisfy the conditions $R_0(\alpha q^{-j}) \neq 0$ for any integer $j \ge m$. Then the following dimension estimate holds*

$$\dim_K \left\{ K + \sum_{\kappa=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\kappa)}(\alpha q^{-\mu}) \right\} \ge (1-\eta) C(k, \ell, m),$$

*where*

$$C(k, \ell, m) := \frac{km((km+1)^2 - k\ell m) + \sqrt{\Delta}}{2k\ell m(km + 2 + 6\pi^{-2}(k-1)m))} \tag{2}$$

*with*

$$\Delta := k^2 m^2 ((km+1)^2 - k\ell m)^2 + 4k\ell m(km+1)^2(km + 2 + 6\pi^{-2}(k-1)m).$$

Note that one may tacitly *always suppose that $u, v$ have only units from $O_K$ as common divisors.* Namely, otherwise the corresponding quotient $\eta'$, say, would satisfy $1 - \eta > 1 - \eta'$ and the dimension estimate would become unnecessarily worse.

After some minor calculation, one sees that expression (2) can be bounded below by

$$C(k, \ell, m) > \frac{km}{\ell(1 + 6\pi^{-2}(1 - 1/k))} - 1. \tag{3}$$

Moreover, it should be pointed out that the choice $k = 1$ and $\eta = 0$ ($\Leftrightarrow q \in O_K$) gives the Main Theorem in [9], whereas, in the case $m = 1$, Theorem 1 yields improvements on Töpfer's Theorems 1, 2 and 3 in [17].

The last remark for the moment refers to a closed form of the entire solution of (1) under the conditions of Theorem 1. Namely, iterating this functional equation, it is easily seen that any solution $f$ of (1) satisfies also equation[1]

$$f(z) = f(zq^{-Jm}) \prod_{j=1}^{J} R_0(zq^{-jm}) + \sum_{j=1}^{J} R_1(zq^{-jm}) \prod_{i=1}^{j-1} R_0(zq^{-im}) \tag{4}$$

[1] As usual, empty products and sums have always to be interpreted as 1 or 0, respectively.

for any $J \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By the assumptions of Theorem 1, one can write $R_0(z) = q^{-t}P(zq^m)$ with $P \in K[z]$, $P(0) = 1$, $t \in \mathbb{N}_0$, where $t = 0$ if and only if $R_1(0) = 0$. Letting $J \to \infty$, (4) leads to

$$f(z) = \prod_{j=0}^{\infty} P(zq^{-jm}) + \sum_{j=0}^{\infty} R_1(zq^{-(j+1)m}) \prod_{i=0}^{j-1} P(zq^{-im})$$

if $R_1(0) = 0$, whereas in case $R_1(0) \neq 0$ to

$$f(z) = \sum_{j=0}^{\infty} R_1(zq^{-(j+1)m}) q^{-jt} \prod_{i=0}^{j-1} P(zq^{-im}).$$

At the beginning of the next section, some explicit examples of functions of the type $f$ discussed here will follow. These functions appeared here and there in irrationality investigations during the last fifteen years, and the main aim of the section is to apply Theorem 1 to these special functions, and to compare the results with the preceding ones. In the third section, the particular case $R_1 = 0$ of equation (1) will be studied in more detail: In this case, linear independence assertions can be obtained under suitable additional conditions, even in a quantitative form. The last section will be devoted to a short sketch of the rather technical proof of Theorems 1 and 2 which will appear in full elsewhere.

## 2. Some 'irrationality' questions

Suppose $P \in K[z]$ with $P(0) = 1$ and $\deg P = \ell \in \mathbb{N}$, and let $m \in \mathbb{N}$. Then define the infinite product

$$f_{0,m}(z) := \prod_{j=0}^{\infty} P(zq^{-jm}), \tag{5}$$

and, for $h \in \mathbb{N}$, the infinite series

$$f_{h,m}(z) := \sum_{j=0}^{\infty} q^{-hmj} \prod_{i=0}^{j-1} P(zq^{-im}). \tag{6}$$

All these functions are entire and satisfy the functional equation (1) with $R_0(z) = q^{-hm}P(zq^m), R_1(z) = 1 - \delta_{h,0}$ for every $h \in \mathbb{N}_0$, $\delta$ denoting Kronecker's symbol. By an easy degree consideration, it follows from (1) that none of the $f_{h,m}$ can be a polynomial.

Taking, in particular, $P(z) = 1 + z$, the infinite product $f_{0,1}$ in (5) becomes the so-called $q$-exponential function

$$E_q(z) := \prod_{j=0}^{\infty}(1 + zq^{-j}), \tag{7}$$

one of the most prominent $q$-functions. On $E_q$, Lototsky [12] proved already in 1943 that, for every $\alpha \in K^\times$ with $-\alpha \notin q^{\mathbb{N}_0}$, one has $E_q(\alpha) \notin K$. Whereas $P(z) = 1 + az$ with $a \in K^\times$ gives nothing really new, it is obvious that $f_{0,1}$ provides interesting new objects for irrationality investigations if $\deg P = \ell \geq 2$.

But already in the smallest case $\ell = 2$, hence $P(z) = 1 + a_1 z + a_2 z^2$ with $a_1, a_2 \in K^\times$, one feels serious difficulties. Namely, generalizing work of Zhou and Lubinsky [23], one of the present authors [5] could demonstrate $f_{0,1}(1) \notin K$ if $q \in O_K$ satisfies the 'natural' conditions $P(q^{-j}) \neq 0$ for any $j \in \mathbb{N}_0$ *but additionally* the 'technical' condition $|q| > q_0(a_1, a_2)$, where the dependence on $a_1$, $a_2$ is effective. Alternatively one obtained an exclusion result of the type that not both of the numbers $f_{0,1}(1), f_{0,1}(-1)$ are in $K$ if $q \in O_K$ with $|q| > 1$ satisfies $P(q^{-j})P(-q^{-j}) \neq 0$ for any $j$ as above. Notice that, in spite of the paper [20], the problem $f_{0,1}(1) \notin K$ if $q \in O_K$ satisfies only the 'natural' conditions remains open.

Whereas the method used in [23] and [20] arose from explicit formulae for multivariate Padé approximants, the proofs of [5] rest on Newton's interpolation series. The reader may ask what happens when applying Theorem 1 to the problem of getting Lototsky-type analogues for products over polynomials $P$ of degree $\ell$ $(\geq 2)$. So, assume $q \in O_K$ $(\Leftrightarrow \eta = 0), |q| > 1$, and $P(q^{-j}) \neq 0$ for any $j \in \mathbb{N}_0$ which, by $R_0(\alpha q^{-j}) = q^{-hm}P(\alpha q^{m-j})$, just ensures the hypotheses on $\alpha$ in Theorem 1 if choosing $\alpha = 1$. Applying Theorem 1 with $k = 1, m = 1$, the following inequality

$$\dim_K K + Kf_{0,1}(1) \geq C(1, \ell, 1) \tag{8}$$

is obtained. Here $C(1, 1, 1) = (3 + \sqrt{57})/6 = 1.758\ldots$ yields $f_{0,1}(1) \notin K$, an equivalent version of Lototsky's result. The fact $C(1, 2, 1) = 1$ is particularly annoying since it means that inequality (8) *just* fails to give $f_{0,1}(1) \notin K$ for $\ell = 2$. Since $C(1, \ell, 1)$ is strictly decreasing, inequality (8) does not yield an 'irrationality' assertion for $\ell \geq 3$ either. *Even worse:* As the following result shows, doubts about the validity of a Lototsky-type analogue for products over higher degree polynomials are advisable.

**Proposition 1.** *With $q \in K$, $|q| > 1$, let $P(z)$ denote the polynomial $(1+z)(1 - qz^2)$ of degree 3. Then the 'natural' conditions $P(q^{-j}) \neq 0$ for any $j \in \mathbb{N}_0$*

*are satisfied but, for the corresponding* $f_{0,1}$ *from* (5), *one has* $f_{0,1}(1) \in K$. *In fact,* $f_{0,1}(1) = P(1) = 2(1-q)$ *holds.*

Proof. A glance at Aufgabe 19 on p. 3 of [15] shows the Eulerian identity

$$\prod_{j=1}^{\infty}(1+q^{-j}) = \prod_{j=1}^{\infty}(1-q^{1-2j})^{-1}$$

for any $q \in \mathbb{C}$ with $|q| > 1$, and this is equivalent to $\prod_{j=1}^{\infty} P(q^{-j}) = 1$, whence the assertion. □

Feeling the difficulties, alluded to above, to get definite 'irrationality' results on $f_{0,1}$ from (5) if $P \in K[z]$ with $P(0) = 1$ satisfies $\deg P \geq 2$, several authors proved exclusion results of the subsequent type which will turn out to be simple corollaries to Theorem 1.

**Proposition 2.** *Let* $q \in O_K$, $|q| > 1$, *and assume that* $P \in K[z]$ *with* $\deg P = \ell \geq 2$ *satisfies* $P(0) = 1$, $P(q^{-j}) \neq 0$ *for any* $j \in \mathbb{N}_0$. *If* $h, m \in \mathbb{N}_0$, $m \geq 2(\ell-1)$, *then at least one among the numbers*

$$f_{h,m}(1), f_{h,m}(q^{-1}), \ldots, f_{h,m}(q^{-(m-1)}) \tag{9}$$

*does not belong to* $K$.

Proof. Notice first that $m \geq 2(\ell-1)$ implies

$$m + \frac{1}{m} > 2(\ell-1). \tag{10}$$

Application of Theorem 1 with $q \in O_K$ (hence $\eta = 0$), $\alpha = 1, k = 1$ leads to

$$\dim_K \left\{ K + \sum_{\mu=0}^{m-1} K f(q^{-\mu}) \right\} \geq \frac{m\big((m+1)^2 - \ell m\big) + \sqrt{\Delta_0}}{2\ell m(m+2)}. \tag{11}$$

with $\Delta_0 := m^2\big((m+1)^2 - \ell m\big)^2 + 4\ell m(m+1)^2(m+2)$. Here a careful but elementary calculation shows that the right-hand side of (11) is greater than 1 if and only if (10) holds, whence the claim of the proposition. □

Under the stronger hypotheses $q \in \mathbb{N}, q > 1, P \in \mathbb{Q}_+[z]$ (fairly implying $P(q^{-j}) \neq 0$ for any $j \in \mathbb{N}_0$), $P(0) = 1$, and $m \geq \ell^2 - 2$, Zhou [22] proved via Padé approximations that, in case $h = 0$, at least one among the real numbers (9) is irrational. By simply applying his old analytic 'irrationality' criterion [4] based on Newton's interpolation series, the first-named author [6] sharpened Zhou's result to $m \geq \ell(\ell-1)$ under the assumptions of Proposition 2 on $q$ and $P$. The results there concern the 'product' case $h=0$ as well as the 'series' case $h=1$ from (6). Again for $h=1$ and polynomials $P$ of degree 2, the corresponding result under the hypothesis $m \geq 2$ is due to Zhou [21], who comments briefly (p. 439) on the incomplete proof in [3] of the irrationality of $f_{1,1}(1)$ if $q \in \mathbb{N}, q>1$, and $P \in \mathbb{Q}_+[z]$ with $P(0)=1$ has degree 2.

To conclude this section, it should be also noted that Choi and Zhou [11] proved a lower bound for

$$\dim_{\mathbb{Q}}\left\{\mathbb{Q}+\sum_{\mu=0}^{m-1}\mathbb{Q}f(q^{-\mu})\right\}$$

similar to (11) if $q \in \mathbb{Q}$, $q>1$, $P \in \mathbb{Q}_+[z]$, $P(0)=1$, $\deg P=\ell \geq 2$. In the particular case $q \in \mathbb{N}$, their bound becomes non-trivial (i.e., greater than 1) only for $m \geq \max(2,(\ell^2-2+\ell\sqrt{\ell^2-4})/2)$. Here the lower bound for $m$ is again quadratic in $\ell$, in contrast to the corresponding inequality of Proposition 2.

## 3. The homogeneous case

In the homogeneous case $R_1=0$ of the functional equation (1), one can get a better lower bound for the dimension of a slightly smaller $K$-vector space than the one from Theorem 1.

**Theorem 2.** *If the assumptions of Theorem 1 are valid and, moreover,* $R_1=0$ *holds, then one has the dimension estimate*

$$\dim_K\left\{\sum_{\kappa=0}^{k-1}\sum_{\mu=0}^{m-1}Kf^{(\kappa)}(\alpha q^{-\mu})\right\} \geq (1-\eta)\widetilde{C}(k,\ell,m),$$

*where*

$$\widetilde{C}(k,\ell,m)=\frac{k^2m-k\ell+\sqrt{(k^2m-k\ell)^2+4k\ell(k+6\pi^{-2}(k-1))}}{2\ell(k+6\pi^{-2}(k-1))}. \tag{12}$$

Note that, in the case $m=1$, Theorem 2 improves the homogeneous parts of Theorems 1 and 2 in [17]. Note also that one can obtain for $\widetilde{C}(k,\ell,m)$ the same lower bound

$$\widetilde{C}(k,\ell,m) > \frac{km}{\ell(1+6\pi^{-2}(1-1/k))}-1 \tag{13}$$

as for $C(k,\ell,m)$ in (3).

Since $\widetilde{C}(2,1,1) = 1.339\ldots$, by (12), one can deduce from Theorem 2 that $E_q(\alpha), E'_q(\alpha)$ are linearly independent over $K$ if $q \in O_K$, $|q| > 1$, $\alpha \in K^\times$, $-\alpha \notin q^{\mathbb{N}_0}$. Defining the $q$-logarithmic function $L_q$ as the logarithmic derivative of the $q$-exponential function $E_q$ from (7), i.e., as

$$L_q(z) := \sum_{j=0}^{\infty} \frac{1}{q^j + z},$$

then the before-mentioned result means $L_q(\alpha) \notin K$ if $q$ and $\alpha$ satisfy the above conditions. Clearly, in the special case $K = \mathbb{Q}$, this is just Borwein's [2] famous result.

Here it is remarkable to take a step more. Namely, by $C(2,1,1) = 2.005\ldots$, one can deduce from Theorem 1 even that $1, E_q(\alpha), E'_q(\alpha)$ are linearly independent over $K$ under the above assumptions on $q$ and $\alpha$. But it should be noticed, that a result of Bézivin [1, Corollaire 1] contains the $K$-linear independence of

$$1, E_q(\alpha_1), \ldots, E_q^{(k-1)}(\alpha_1), \ldots, E_q(\alpha_m), \ldots, E_q^{(k-1)}(\alpha_m)$$

for arbitrary $k \in \mathbb{N}$, where $\alpha_1, \ldots, \alpha_m$ are $m \in \mathbb{N}$ distinct numbers from $K^\times$, not in $-q^{\mathbb{N}_0}$ and satisfying an additional but very 'natural' condition which is void for $m = 1$. By the way, Bézivin's method, entirely different from the one used to prove Theorems 1, 2 (and 3 below), highly depends on the particular form

$$\sum_{j=0}^{\infty} \frac{q^j z^j}{\prod_{i=1}^{j}(q^i - 1)}$$

of the Taylor series of $E_q(z)$ about the origin.

Whereas the last few applications concerned small values of $k, \ell, m$, the following one relates to arbitrary $m \in \mathbb{N}$. Namely, (13) implies $\widetilde{C}(1,1,m) > m-1$, whence, by Theorem 2,

$$\dim_K \left\{ \sum_{\mu=0}^{m-1} K f(\alpha q^{-\mu}) \right\} > m - 1$$

if $\eta = 0$. Thus, if $q \in O_K$ and $k = \ell = 1$ in this homogeneous case, one obtains the linear independence over $K$ of the $m$ numbers $f(\alpha q^{-\mu}), 0 \le \mu < m$. By a different method, the second-named author [19] even obtained the $K$-linear independence of all numbers 1 and $f(\alpha q^{-\mu}), 0 \le \mu < m$, and moreover, a linear independence measure for them in the general case of (1) with $k$, $\ell$, $\eta$ as before. Here the fact that, in the case $\ell = 1$, the solutions $f$ of (1) are somehow related to the solutions of functional equations of type

$$z^s f(z) = S_0(z) f(qz) + S_1(z) \tag{14}$$

with $s \in \mathbb{N}$ and polynomials $S_0$, $S_1$ is used. However, no similar interrelation is known for $\ell \geq 2$. Generally speaking, the arithmetic behavior of the solutions of type (1) functional equations seems to be more complicated than the one of solutions of type (14). In this second case, very general and precise results are available. For a survey, the reader may be referred to [18].

The main tool in the proofs of Theorems 1 and 2 is Nesterenko's dimension estimate [14] and its generalization to arbitrary algebraic number fields due to Töpfer [16], compare Section 4. Whereas these dimension estimates led only to qualitative results, Töpfer and the first-named author [8], [16] showed the following. Essentially in all situations, where one can deduce from Nesterenko-type estimates linear independence of numbers over the algebraic number field under consideration, one can write down, with not much additional expense, measures for this linear independence.

An example is the following quantitative version of the last-mentioned linear independence result.

**Theorem 3.** *Let the assumptions of Theorem 1 be valid, and assume furthermore* $\deg R_0 = 1$, $R_1 = 0$, $q \in O_K, m \geq 2$. *Then, for every* $\varepsilon \in \mathbb{R}_+$, *there exists a constant* $C_0 \in \mathbb{R}_+$ *depending at most on* $q$, $m$, $R_0'(0)$, $\alpha$, $\varepsilon$ *such that, for every* $\underline{\Lambda} := (\lambda_0, \dots, \lambda_{m-1}) \in O_K^m$ *with* $|\underline{\Lambda}| := \max(|\lambda_0|, \dots, |\lambda_{m-1}|) \geq C_0$, *the following inequality holds*

$$\big| \sum \lambda_\mu f(\alpha q^{-\mu}) \big| \geq |\underline{\Lambda}|^{-(\psi(m)+\varepsilon)} \tag{15}$$

*where* $\psi(m) := (m-1)\big(\sqrt{(m-1)^2+4} + m - 1\big)/2$.

Notice that if all hypotheses of Theorem 3 hold and, moreover, $m = 2$, then the quotient $f(\alpha)/f(\alpha/q)$ does not belong to $K$, and has 'irrationality' exponent not greater than $1 + \psi(2) = (3 + \sqrt{5}\,)/2 = 2.618\dots$.

It should be pointed out that, in a recent paper, Matala-aho [13] obtained very strong quantitative linear independence results on three special functions. The third one of these implies a measure for 1 and the $E_{q^m}(\alpha q^{-\mu})$, $0 \leq \mu < m$, for the $q$-exponential function $E$ from (7) and every $\alpha \in K^\times \setminus q^{m\mathbb{Z}}$. Here not only the number 1 is included in contrast to the situation in Theorem 3 but also the exponent $\psi(m)$ in (15) is asymptotically $m^2$, whereas Matala-aho's is linear in $m$.

## 4. Sketch of proofs

As already indicated in the preceding section, the basic ingredient of the proofs of Theorems 1 and 2 is the following

**Nesterenko-type dimension estimate.** *Let $\mathbb{E}$ be $\mathbb{R}$ or $\mathbb{C}$ according as $K$ is $\mathbb{Q}$ or an imaginary quadratic field. Further, let $d \in \mathbb{N}$, $d \geq 2$ and $\underline{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{E}^d \setminus \{\underline{0}\}$. Finally, assume that there exist $N_0 \in \mathbb{N}$, $\tau \in \mathbb{R}_+$, an unbounded increasing function $F : \mathbb{N} \to \mathbb{R}_+$, and a sequence $(\Lambda_N)_{N \geq N_0}$ of linear forms over $O_K$ in $d$ variables with*

(i) $\limsup_{N\to\infty} F(N+1)/F(N) \leq 1$,

(ii) $\log |\underline{\Lambda}_N| \leq F(N)$ *for every* $N \geq N_0$,

(iii) $\log |\Lambda_N(\underline{\omega})| = -(\tau + \mathrm{o}(1))F(N)$ *for every large* $N \geq N_0$,

*where $|\underline{\Lambda}_N|$ denotes the maximum norm of the coefficient vector of the linear form $\Lambda_N$. Then the following dimension estimate holds*

$$\dim_K K\omega_1 + \cdots + K\omega_d \geq 1 + \tau.$$

In the case $K = \mathbb{Q}$, and hence $\mathbb{E} = \mathbb{R}$, this result is Nesterenko's [14]. If $K$ is imaginary quadratic, it is Töpfer's Korollar 2 in [16] combined with his remark after Korollar 7 that, in case of such particular algebraic number fields, one may replace the field degree $[K : \mathbb{Q}] = 2$ by 1.

It is obviously enough to prove Theorems 1, 2 (and 3) only for $\alpha = 1$. To apply the above dimension estimate, one has to construct an infinite sequence of linear forms over $O_K$ in[2]

$$1, f(1), \ldots, f^{(k-1)}(1), f(q^{-1}), \ldots, f^{(k-1)}(q^{-1}), \ldots,$$
$$f(q^{-(m-1)}), \ldots, f^{(k-1)}(q^{-(m-1)}) \quad (16)$$

(indexed by $N$, say), whose absolute values tend sufficiently rapidly to zero as $N \to \infty$, whilst the maximum of the absolute values of their coefficients do not increase too quickly with $N$.

Before indicating the construction of such a sequence, the following easy consequence of (1) should be noticed for the Taylor series $\sum_{n=0}^{\infty} c_n z^n$ of $f(z)$. Since $f$ is not a polynomial, for every integer $n > \max(\ell, \deg R_1)$, not all $c_{n-1}, \ldots, c_{n-\ell}$ can vanish. But much more than this fact can be deduced from the well-known growth behavior of $|f|_r := \max_{|z|=r} |f(z)|$ for entire transcendental solutions $f$

[2] For Theorem 2, the number 1 has to be omitted.

of (1). Namely, for any large integer $n$, there is an $\tilde{n} \in \{n-1, \ldots, n-\ell\}$ such that $|c_{\tilde{n}}|$ essentially reaches its largest possible size allowed by the size of $|f|_r$.

To construct now a sequence suitable for the present purposes, one considers complex integrals of type

$$I(N) := \frac{1}{2\pi i} \oint\limits_{\Gamma(N)} \frac{f(q^G z)dz}{z^L \prod_{j=0}^{M+\beta_N+1} (z-q^j)^{k_j}}, \tag{17}$$

where $L \geq 0$, $M > 0$ and $G$ are integer parameters depending on $N$, suitable $\beta_N \in \{0, \ldots, \ell-1\}$ having to do with the previous remark on $\ell$ successive Taylor coefficients of $f$, and with $k_j = k$ (the derivation order of $f$) if $0 \leq j \leq M$, and $k_j = 1$ if $M < j \leq M + \beta_N + 1$. Of course, $\Gamma(N)$ denotes a sufficiently large circle centered at the origin.

For the asymptotic evaluation of $|I(N)|$, a very careful analysis is necessary. The basic point here is to show that the absolute values of $I(N)$ and of

$$\frac{1}{2\pi i} \oint\limits_{\Gamma(N)} \frac{f(q^G z)dz}{z^{L+k(M+1)+\beta_N+1}} = q^{G(L+k(M+1)+\beta_N)} c_{L+k(M+1)+\beta_N}$$

are asymptotically equal. So far the analytic preparations.

For the arithmetic parts of the proof, one first evaluates $I(N)$ from (17) by the residue theorem, and then expresses the arising derivatives of $f$ of orders less than $k$ at integral powers of $q$ by the $k \cdot m$ numbers $f^{(\kappa)}(q^{-\mu})$ $(0 \leq \kappa < k, 0 \leq \mu < m)$ via iteration and differentiation of (1). Thus, one is led to linear forms in 1 and these $f^{(\kappa)}(q^{-\mu})$ with explicitly known coefficients in $K$. These have to be transformed into $O_K$-linear forms on multiplying by appropriate $\Omega(N) \in O_K \setminus \{0\}$ whose growth with $N$ has to be precisely controlled. Checking all details it comes out that the constructed sequence of $O_K$-linear forms $\Omega(N)I(N)$ in the numbers (16) tends to zero very rapidly, and thus all conditions of the Nesterenko-type dimension estimate are fulfilled yielding Theorems 1 and 2.

## References


[1] J.-P. Bézivin, Indépendance linéaire des valeurs des solutions transcendantes de certaines équations fonctionnelles, *Manuscripta Math.* **61** (1988), 103–129.

[2] P. B. Borwein, On the irrationality of $\sum(1/(q^n + r))$, *J. Number Theory* **37** (1991), 253–259.

[3] P. B. Borwein and P. Zhou, On the irrationality of $\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i}(1+q^{-j}r+q^{-2j}s)$, in: Approximation Theory IX, Vol. 1, pp. 51–58. Innov. Appl. Math., *Vanderbilt Univ. Press, Nashville, TN*, 1998.

[4] P. Bundschuh, Ein Satz über ganze Funktionen und Irrationalitätsaussagen, *Invent. Math.* **9** (1970), 175–184.

[5] P. Bundschuh, Again on the irrationality of a certain infinite product, *Analysis (Munich)* **19** (1999), 93–101.

[6] P. Bundschuh, Note on the irrationality of certain multivariate $q$-functions, *Taiwanese J. Math.* **10** (2006), 603–611.

[7] P. Bundschuh, Arithmetical results on certain $q$-series, I, *Int. J. Number Theory* **4** (2008), 25–43.

[8] P. Bundschuh and T. Töpfer, Über lineare Unabhängigkeit, *Monatsh. Math.* **117** (1994), 17–32.

[9] P. Bundschuh and K. Väänänen, Arithmetical results on certain $q$-series, II, *Int. J. Number Theory* **5** (2009), 1231–1245.

[10] P. Bundschuh and K. Väänänen, An application of Nesterenko's dimension estimate to $p$-adic $q$-series, *Int. J. Number Theory* **7** (2011), 431–447.

[11] S. Choi and P. Zhou, On linear independence of a certain multivariate infinite product, *Canad. Math. Bull.* **51** (2008), 32–46.

[12] A. V. Lototsky, Sur l'irrationalité d'un produit infini, *Mat. Sbornik* **12**(54) (1943), 262–271.

[13] T. Matala-aho, On $q$-analogues of divergent and exponential series, *J. Math. Soc. Japan* **61** (2009), 291–313.

[14] Yu. V. Nesterenko, On the linear independence of numbers, *Vestn. Mosk. Univer. Ser. I* **40** (1985), 46–49, Engl. transl.: *Mosc. Univ. Math. Bull.* **40** (1985), 69–74 (in *Russian*).

[15] G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis, I, 4. Aufl., *Springer, Berlin – Heidelberg – New York*, 1970.

[16] T. Töpfer, Über lineare Unabhängigkeit in algebraischen Zahlkörpern, *Results Math.* **25** (1994), 139–152.

[17] T. Töpfer, Arithmetical properties of functions satisfying $q$-difference equations, *Analysis* **15** (1995), 25–49.

[18] K. Väänänen, On arithmetic properties of $q$-series, in: Analytic Number Theory and Related Areas, (by K. Kawada, ed.), *RIMS Kokyuroku* ***1665***, 2009, 47–59.

[19] K. Väänänen, Remarks on linear independence of certain $q$-series, *Fundam. Prikl. Mat.* **16** (2010), 41–47.

[20] P. Zhou, On the irrationality of the infinite product $\prod_{j=0}^{\infty}(1+q^{-j}r+q^{-2j}s)$, *Math. Proc. Camb. Phil. Soc.* **126** (1999), 387–397, Erratum: **139** (2005), 563.

[21] P. Zhou, On the irrationality of a certain series of two variables, in: Approximation Theory XI, Modern Methods Math., *Nashboro Press, Brentwood, TN*, 2005.

[22] P. Zhou, On the irrationality of a certain multivariate infinite product, *Quaest. Math.* **29** (2006), 351–365.

[23] P. Zhou and D. S. Lubinsky, On the irrationality of $\prod_{j=0}^{\infty}(1 \pm q^{-j}r + q^{-2j}s)$, *Analysis* **17** (1997), 129–153.

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