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The least nonzero digit of $n!$ in base 12

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To Kálmán Győry, Attila Pethő, János Pintz and András Sárközy, with respect and friendship, for their joint 260*th birth anniversary*

Abstract. We positively answer a question raised by the first author and prove that, for $1 \le a \le 11$, the sequence $\{n : \ell_{12}(n!) = a\}$ has an asymptotic density, which is $1/2$ if $a = 4$ or $a = 8$ and 0 otherwise; here $\ell_b(m)$ denotes the least nonzero digit of m in base b.

1. Introduction

In [3], Dresden has studied, among others, the sequence of the least nonzero digit of $n!$ in base 10, and showed that the decimal number written in base 10 by concatenating those digits one after the other is transcendental. More precisely, if we write a positive integer m in the base b as

$$m = \sum_{k \le \log_b m} \epsilon_k(m) b^k, \quad \text{with} \quad \epsilon_k(m) \in \{0, 1, \dots b-1\}, \tag{1}$$

the *least nonzero digit of* m *in the base* b, denoted by $\ell_b(m)$, is the number $\epsilon_h(m)$, where $\epsilon_h(m) \neq 0$, whereas $\epsilon_k(m) = 0$ for any $k < h$. For example, $\ell_{10}(403000) = 3$.

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Although Dresden does not phrase it this way, the key point in his proof that the number $\sum \ell_{10}(n!)/10^n$ is transcendental is the fact that the sequence $(\ell_{10}(n!))$ is 5-*automatic*, but not periodic. We refer the reader to the excellent monograph [1] by ALLOUCHE and SHALLIT for the definition of automatic sequences and their properties, including the transcendence of power series they generate.

One important feature of the sequence $(n!)$ used by Dresden is the fact that

$$\text{for } n \geq 2, \quad 5^k \mid n! \Rightarrow 2^{k+1} \mid n!. \tag{2}$$

Due to analogues of this property, Dresden's result extends to a majority of bases; for example, if b is squarefree and p is the largest prime factor of b, then the sequence $(\ell_b(n!))$ is p-automatic.

In this paper, we study a special case when we cannot guarantee the automaticity of the sequence $(\ell_b(n!))$, namely the case $b = 12$. In this case, there is no systematic relation like (2); indeed, for $n = 3^m$ and $k = (n-1)/2$, we have $3^k \mid n!$ and $4^k \nmid n!$, but for $n = 4^m$ and $k = \lfloor (n-1)/2 \rfloor$ we have $4^k \mid n!$ but $3^k \nmid n!$.

Although the sequence $(\ell_{12}(n!))$ seems not to be automatic, it turns out that it coincides almost everywhere (i.e. on a set of asymptotic density 1; this classical notion is defined a few lines below Lemma 1) with a 3-automatic sequence, and this permits to solve the question raised by the first-named author during the conference "Analytic and Combinatorial Number Theory" held in September 2010 in IMSc, Chennai (cf. [4]). More precisely, we have the following result.

Theorem. *Let* $0 \leq a \leq 11$. *The sequence* $\{n : \ell_{12}(n!) = a\}$ *has an asymptotic density, which is* $1/2$ *if* $a = 4$ *or* $a = 8$ *and* 0 *otherwise.*

It seems to be difficult to find the order of magnitude of the counting functions of these sets in the case $a \neq 4, 8$. We can show that

$$|\{n \leq x : \ell_{12}(n!) = a\}| = O(x^c)$$

with some $c < 1$ whenever $a \neq 4, 8$. From the other side we cannot even prove that they are all infinite, which seems to be likely.

2. Proof of the main result

We use $v_q(n)$ to denote the largest integer k such that q^k divides n.

The following lemma, which is due to Legendre, is a direct consequence of Theorem 3.2.1 of [1].

Lemma 1. *Let p be a prime, let $a \geq 1$ and $n \geq 0$ be integers. We have*

$$v_p(n!) = \frac{n - s_p(n)}{p-1} \quad \text{and} \quad v_{p^a}(n!) = \left\lfloor \frac{n - s_p(n)}{a(p-1)} \right\rfloor. \tag{3}$$

Our second basic lemma is also fairly classical: it follows from the fact that $s_b(n)$ behaves like the sum of $\log_b n$ independent random variables which are uniformly distributed in $\{0, 1, \ldots, b-1\}$. We could not find out who first explicitly expressed and proved it. More general results abound in the literature; Bassily [2] gives a simple proof (and generalizations which are irrelevant for our purposes).

We recall that a sequence of non-negative integers $\mathcal{A}$ is said to have *asymptotic density* (or simply *density*) α if $|\{a \in \mathcal{A} : a \leq x\}|$ is asymptotically equal to αx as x tends to infinity.

Lemma 2. *Let $\delta > 0$ and $b \geq 2$. The set*

$$\mathcal{S}_b(\delta) := \left\{ n : \left| s_b(n) - \frac{b-1}{2} \log_b n \right| \geq \delta \log_b n \right\} \tag{4}$$

has asymptotic density 0.

Our next lemma is a direct consequence of the previous one.

Lemma 3. *Let $p < q$ be prime numbers. There exists a positive δ such that the set of the integers n satisfying*

$$s_p(n) \leq s_q(n) - \delta \log n \tag{5}$$

has asymptotic density 1.

Proof. The function $x \mapsto (x-1)/\log x$ is strictly increasing on $[2, +\infty)$. We let

$$\delta = \frac{q-1}{4 \log q} - \frac{p-1}{4 \log p},$$

which is positive, and we consider the sets

$$\mathcal{T}^+_{q,\delta} = \left\{ n : s_q(n) \geq \frac{q-1}{2} \log_q n - \frac{\delta}{2} \log n \right\}$$

and

$$\mathcal{T}^-_{p,\delta} = \left\{ n : s_p(n) \leq \frac{p-1}{2} \log_p n + \frac{\delta}{2} \log n \right\}.$$

By Lemma 2, the sets $\mathcal{T}^+_{q,\delta}$ and $\mathcal{T}^-_{p,\delta}$ have density 1. Now if n is in their intersection, which still has density 1, we have

$$\begin{aligned} s_p(n) &\leq \left(\frac{p-1}{2 \log p} + \frac{\delta}{2} \right) \log n \\ &\leq \left(\frac{q-1}{2 \log q} - \frac{\delta}{2} \right) \log n - \delta \log n \leq s_q(n) - \delta \log n. \end{aligned}$$

□

We now show that $\ell_{12}(n!)$ and $4\ell_3(n!)$ are equal on a set of density 1.

Proposition 1. *Let n be an integer such that $s_2(n) \leq s_3(n) - 3$; then $\ell_{12}(n!) = 4\ell_3(n!)$.*

Proof. Let us consider such an integer n and let us write it in the bases 3 and 12 as

$$n! = \sum_{k \geq k_0} \eta_k^{(12)} 12^k = \sum_{h \geq h_0} \eta_h^{(3)} 3^h,$$

where $\eta_{k_0}^{(12)} = \ell_{12}(n!)$ and $\eta_{h_0}^{(3)} = \ell_3(n!)$.

We have

$$\begin{aligned} v_4(n!) = \lfloor (1/2)(n - s_2(n)) \rfloor &\geq (1/2)(n - s_2(n) - 1) \\ &\geq (1/2)(n - s_3(n) + 2) = v_3(n!) + 1 = h_0 + 1. \end{aligned}$$

This implies that $4^{h_0+1} \mid n!$; since $3^{h_0} \mid n!$, $3^{h_0+1} \nmid n!$, we have $k_0 = h_0$ and $\eta_{k_0}^{(12)} \equiv 0 \pmod 4$. We may thus write $n! = 3^{k_0}(\eta_{k_0}^{(3)} + 3r) = 12^{k_0}(\eta_{k_0}^{(12)} + 12s)$, and so $\eta_{k_0}^{(3)} \equiv 4^{k_0}\eta_{k_0}^{(12)} \equiv \eta_{k_0}^{(12)} \pmod 3$. We thus have the two congruences $\eta_{k_0}^{(12)} \equiv 0 \pmod 4$ and $\eta_{k_0}^{(12)} \equiv \eta_{k_0}^{(3)} \pmod 3$, whence $\eta_{k_0}^{(12)} = 4\eta_{k_0}^{(3)}$. □

We now give a direct proof that the sequence $(\ell_3(n!))$ is automatic.

Proposition 2. *The sequence $(\ell_3(n!))_{n \geq 0}$ is the fixed point, starting with 1, of the substitution given by*

$$1 \mapsto 1\,1\,2\,2\,2\,1\,2\,2\,1 \quad \textit{and} \quad 2 \mapsto 2\,2\,1\,1\,1\,2\,1\,1\,2.$$

Proof. We easily notice that $\ell_3(nm) \equiv \ell_3(n)\ell_3(m) \pmod 3$ and in particular, we have $\ell_3(9n) = \ell_3(n)$. Let $\alpha = \ell_3((9n)!)$ and let $\beta \equiv 2\alpha \pmod 3$. We have $\ell_3((9n+1)!) = \alpha\ell_3(9n+1) = \alpha$, and we find in the same way that we have successively $\ell_3((9n+2)!) = \beta, \ell_3((9n+3)!) = \beta, \ell_3((9n+4)!) = \beta, \ell_3((9n+5)!) = \alpha, \ell_3((9n+6)!) = \beta, \ell_3((9n+7)!) = \beta$ and $\ell_3((9n+8)!) = \alpha$. This proves that the sequence of the nine values $(\ell_3((9n+i)!))_{0 \leq i \leq 8}$ is one of the two sequences given in the proposition.
In order to prove the proposition, it is enough to prove that $\ell_3(0!) = 1$, which is obvious, and that for any n, we have $\ell_3((9n)!) = \ell_3(n!)$, an assertion which we shall prove by induction. We assume that for some k, we have $\ell_3((9k)!) = \ell_3(k!)$; again, by the previous computation, we have

$$\begin{aligned} \ell_3((9k+9)!) &\equiv \ell_3(9k+9)\ell_3(9k+8)!) \equiv \ell_3(k+1)\ell_3(9k)!) \\ &\equiv \ell_3(k+1)\ell_3(k!) \equiv \ell_3((k+1)!) \pmod 3. \end{aligned}$$

This ends the proof of the proposition. □

We are now equipped to prove the main result. The automaticity of a sequence does not imply the existence of its density (cf. the example of the numbers starting with a 1 in the base 10), and a little extra work is still needed. Let us consider the sequence $(\alpha(n))_{n\geq 0}$ which is the fixed point, starting with a 4, of the substitution given by

$$4 \mapsto 4\,4\,8\,8\,8\,4\,8\,8\,4 \quad \text{and} \quad 8 \mapsto 8\,8\,4\,4\,4\,8\,4\,4\,8.$$

Let $1 \leq a < b$ be two integers, h a non-negative integer and let us write, for i being 4 or 8

$$\pi_i(h) = \frac{1}{(b-a)9^h} \left|\left\{n \in [a \cdot 9^h, b \cdot 9^h) : \alpha(n) = i\right\}\right|.$$

For $h \geq 0$, we have

$$\begin{pmatrix} \pi_4(h+1) \\ \pi_8(h+1) \end{pmatrix} = M \begin{pmatrix} \pi_4(h) \\ \pi_8(h) \end{pmatrix},$$

where $M = \left(\begin{smallmatrix} 4/9 & 5/9 \\ 5/9 & 4/9 \end{smallmatrix}\right)$.

One readily sees that $M^h \to \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$ as h tends to infinity, for example because M is positive and bi-stochastic. This implies that the sequence $\mathcal{D}_4 = \{n : \alpha(n) = 4\}$ has the property that

$$\forall a < b : \lim_{h\to\infty} \frac{1}{(b-a)9^h} \left|\mathcal{D}_4 \cap [a \cdot 9^h, b \cdot 9^h)\right| = \frac{1}{2}.$$

This implies in turn that the sequence $\mathcal{D}_4$ has density 1/2. By Proposition 5, the set $\mathcal{D}_4 \cap \{n : s_2(n) \leq s_3(n) - 3\}$ has density 1/2, and so $\{n : \ell_{12}(n!) = 4\}$ has also density 1/2. For the same reason, the set $\{n : \ell_{12}(n!) = 8\}$ has also density 1/2; this furthermore implies that for any $u \in \{1, \ldots, 11\} \backslash \{4, 8\}$, the set $\{n : \ell_{12}(n!) = u\}$ has density 0. This completes the proof of Theorem 1.

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