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Two results on Beurling generalized numbers

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Dedicated to Professors Győry, Pethő, Pintz, and Sárközy
on their birthday milestones

Abstract. A sequence of Beurling generalized primes (g-primes) is an unbounded sequence of real numbers $\mathcal{P} = \{p_i\}$ satisfying $1 \le p_1 \le p_2 \le \dots$. The multiplicative semigroup generated by $\mathcal{P}$ along with 1 is designated as the corresponding collection of g-integers $\mathcal{N}$. Here we give a brief survey of Beurling numbers and then describe two achievements of recent years: the L^2 prime number theorem of Kahane and the oscillation result of Diamond, Montgomery, and Vorhauer.

1. Introduction

Let $\mathcal{P} = \{p_i\}$ be an unbounded sequence of real numbers satisfying $1 < p_1 \le p_2 \le \dots$. We call $\mathcal{P}$ a sequence of Beurling generalized primes (g-primes), and the multiplicative semigroup generated by $\mathcal{P}$ and 1 is designated as the corresponding collection $\mathcal{N}$ of g-integers. Note that it is not assumed that either collection lies in the positive integers nor that such staples of classical number theory as unique prime factorization are in force.

We define counting functions

$$\pi(x) = \pi_{\mathcal{P}}(x) = \#\{\mathcal{P} \cap [1, x]\} \quad \text{and} \quad N(x) = N_{\mathcal{P}}(x) = \#\{\mathcal{N} \cap [1, x]\},$$

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with the understanding that counts are made with appropriate multiplicity. Usually, we have hypotheses on the distribution of one of N and π and we seek to draw conclusions about the other one.

A simple example of g-primes is the sequence of natural primes exceeding 2. Here $\mathcal{N}$ is just the collection odd numbers, and $N(x) = (1/2)x + O(1)$. Note that omission of the natural prime 2 had a very small effect on the prime count, but it cut the integer density in half.

The subject of generalized numbers, in the present form, was initiated by Beurling in a paper [Beur] in which he assumed that

$$N(x) = cx + O\big(x \log^{-\gamma} x\big) \tag{1}$$

with $c > 0$ and $\gamma > 3/2$ and showed that

$$\pi(x) \sim x/\log x, \tag{2}$$

i.e. the analogue of the prime number theorem (PNT) holds for this g-number system. Moreover, he showed the result to be optimal in the sense that there exist examples which satisfy (1) with $\gamma = 3/2$ but for which (2) does not hold.

Beurling's result can be regarded as an abstraction and sharpening of an earlier article of E. Landau [Land] which established the prime ideal theorem for algebraic number fields. Like g-integers, norms of integral ideals do not have an additive structure, and Landau's arguments deal with only the counting functions of the integral ideals and prime ideals having norms at most a certain size.

In classical prime number theory we have the following notation and relations which translate directly to the Beurling context:

$$N(x) := \sum_{n \le x} 1 = [x] \text{ and } \pi(x) := \sum_{p \le x} 1, \quad x > 0,$$

the counting functions of natural numbers and primes respectively, and

$$\Pi(x) := \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots = \sum_{p^\alpha \le x} \frac{1}{\alpha}, \tag{3}$$

the weighted prime counting function.

The connection between N and Π can be expressed in a few ways. In analytic terms, via the Riemann zeta function

$$\zeta(s) = \int_{1-}^{\infty} x^{-s} dN(x), \quad \sigma := \Re s > 1, \tag{4}$$

we have

$$\log \zeta(s) = \int_1^\infty x^{-s} d\Pi(x), \quad \sigma > 1. \tag{5}$$

In "elementary" terms, the key relation between the counting functions of integers and (weighted) primes can be expressed as

$$dN = \delta_1 + d\Pi + \frac{1}{2!} d\Pi \star d\Pi + \frac{1}{3!} d\Pi \star d\Pi \star d\Pi + \cdots =: \exp d\Pi, \tag{6}$$

where δ_1 = Dirac point mass at 1, $\star$ denotes multiplicative convolution, and convergence is in the sense of uniform convergence on compacta. Comparing the two formulations, we obtain an interesting equivalent representation for $\zeta(s)$:

$$\int_{1-}^\infty x^{-s} \{\exp d\Pi\}(x) = \exp \int_1^\infty x^{-s} d\Pi(x), \quad \sigma > 1.$$

In Beurling theory, analogous relations are assumed to hold between the counting functions of the g-primes and g-integers. Moreover, we have the surprising feature that we do not need actual g-primes and g-integers — instead we can use mass distributions. Indeed, Beurling's example of a failure of the PNT was presented in this form. The only assumptions are that the (weighted) "prime density" has to be non-negative and supported on $[1, \infty)$, and the "integer density" is connected to the g-prime density by (6).

Here is an important example of such a pair of mass distributions.

Lemma 1.1. *Let*

$$d\Pi^*(x) = \frac{1 - x^{-1}}{\log x}\, dx, \quad x > 1.$$

Then the associated g-integer distribution satisfies $dN^*(x) = \delta_1 + dx$, i.e.

$$\delta_1 + dx = \exp\left\{\frac{1 - x^{-1}}{\log x}\, dx\right\}, \quad x > 1. \tag{7}$$

Proof. We establish this formula using Mellin transforms. Starting with

$$\zeta^*(s) := \int_{1-}^\infty x^{-s}(\delta_1 + dx) = 1 + \frac{1}{s-1} = \frac{s}{s-1}, \quad \sigma > 1,$$

we show that

$$\int_1^\infty x^{-s}\left\{\frac{1 - x^{-1}}{\log x}\, dx\right\} = \log \frac{s}{s-1} = \log \zeta^*(s). \tag{8}$$

Indeed, as $\sigma \to \infty$, each side of the last formula goes to 0. Also, the derivatives of the two sides of (8) give the valid formula

$$-\int_1^\infty x^{-s}(1-x^{-1})\,dx = \frac{1}{s} - \frac{1}{s-1}, \quad \sigma > 1.$$

Thus (8) holds, and dN^* is the g-integer distribution associated with the g-prime distribution $d\Pi^*$. □

The integrals of the measures in (7) have a further interesting property. Regarding $N^*(x)$ as a continuous integer distribution, we have

$$N^*(x) = \int_{1-}^x \delta_1 + dt = 1 + (x-1) = [x] + O(1)$$

and the PNT holds for Π^*:

$$\Pi^*(x) = \int_1^x \frac{1-t^{-1}}{\log t}\,dt \sim \frac{x}{\log x}. \tag{9}$$

2. Kahane's L^2 theorem

In a 1969 survey article [BaD], P. T. Bateman and the author conjectured that the condition

$$\int_1^\infty \Big|\frac{N(x)-cx}{x}\Big|^2 \log^2 x\,\frac{dx}{x} < \infty \tag{10}$$

would imply the PNT for a g-number system. Condition (10) is a bit weaker than Beurling's hypothesis (despite the optimality of Beurling's result among g-number systems satisfying (1)).

The conjecture was probably motivated by the big role L^2 estimates played in Beurling's article and the fact that with condition (10) we could carry out most of the proof of the PNT in our survey. To finish the argument, it remained to show that the zeta function of the system is nonvanishing on the line $\{s \in \mathbb{C} : \Re s = 1\}$.

J.-P. Kahane studied this conjecture, and, in a lecture in Montreal in 1996, he expressed doubts about its truth. However, as he wrote up the lecture, he had doubts about his doubts, and he went on to prove the conjecture [Kah1].

At worst, ζ has at most a single pair of conjugate zeros, each of order 1/2, at points $1 \pm it_0$; further zeros are ruled out by a de la Vallée Poussin-style argument. Kahane assumed, without loss of generality, that ζ had such a pair of zeros at $1 \pm i$. He took Beurling's critical example

$$\Pi_0(x) = \int_1^x (1-\cos(\log t))\,dt/\log t,$$

whose associated zeta function is

$$\sqrt{(s-1)^2+1}\,/(s-1),$$

and set $R(x) := P(x) - \Pi_0(x)$, with P the counting function of g-primes.

Kahane studied the behavior of $R(x)$ by delicate Fourier analysis. Using the fact that dP is everywhere nonnegative and in particular for t near $\exp 2\pi n$, $n = 1, 2, 3, \ldots$, where P has density zero, he was able to show that the average value of $|\zeta'/\zeta(1+it)|$ is too large and obtain a contradiction. A key step was to show that hypothesis (10) implies that the zeta function goes to zero more slowly at $1 \pm i$ than does Beurling's critical example.

This argument is subtle in both its structure and details, particularly in the use of distribution techniques to treat measures.

In fact, Kahane established the PNT for g-numbers under a weaker hypothesis than (10). The elementary formulation of his result [Kah2] is as follows: Suppose that $f \in L^1(\mathbf{R})$ and its Fourier transform satisfies $\hat{f}(t) \gg \exp(-|t|^\alpha)$ for some $\alpha < 2$. If

$$f \star (N(t) - ct)t^{-1} \log t \in L^2(1, \infty), \tag{K^α}$$

then the PNT holds. Further, Kahane showed that the condition (K^2) does not guarantee the PNT.

3. Oscillation example

In 1899, Ch. J. de la Vallée Poussin [dlVP] proved that

$$\zeta(s) \neq 0 \quad \text{for } \sigma > 1 - c/\log \tau, \tag{11}$$

where $s = \sigma + it$ and $\tau = |t| + 4$, the "classical" zero-free region of the Riemann zeta function. From this the PNT was deduced with an error term

$$\pi(x) = \operatorname{li} x + O\bigl(x \exp\{-c\,(\log x)^{1/2}\}\bigr). \tag{12}$$

Today better PNT error terms are known, e.g. that Vinogradov–Korobov, with any exponent less than 3/5, but these improvements are made by special arguments using the well-spacing of the rational integers.

It has long been suspected that (12) may be optimal for g-numbers, and we show just this: we give an example [DMV] in which the zero-free region (11) and the PNT error term (12) are exactly realized for a Beurling g-integer sequence. Two further examples, with sharp constants, were given by W.-B. Zhang [Zha].

Theorem 3.1. *There is a Beurling g-integer sequence $\mathcal{N}$ satisfying*

$$N(x) = cx + O\big(x^\theta\big) \tag{13}$$

with $c > 0$ and $\theta < 1$ having precisely the zero-free region and PNT error term of de la Vallée Poussin.

3.1. Approach. First, construct a continuous g-prime counting function $\Pi_C(x)$ that is connected by (5) to an integer counting function $N_C(x)$ that is continuous on $(1, \infty)$, and show that $\zeta_C(s)$, $N_C(x)$, and $\Pi_C(x)$ have the desired properties. Continuity of the prime and integer distribution is somewhat of a cosmetic defect, so we change the example into a discrete one. The conversion is carried out by a probabilistic construction.

3.2. Continuous prime distribution. Let $\chi(u)$ denote the indicator function of $[e, e^2]$ and $\chi^{\star n}$ the n-fold multiplicative convolution of χ with itself. Let

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{n} \chi^{\star n}(u), \quad u \geq 1.$$

Beyond the initial interval, this function becomes increasingly complicated, but one can show that $f(u) \log u < 3$ holds for all $u > 1$.

Our continuous Beurling prime counting measure is

$$d\pi_0(x) = \Big\{ \frac{1 - x^{-1}}{\log x} - 2 \sum_{k \geq 1} \frac{f\big(x^{1/(\alpha \log t_k)}\big)}{\alpha \log t_k} x^{-\beta/(\alpha \log t_k)} \cos(t_k \log x) \Big\}\, dx$$

for $x > 1$. Here $\{t_k\}$ is a rapidly increasing positive sequence and α and β are suitable constants.

The main contribution to $d\pi_0$ comes from the first term, which appeared in Lemma 1.1. The trigonometric part provides the required wobble. Important properties of this measure are nonnegativity, $d\pi(x) \leq 2\, dx/\log x$, and

$$1/2 \leq \pi_0(x)/\{x/\log x\} \leq 2, \quad x \geq 2.$$

The associated zeta function is expressed in terms of the function

$$G(z) = 1 - \frac{e^{-z} - e^{-2z}}{z}. \tag{14}$$

G is connected with f by the identity

$$\log G(z) = -\int_1^\infty f(u) u^{-z-1} du,$$

valid for $\Re z > 0$.

The function G is entire with $G(0) = 0$ and infinitely many other zeros $\{z_j\}$ with $\Re z_j < 0$; having many zeros here is a Good Thing. The zeta function is

$$\zeta_C(s) = \frac{s}{s-1} \prod_{k \neq 0} G(\ell_k \{s - \rho_k\}), \quad \sigma > 1.$$

Here

$$\rho_k = 1 - a/\log|\gamma_k| + i\gamma_k, \quad k = 1, 2, \dots,$$

and $\{\ell_k\}$ and $\{\gamma_k\}$ are sequences going swiftly to infinity.

The factor $s/(s-1)$ will give the main contribution to the "integer" and "prime" counting functions, and the product of G's will provide the desired zeta zeros and fluctuation in the prime count. Since $G(0) = 0$, it follows that $\zeta_C(s)$ has zeros at the points $s = \rho_k$. Importantly, $\zeta_C(s)$ has other zeros too, points of the form $s = \rho_k + z_j/\ell_k$, at the edge of de la Vallée Poussin's zero-free region.

3.3. Probabilistic construction of primes. Let $1 = x_0 < x_1 < x_2 < \dots$ be a sequence of real numbers tending slowly to infinity. For $k = 1, 2, \dots$ let X_k be independent Bernoulli variables with parameters

$$p_k = \int_{x_{k-1}}^{x_k} 1 \, d\pi_0(u),$$

i.e.

$$X_k = \begin{cases} 1 & \text{with probability } \; p_k, \\ 0 & \text{with probability } \; 1 - p_k. \end{cases}$$

(One condition on the x_k is that they must increase sufficiently slowly to ensure that $p_k < 1$.) At any given point ω of our probability space, we let $P(\omega)$ be the set of those x_k for which $X_k = 1$. This is a set of candidates for our g-primes. For $x \geq 2$ let K be determined by $x_K \leq x < x_{K+1}$. For a given element ω, define a discrete prime counting function by $\pi_1(x) = \sum_{k=1}^{K} X_k = X$, with expectation

$$E(X) = \sum_{k=1}^{K} p_k = \pi_0(x_K) \leq \pi_0(x).$$

Using a form of Kolmogorov's inequality, we show that most of the sets $P(\omega)$ determine a measure $d\pi_1(x)$ that is sufficiently close to $d\pi_0(x)$ to inherit the wobble and other properties of the continuous "prime" and "integer" counting functions. Specifically, for most $P(\omega)$ the key relation

$$\int_1^x u^{-it} \, d\pi_1(u) = \int_1^x u^{-it} \, d\pi_0(u) + O\big(\sqrt{x \log(|t| + 2)}\,\big)$$

holds uniformly for $x \geq 1$ and all real t. In particular, for $t = 0$ this formula asserts that

$$\pi_1(x) = \pi_0(x) + O\big(\sqrt{x}\big)$$

for $x \geq 1$. Also, the discrete integer counting function is shown to lie close to the continuous one, which completes the construction.

4. Concluding remarks

We conclude with a moral and an unsolved problem.

4.1. A moral for classical prime number theory. We constructed an example of Beurling numbers whose zeta function has zeros on the boundary of de la Vallée Poussin's zero-free region. There exist other g-integer examples having counting function very close to the classical $[x]$ but with a zeta function whose zeros lie arbitrarily close to the line $\sigma = 1$. The message of such examples is that a successful proof of the Riemann Hypothesis — or any further improvement in estimates of the zero-free region of zeta — will need to exploit what is lacking in Beurling theory, namely the additive structure of the integers.

4.2. An unsolved problem of Beurling. If $|N_{\mathcal{P}}(x) - [x]|$ is sufficiently small, then $\mathcal{P}$ is the set of classical primes. What is sufficiently small? The conclusion is trivial if $|N_{\mathcal{P}}(x) - [x]| < 1$; it need not be true if $|N_{\mathcal{P}}(x) - [x]| < C \log x$ for some C. What is the cut point?

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