

On periodogram based least squares estimation of the long memory parameter of FARMA processes

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Abstract. In [1] a periodogram based least squares regression method was proposed to get an estimation $\hat{\delta}$ of the long memory parameter $\delta \in (-\frac{1}{2}, \frac{1}{2})$ of the FARMA process (3). It was suspected for the case $\delta > 0$ and was proved for the case $\delta < 0$ that there exists a sequence $\hat{\delta}_n$ of estimators under consideration for which $\hat{\delta}_n$ is asymptotically normal, namely $\hat{\delta}_n \rightsquigarrow \mathcal{N}(\delta, a_n)$ holds, where $\lim_{n \rightarrow \infty} a_n = 0$. However we shall state that the large-sample behaviour of the FARMA's periodogram for very low frequencies is unusual, so we shall have to improve the considerations of [1]. In this paper a sufficient condition for the asymptotical "goodness" of the periodogram, will be given. For Gaussian FARMA processes by each $\delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ but one at most, it happens to be necessary condition too. With the aid of this condition we shall prove (both for $\delta < 0$ and for $\delta > 0$) the asymptotical normality of $\hat{\delta}_n$ arising from some modification of the estimation procedure of [1].

§ 1. Introduction

Sometimes the periodogram of the time series examined has a high peak at some frequency. One can consider this symptom as an indication of a deterministic trend or seasonal component. However there are data sets coming from hidrology, economy and astronomy, which have the former feature and which can be rather explained by stationary models, called long memory (or strongly dependent) time series models. The characteristic property of these various processes is the long-range dependence. Namely, the autocovariance series, $R(k)$ decreases so slowly as $k \rightarrow \infty$, that $\sum |R(k)|$ diverges. Equivalently, the spectral density function is unbounded, and so it has one or more infinitely high peak.

In [2] there are different linear models for long memory processes and [3] studies strongly dependent bilinear models. We shall deal only with the

case in which the spectral density of the underlying linear process has one peak and it is at zero. GRANGER and JOYEUX ([4]) and HOSKING ([5]) have proposed the use of fractional ARMA (FARMA) model to describe this long memory phenomenon. The originality of this approach lies in avoiding overdifferencing.

The FARMA (or ARIMA(p, δ, q)) process can be defined with the “fractional difference equation”

$$(1) \quad \phi(B)(I - B)^\delta X_t = \theta(B)Z_t, \quad t \in \mathbb{Z}$$

where ϕ and θ are polynomials of degrees p and q respectively, I is the identity and B is the back-shift operator and Z_t is white noise with mean zero and variance $\sigma^2 = 1$, for simplicity. δ is a real value, called the long memory parameter and $(I - B)^\delta$ is the fractional difference operator, defined for $\delta > -1$ by the binomial expansion

$$(I - B)^\delta = \sum_{k=0}^{\infty} c_k B^k,$$

$$c_0 = 1, \quad c_k = \frac{\Gamma(k - \delta)}{\Gamma(k + 1)\Gamma(-\delta)} = \prod_{j=1}^k \frac{j - 1 - \delta}{j}, \quad k = 1, 2, \dots,$$

where Γ is the gamma function.

From Stirling's formula it follows that

$$(2) \quad c_k = \mathcal{O}(\Gamma(-\delta)k^{-\delta-1}).$$

(All over this article $a_k = \mathcal{O}(b_k)$ means $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = 1$.) Thus for $\delta < \frac{1}{2}$,

$(1 - e^{-i\lambda})^\delta = \sum_{k=0}^{\infty} c_k e^{-ik\lambda}$ is mean square (i.e. $L^2[0, 1]$) convergent, while

for $0 < \delta$ it is even uniformly (i.e. $C[0, 1]$) convergent. So, according to the theorem about the composition of linear filters, it follows that if

a) $-\frac{1}{2} < \delta < \frac{1}{2}$ and

b) ϕ and θ have no common zeroes and $\phi(z) \neq 0$ on the complex unit circle

then there exists a unique stationary solution of (1), given by

$$(3) \quad X_t = \psi(B)(I - B)^{-\delta} Z_t, \quad \text{where } \psi(z) = \frac{\theta(z)}{\phi(z)}.$$

So X_t can be thought of as an ARMA(p, q) sequence with innovation process $(I - B)^{-\delta} Z_t$, called fractional white noise.

The spectral density function of the FARMA process (3) is of the form

$$(4) \quad f^{(X)}(\lambda) = |1 - e^{-i\lambda}|^{-2\delta} f^{(Y)}(\lambda) = (2 \sin \frac{\lambda}{2})^{-2\delta} f^{(Y)}(\lambda), \\ \lambda \in [-\pi, \pi], \quad \lambda \neq 0 \text{ if } \delta > 0,$$

where $f^{(Y)}(\lambda)$ is the spectral density of $Y_t = \psi(B)Z_t$, i.e.,

$$f^{(Y)}(\lambda) = \frac{1}{2\pi} \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2, \quad \lambda \in [-\pi, \pi].$$

Let us suppose that

c) $\theta(1) \neq 0$,

then $f^{(Y)}(\lambda)$ is strictly positive at some neighbourhood of $\lambda = 0$, so

$$(5) \quad f^{(X)}(\lambda) = \mathcal{O} \left(\frac{1}{2\pi} \left(\frac{\theta(1)}{\phi(1)} \right)^2 \lambda^{-2\delta} \right) \quad \text{as } \lambda \rightarrow 0,$$

whereby $f^{(X)}(\lambda)$ really has a peak at $\lambda = 0$ if $\delta > 0$. In the case $\delta < 0$, $f^{(X)}(0) = 0$ and $f^{(X)}(\lambda)$ is continuous on $[-\pi, \pi]$, so X_t has not the long memory property. In such a case X_t is called intermediate memory process since for its autocovariance series $R(k) = \text{const} \mathcal{O}(k^{2\delta-1})$ holds (as for $\delta > 0$ too), while processes with $R(k) = \text{const} \mathcal{O}(r^{k-1})$, where $0 < r < 1$, e.g. ARMA processes, are defined to be short memory.

The above mentioned things and other properties of FARMA processes can be found in [6], ch.12.

Throughout the paper X_t denotes the FARMA process defined by (3) and in what follows we shall suppose assumption (a), (b) and (c).

There are a number of approaches to parameter estimation of FARMA processes. The most attractive and computationally simple method was invented by GEWEKE and PORTER-HUDAK in [1]. It has the virtue of permitting estimation of δ without knowledge of p and q . Once the long memory parameter is estimated at $\hat{\delta}$, the series X_t can be transformed (applying $(I - B)^{\hat{\delta}}$ to (3)) to obtain the series \hat{Y}_t . After this step standard identification methods developed for ARMA processes can be used to estimate p , q and ϕ , θ .

This method is based on the assumption that the random variables 'periodogram/spectral density' at different frequencies are asymptotically identically exponentially distributed and independent. It is well known (see e.g. [6], Theorem 10.3.2 and [8], §2) that the processes occurring most frequently in the literature, have this feature. But we shall see in §3 that FARMA processes have not this good asymptotic property. So, the estimation method of [1] calls for modification. In §4 the asymptotic normality of the modified estimator $\hat{\delta}$ will be proved.

§ 2. The estimation of δ and the problem associated with it

Now, let us see the estimation procedure of δ , first as described in [1]. The periodogram of X_0, \dots, X_{n-1} will be denoted by

$$I_n^{(X)}(\omega_k^{(n)}) = \frac{1}{2\pi n} \left| \sum_{j=0}^{n-1} X_j e^{-ij\omega_k^{(n)}} \right|^2$$

at Fourier frequencies $\omega_k^{(n)} = 2\pi k/n$ ($k = 0, 1, \dots, [(n+1)/2]$). Take logarithms in (4), replace λ by the Fourier frequencies $\omega_k^{(n)} \in [0, \pi]$ and add $\log(I_n^{(X)}(\omega_k^{(n)}))$ to both sides, one obtains

$$(6) \quad \begin{aligned} \log(I_n^{(X)}(\omega_k^{(n)})) &= \log(f^{(Y)}(0)) - \delta \log\left(|1 - e^{-i\omega_k^{(n)}}|^2\right) + \\ &+ \log\left(\frac{I_n^{(X)}(\omega_k^{(n)})}{f^{(X)}(\omega_k^{(n)})}\right) + \log\left(\frac{f^{(Y)}(\omega_k^{(n)})}{f^{(Y)}(0)}\right), \quad k = 1, 2, \dots, \ell(n), \end{aligned}$$

where $\ell : \mathbb{N} \rightarrow \mathbb{N}$, $\ell(n) < n/2$ for all $n \geq 3$. ($\mathbb{N} = \{1, 2, 3, \dots\}$) If we choose the sequence ℓ so that $\lim_{n \rightarrow \infty} 2\pi\ell(n)/n = 0$, then the last term in (6) converges to zero. So, for large n we can consider (6) as the linear regression equation

$$(7) \quad y_k = a + \delta x_k + \varepsilon_k, \quad k = 1, 2, \dots, \ell(n),$$

where $y_k = \log(I_n^{(X)}(\omega_k^{(n)}))$, the intercept parameter $a = \log(f^{(Y)}(0))$, $x_k = -\log(|1 - e^{-i\omega_k^{(n)}}|^2)$ and the error variables are

$$\varepsilon_k = \log\left(\frac{I_n^{(X)}(\omega_k^{(n)})}{f^{(X)}(\omega_k^{(n)})}\right).$$

This suggests estimating δ by least squares regression.

The usability of this method depends on the large-sample behaviour of the joint distribution of the regression errors. In [1] the following theorem has been stated and proved.

Let X_t be a FARMA process with $\delta < 0$. Then there exists a sequence $\ell : \mathbb{N} \rightarrow \mathbb{N}$ for which the least squares estimator $\hat{\delta}$ of δ in (7) is asymptotically Gaussian, namely

$$\hat{\delta} \rightsquigarrow \mathcal{N}\left(\delta, \pi^2 / \left(6 \sum_{i=1}^{\ell(n)} (x_i - \bar{x})^2\right)\right) \quad \text{as } n \rightarrow \infty,$$

where x_i is the same as in (7) and $\bar{x} = \sum_{i=1}^{\ell(n)} x_i / \ell(n)$. Moreover $\widehat{\delta}$ is consistent in the sense that $\lim_{n \rightarrow \infty} \sum_{i=1}^{\ell(n)} (x_i - \bar{x})^2 = \infty$.

In [1] the proof is based on the statement that

$$(8) \quad \left\{ \begin{array}{l} \text{for } \delta < 0 \text{ and for any fixed } k \in \mathbb{N} \text{ the joint distribution of} \\ \frac{I_n^{(X)}(2\pi/n)}{f^{(X)}(2\pi/n)}, \frac{I_n^{(X)}(4\pi/n)}{f^{(X)}(4\pi/n)}, \dots, \frac{I_n^{(X)}(2\pi k/n)}{f^{(X)}(2\pi k/n)} \\ \text{converges weakly to the one of a } k\text{-dimensional random} \\ \text{vector with independent and mean 1 exponentially distri-} \\ \text{buted components.} \end{array} \right.$$

If this was so, it would follow (see [7], Theorem 5.1) that the asymptotical finite dimensional joint distribution of the errors in (6) and (7) is the same as the joint distribution of independent random variables with distribution function $1 - \exp(-e^x)$, $x \in \mathbb{R}$. It is to be noted that the expectation of this latter distribution, $-\gamma$ (γ is Euler's constant), can be included in the intercept term in (7), while the variance of it is $\pi^2/6$ (see [1]).

Nevertheless we shall see in §3, that for each $\delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, except one at most, there exist FARMA processes with long memory parameter δ , not meeting the statement of (8). The cause of this unexpected behaviour of the periodogram lies in property (5) of the spectral density, and so in the property of long or intermediate memory after all. However we shall prove that taking the frequencies $(2\pi h(n)/n, 2\pi(h(n) + 1)/n, \dots, 2\pi(h(n) + k - 1)/n)$ instead of the ones in (8), where $h : \mathbb{N} \rightarrow \mathbb{N}$, $\lim_{n \rightarrow \infty} (h(n)/n) = 0$ and $\lim_{n \rightarrow \infty} h(n) = \infty$, the assertion respective to (8), becomes true. This is valid for both $\delta < 0$ and $\delta > 0$.

In §4 it will turn out to be true that $\widehat{\delta}$ arising from the equations which remain after dropping the first $h(n) - 1$ equations in (7), is asymptotically normal.

§ 3. Asymptotical behaviour of the FARMA process's periodogram in frequency zero

Let us denote the discrete Fourier transform of X_0, \dots, X_{n-1} and of

Z_0, \dots, Z_{n-1} at frequency $\omega_k^{(n)} = 2\pi k/n$ by

$$J_n^{(X)}(\omega_k^{(n)}) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} X_j e^{-ij\omega_k^{(n)}}, \quad J_n^{(Z)}(\omega_k^{(n)}) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} Z_j e^{-ij\omega_k^{(n)}},$$

respectively. Moreover, φ , $\varphi^{(Y)}$ and $\varphi^{(X)}$ will be the transfer function of the operator $(I - B)^{-\delta}$, $\psi(B)$ and $\psi(B)(I - B)^{-\delta}$ respectively, i.e.

$$\varphi(\lambda) = (1 - e^{-i\lambda})^{-\delta}, \quad \varphi^{(Y)}(\lambda) = \psi(e^{-i\lambda}), \quad \varphi^{(X)}(\lambda) = \varphi^{(Y)}(\lambda)\varphi(\lambda),$$

where ψ has been defined in (3) and let $I_n^{(X)}$ and $f^{(X)} = \frac{1}{2\pi} |\varphi^{(X)}(\lambda)|^2$ be the periodogram and the spectral density of the FARMA process X_t .

h will be a sequence with the properties

$$(9) \quad h : \mathbb{N} \rightarrow \mathbb{N}, \quad \frac{h(n)}{n} < \frac{1}{2} \text{ if } n \geq 3, \quad \lim_{n \rightarrow \infty} \left(\frac{h(n)}{n} \right) = 0.$$

We shall frequently use the notations

$$(10) \quad \mathcal{D}_n = \left\{ \lambda \in \left(-\frac{n\pi}{2h(n)}, \frac{n\pi}{2h(n)} \right), \lambda \neq \pi \text{ if } \delta > 0 \right\},$$

$$A_n(\lambda) = \begin{cases} \frac{\varphi^{(X)}\left(\frac{2h(n)}{n}(\pi - \lambda)\right)}{\varphi^{(X)}\left(\frac{2h(n)}{n}\pi\right)} & \text{if } \lambda \in \mathcal{D}_n \\ 0 & \text{otherwise.} \end{cases}$$

Now we state a lemma, which has only technical importance.

Lemma 1. *If h and $A_n(\lambda)$ are defined by (9) and (10), respectively, then*

a) *there exist $a, b, c \in \mathbb{R}$ for which*

$$|A_n(\lambda)|^2 \leq a \left| \frac{\pi}{\pi - \lambda} \right|^{2\delta}, \quad |A_n(\lambda) - 1|^2 \leq b \left(\left| \frac{\pi}{\pi - \lambda} \right|^\delta + c \right)^2$$

holds for all $n \in \mathbb{N}$ and $\lambda \in \mathcal{D}_n$;

b) *if $0 < \varepsilon < \pi$, then*

$$\lim_{n \rightarrow \infty} \sup_{[-\varepsilon, \varepsilon]} |A_n(\lambda) - 1|^2 \leq \left| \left(\frac{\pi}{\pi - \varepsilon} \right)^\delta - 1 \right|^2;$$

c)

$$\lim_{n \rightarrow \infty} |A_n(\lambda)|^2 = \left| \frac{\pi}{\pi - \lambda} \right|^{2\delta}, \quad \lim_{n \rightarrow \infty} |A_n(\lambda) - 1|^2 = \left(\left| \frac{\pi}{\pi - \lambda} \right|^\delta - 1 \right)^2$$

holds for all $\lambda \in \mathbb{R}$, $\lambda \neq \pi$ if $\delta > 0$.

PROOF. The assertions concerning $|A_n(\lambda) - 1|^2$ can be found in [11] with proof while the ones with respect to $|A_n(\lambda)|^2$ can be verified similarly but more easily. \square

The following three theorems throw light on the interesting phenomenon that the asymptotical distribution of the scaled periodogram ordinates

$$(11) \quad I_n^{(X)} \left(\omega_{h(n)}^{(n)} \right) / f^{(X)} \left(\omega_{h(n)}^{(n)} \right), \quad \omega_{h(n)}^{(n)} = 2\pi \frac{h(n)}{n}$$

depends on how fast the series $\omega_{h(n)}^{(n)}$ approach to zero. It is well known that there is no such subordination for stationary linear processes with both absolutely convergent AR and MA representation. (See [6], Theorem 10.3.2.) In other words, for the last mentioned processes the series of the quantities (11), always has mean 1 exponential limit distribution.

Theorem 1 and 2 are about the sufficient condition $\lim_{n \rightarrow \infty} h(n) = \infty$ which ensure the scaled periodogram ordinates (11) to have mean 1 exponential limit distribution. In [9] and in [10] a stronger condition, $\lim_{n \rightarrow \infty} (h(n)/\sqrt{n}) = \infty$ is given and only for the case of Gaussian X_t .

Theorem 1. *Let h be defined by (9) and $\lim_{n \rightarrow \infty} h(n) = \infty$. Then*

$$\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{J_n^{(X)} \left(\omega_{h(n)}^{(n)} \right)}{\varphi^{(X)} \left(\omega_{h(n)}^{(n)} \right)} - J_n^{(Z)} \left(\omega_{h(n)}^{(n)} \right) \right|^2 = 0.$$

PROOF. Since the spectral representation of Z_t is

$$Z_t = \int_{-\pi}^{\pi} e^{it\lambda} d\mathcal{Z}(\lambda),$$

where $\mathcal{Z}(\lambda)$ is an orthogonal increment process for which $\mathbf{E} |d\mathcal{Z}(\lambda)|^2 = d\lambda/(2\pi)$, thus

$$X_t = \int_{-\pi}^{\pi} \varphi^{(X)}(\lambda) e^{it\lambda} d\mathcal{Z}(\lambda).$$

From this

$$\begin{aligned}
 & \mathbb{E} \left| \frac{J_n^{(X)}(\omega_{h(n)}^{(n)})}{\varphi^{(X)}(\omega_{h(n)}^{(n)})} - J_n^{(Z)}(\omega_{h(n)}^{(n)}) \right|^2 \\
 (12) \quad &= \mathbb{E} \left| \int_{-\pi}^{\pi} \left(\frac{\varphi^{(X)}(\lambda)}{\varphi^{(X)}(\omega_{h(n)}^{(n)})} - 1 \right) \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{ik(\lambda - \omega_{h(n)}^{(n)})} dZ(\lambda) \right|^2 \\
 &= \int_{-\pi}^{\pi} \left| \frac{\varphi^{(X)}(\lambda)}{\varphi^{(X)}(\omega_{h(n)}^{(n)})} - 1 \right|^2 K_n(\omega_{h(n)}^{(n)} - \lambda) d\lambda,
 \end{aligned}$$

where

$$K_n(\mu) = \frac{1}{2\pi n} \left| \sum_{k=0}^{n-1} e^{ik\mu} \right|^2 = \frac{\sin^2(n\mu/2)}{2\pi n \sin^2(\mu/2)}$$

is the Fejér kernel. Now, first making use of that $\varphi^{(X)}$ and K_n has period 2π , we can perform the substitution $\mu = 2h(n)\pi/n - \lambda$. Then the substitution $\lambda = n\mu/(2h(n))$ leads to the equation

$$\mathbb{E} \left| \frac{J_n^{(X)}(\omega_{h(n)}^{(n)})}{\varphi^{(X)}(\omega_{h(n)}^{(n)})} - J_n^{(Z)}(\omega_{h(n)}^{(n)}) \right|^2 = \int_{\mathcal{D}_n} |A_n(\lambda) - 1|^2 \frac{2h(n)}{n} K_n\left(\frac{2h(n)}{n}\lambda\right) d\lambda.$$

Let us introduce the function

$$B_n(\lambda) = \begin{cases} \frac{2h(n)}{n} K_n\left(\frac{2h(n)}{n}\lambda\right) & \text{if } -\frac{n\pi}{2h(n)} < \lambda < \frac{n\pi}{2h(n)} \\ 0 & \text{otherwise.} \end{cases}$$

We mention that

$$(13) \quad B_n(\lambda) \leq \frac{\pi}{4h(n)\lambda^2} \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda \neq 0.$$

Let $\varepsilon \in (0, \pi)$. Then $\int_{-\pi}^{\pi} K_n(\lambda) d\lambda = 1$ implies that

$$(14) \quad \int_{-\varepsilon}^{\varepsilon} B_n(\lambda) d\lambda \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Using (13) and (14) we get

$$(15) \quad \int_{\mathcal{D}_n} |A_n(\lambda) - 1|^2 B_n(\lambda) d\lambda \leq \sup_{[-\varepsilon, \varepsilon]} |A_n(\lambda) - 1|^2 + \frac{\pi}{4h(n)} \int_{\mathcal{D}_n \setminus [-\varepsilon, \varepsilon]} \frac{|A_n(\lambda) - 1|^2}{\lambda^2} d\lambda.$$

Utilizing Lemma 1 (a) and the inequality $-\frac{1}{2} < \delta < \frac{1}{2}$, it follows that the last integral in (15) can be majorized for all $n \in \mathbb{N}$ with the same finite constant. Taking $\lim_{n \rightarrow \infty}$ in (15), applying Lemma 1 (b) and exploiting the continuity of $|(\pi/(\pi - \varepsilon))^\delta - 1|^2$ at $\varepsilon = 0$ we have finished the proof. \square

Theorem 2. *Let $N \in \mathbb{N}$ be a fixed number and $h_i : \mathbb{N} \rightarrow \mathbb{N}$, $i = 1, 2, \dots, N$ sequences for which $h_i(n) \neq h_j(n)$ if $i \neq j$, $0 < 2\pi h_i(n)/n < \pi$, $i = 1, 2, \dots, N$ hold for all $n \in \mathbb{N}$ being large enough. Moreover assume that $\lim_{n \rightarrow \infty} h_i(n)/n = 0$, $i = 1, 2, \dots, N$ and the white noise innovation process Z_t is i.i.d. or mixing (see [8]).*

Then $\lim_{n \rightarrow \infty} \min_{i=1, \dots, N} h_i(n) = \infty$ implies that the joint distribution of

$$I_n^{(X)} \left(\omega_{h_i(n)}^{(n)} \right) / f^{(X)} \left(\omega_{h_i(n)}^{(n)} \right), \quad i = 1, 2, \dots, N$$

converges to the joint distribution of N independent mean 1 exponentially distributed random variables.

PROOF. The vector of the discrete Fourier transforms $J_n^{(Z)}(\omega_{h_i(n)}^{(n)})$, $i = 1, 2, \dots, N$, converges weakly to the N -dimensional complex normal distribution with independent $\mathcal{N}(\underline{0}, \Sigma)$ components where Σ is a diagonal matrix with elements π (see [6], Proposition 10.3.2 and [8]). Since we are in possession of the key Theorem 1, the method of Theorem 10.3.1 and 10.3.2 in [6] can be applicable. \square

The next theorem is about the necessity of the condition $\lim_{n \rightarrow \infty} h(n) = \infty$.

Theorem 3. *Let h be defined by (9) and assume that $h(n) \not\rightarrow \infty$ as $n \rightarrow \infty$. Then for each $\delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ except one δ at most, the FARMA process (3) with long memory parameter δ , satisfies*

$$\mathbb{E} I_n^{(X)} \left(\omega_{h(n)}^{(n)} \right) / f^{(X)} \left(\omega_{h(n)}^{(n)} \right) \not\rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

and so, if X_t is Gaussian, then the series of the distributions of the scaled periodogram ordinates (11) do not converge to the mean 1 exponential distribution.

PROOF. Similarly as we began the proof of Theorem 1,

$$\mathbb{E} I_n^{(X)} \left(\omega_{h(n)}^{(n)} \right) / f^{(X)} \left(\omega_{h(n)}^{(n)} \right) = \int_{-\infty}^{\infty} |A_n(\lambda)|^2 B_n(\lambda) d\lambda.$$

There exists a $H \in \mathbb{N}$ and a subsequence n_k , for which $h(n_k) = H$, for all $k \in \mathbb{N}$. Now,

$$\lim_{k \rightarrow \infty} B_{n_k}(\lambda) = \frac{\sin^2(H\lambda)}{\pi H \lambda^2}.$$

Utilizing Lemma 1 (c), (a), the inequality (13) and $B_{n_k}(\lambda) \leq H/\pi$, we can apply Lebesgue's theorem to conclude that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \mathbb{E} I_{n_k}^{(X)} \left(\omega_H^{(n_k)} \right) / f^{(X)} \left(\omega_H^{(n_k)} \right) \\ (16) \quad &= \int_{-\infty}^{\infty} \left| \frac{\pi}{\pi - \lambda} \right|^{2\delta} \frac{1}{\pi H} \frac{\sin^2(H\lambda)}{\lambda^2} d\lambda = \int_{-\infty}^{\infty} \left| \frac{\pi}{\pi - \frac{\lambda}{H}} \right|^{2\delta} \frac{1}{\pi} \frac{\sin^2 \lambda}{\lambda^2} d\lambda. \end{aligned}$$

Next we show that the limit in (16) is different from 1 for all $\delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ except one δ at most. Let us introduce the notations

$$K(\lambda) = \frac{\sin^2 \lambda}{\pi \lambda^2}, \quad D(\delta) = \int_{-\infty}^{\infty} \left| \frac{\pi}{\pi - \frac{\lambda}{H}} \right|^{2\delta} K(\lambda) d\lambda.$$

Since $K(\lambda)$ is the limit of the pulled out version of $K_n(\lambda)$ to $[-\frac{n\pi}{2}, \frac{n\pi}{2}]$, it integrates to 1. From the Hölder inequality,

$$(17) \quad 1 = \int_{-\infty}^{\infty} K(\lambda) d\lambda \leq \sqrt{D(\delta)} \sqrt{D(-\delta)}.$$

Application of the strict Jensen inequality leads to

$$(18) \quad (D(\delta))^r > D(\delta r), \quad \delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}, \quad 0 < r < 1.$$

So, if for some $\tilde{\delta} \in (0, \frac{1}{2})$, $D(\tilde{\delta}) = 1$, then the function $D < 1$ on $(0, \tilde{\delta})$ and $D > 1$ on $(\tilde{\delta}, \frac{1}{2})$, because of (18). Moreover, (17) ensures that $D > 1$ on $(-\tilde{\delta}, 0)$. Using (18) again, we get that $D > 1$ holds on $(-\frac{1}{2}, 0)$ too. So, $D(\delta) - 1$ has one zero at most, in addition to $\delta = 0$. The case $\tilde{\delta} < 0$ can be fixed up similarly. We have proved the first statement of the theorem.

In connection with the second assertion we remark that for Gaussian X_t processes the distribution of the scaled periodogram ordinate (11) is

the same as the one of the random variable $a_n^2 \xi_n^2 + b_n^2 \eta_n^2$, where (ξ_n, η_n) is Gaussian with mean zero and $D^2 \xi_n = D^2 \eta_n = 1$. For the previously used subsequence n_k ($h(n_k) = H$), the series of the expectations of the scaled periodogram ordinates is convergent, so it is bounded. Therefore $a_{n_k}^2 + b_{n_k}^2$ is bounded sequence. So, the sequence of the second order moments of the random variables $I_{n_k}^{(X)}(\omega_H^{(n_k)}) / f^{(X)}(\omega_H^{(n_k)})$, $k \in \mathbb{N}$, is bounded. Thus, the weak convergence of the distributions of (11) to the mean 1 exponential distribution would imply (see [7], exercise 4 of §7) the convergence of the means to 1. But the latter is possible for one δ at most, in the set $(-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. \square

Remark 1. Theorem 2 in [11], corresponding to the latter Theorem 3 here, is faulty.

Remark 2. It comes from the previous proof that for fixed $H \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \mathbb{E} I_n^{(X)}(\omega_H^{(n)}) / f^{(X)}(\omega_H^{(n)}) = \int_{-\infty}^{\infty} \left| \frac{\pi}{\pi - \frac{\lambda}{H}} \right|^{2\delta} \frac{1 \sin^2 \lambda}{\pi \lambda^2} d\lambda = D_H(\delta).$$

Numerical computation of D_H shows that for each $H \in \mathbb{N}$, there exists a $\tilde{\delta}_H$ for which $D_H(\tilde{\delta}_H) = 1$. Moreover, $\tilde{\delta}_1 \approx .06$ and $0 < \tilde{\delta}_{H+1} < \tilde{\delta}_H$.

Remark 3. Theorem 3 suggests handling periodogram based tests with care for FARMA processes because the scaled periodogram ordinates $I_H^{(X)}(\omega_H^{(n)}) / f^{(X)}(\omega_H^{(n)})$, for $H \ll n$, may behave unusually.

§ 4. The asymptotical distribution of the estimation of δ

The following lemma will be used in the proof of the main theorem, Theorem 4.

Lemma 2. Let $g : \mathbb{N} \rightarrow \mathbb{N}$, $\lim_{n \rightarrow \infty} (g(n)/n) = 0$, $\lim_{n \rightarrow \infty} g(n) = \infty$, $\ell \in \mathbb{N}$ be fixed and $k(n) = g(n) + k'$, $n \in \mathbb{N}$, where $k' \in \{0, 1, \dots, \ell\}$ is arbitrary. Moreover let $x_k = -\log |1 - \exp(-2k\pi i/n)|^2$, $k \in \mathbb{N}$.

Then

$$\lim_{n \rightarrow \infty} \left(\sum_{j=g(n)}^{g(n)+\ell} (x_j - \bar{x})^2 \right)^{-\frac{1}{2}} (x_{k(n)} - \bar{x}) = \frac{\sqrt{3}(\ell - 2k')}{\sqrt{\ell(\ell+1)(\ell+2)}},$$

where $\bar{x} = \frac{1}{\ell+1} \sum_{j=g(n)}^{g(n)+\ell} x_j$.

PROOF. See [11]. \square

Let us consider the regression equations

$$(19) \quad y_k = a + \delta x_k + \varepsilon_k, \quad k = g(n), g(n) + 1, \dots, g(n) + \ell(n),$$

where $g, \ell : \mathbb{N} \rightarrow \mathbb{N}$, $g(n) + \ell(n) < n/2$ for all $n \geq 5$ and y_k, x_k, ε_k and a denotes the same as in (7).

Theorem 4. *Let X_t be the FARMA process (3) (whether $\delta < 0$ or $\delta > 0$) with white noise innovation process Z_t . Assume that Z_t is either i.i.d. or mixing. Moreover let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a sequence for which $g(n) < n/2$ for all $n \geq 3$, $\lim_{n \rightarrow \infty} (g(n)/n) = 0$, $\lim_{n \rightarrow \infty} g(n) = \infty$ hold.*

Then there exists a sequence $\ell : \mathbb{N} \rightarrow \mathbb{N}$ for which the least squares estimator $\hat{\delta}$ of δ in (19) is asymptotically Gaussian, namely

$$\hat{\delta} \rightsquigarrow \mathcal{N} \left(\delta, \pi^2 / \left(6 \sum_{i=g(n)}^{g(n)+\ell(n)} (x_i - \bar{x})^2 \right) \right) \quad \text{as } n \rightarrow \infty,$$

where x_i is the same as in (7) and $\bar{x} = \frac{1}{\ell(n) + 1} \sum_{k=g(n)}^{g(n)+\ell(n)} x_k$.

PROOF. The least squares estimator obtained from (19) is

$$\hat{\delta} = \frac{\sum_{g(n)}^{g(n)+\ell(n)} (x_k - \bar{x})(y_k - \bar{y})}{\sum_{g(n)}^{g(n)+\ell(n)} (x_k - \bar{x})^2} = \delta + \frac{\sum_{g(n)}^{g(n)+\ell(n)} (x_k - \bar{x})\varepsilon_k}{\sum_{g(n)}^{g(n)+\ell(n)} (x_k - \bar{x})^2}.$$

If the sequences g and ℓ are chosen to satisfy

$$\lim_{n \rightarrow \infty} g(n) = \infty, \quad \lim_{n \rightarrow \infty} (g(n)/n) = 0, \quad \ell(n) = \ell = \text{const.},$$

then the conditions of Theorem 2 are fulfilled. Moreover, since the discontinuities of the logarithm function has measure zero with respect to the exponential distribution, it follows (see [7], Theorem 5.1) that the error variables ε_k are asymptotically i.i.d. random variables with double exponential distribution. This distribution has expectation $-\gamma$, where γ is the Euler constant and the variance of it is $\pi^2/6$ (see [1]). Let us replace for all $k = g(n), g(n) + 1, \dots, g(n) + \ell$, ε_k with $\varepsilon_k + \gamma$. From this replacement $\hat{\delta}$ will not be changed.

The formal standardization of $\widehat{\delta}$ is

$$(20) \quad \begin{aligned} & (\widehat{\delta} - \delta) \frac{\sqrt{6}}{\pi} \left(\sum_{g(n)}^{g(n)+\ell} (x_k - \bar{x})^2 \right)^{\frac{1}{2}} \\ &= \frac{\sqrt{6}}{\pi} \left(\sum_{g(n)}^{g(n)+\ell} (x_k - \bar{x})^2 \right)^{-\frac{1}{2}} \sum_{g(n)}^{g(n)+\ell} (x_k - \bar{x}) \varepsilon_k. \end{aligned}$$

Using Lemma 2 too, we get that for fixed $\ell \in \mathbb{N}$ the distribution of the right of (20) converges weakly as $n \rightarrow \infty$ to the one of

$$(21) \quad \frac{\sqrt{6}}{\pi} \sum_{i=0}^{\ell} \frac{(\ell - 2i)\sqrt{3}}{\sqrt{\ell(\ell^2 + 3\ell + 2)}} (\tilde{\varepsilon}_i + \gamma),$$

where $\tilde{\varepsilon}_0, \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_\ell$ are independent double exponentially distributed random variables. Moreover, since (21) is linear combination of i.i.d. random variables, to satisfy the Lindeberg condition as $\ell \rightarrow \infty$, it is enough to prove that

$$\lim_{\ell \rightarrow \infty} \max_{0 \leq i \leq \ell} \frac{(\ell - 2i)^2}{\ell(\ell^2 + 3\ell + 2)} = \lim_{\ell \rightarrow \infty} \frac{\ell^2}{\ell(\ell^2 + 3\ell + 2)} = 0,$$

which trivially holds. Thus the distribution of (21) converges weakly as $\ell \rightarrow \infty$ to the standard normal distribution.

Since the weak convergence of distributions is metrizable (e.g. with the Lévy–Prohorov distance), we get from the above two weak convergences that there exists a sequence $\ell(n)$, for which $\lim_{n \rightarrow \infty} \ell(n) = \infty$ holds and the distribution of (20) (with $\ell(n)$ in it instead of ℓ) converges weakly to the standard normal distribution. Here the condition $\lim_{n \rightarrow \infty} ((g(n) + \ell(n))/n) = 0$ (which is needed because in the regression equations the largest frequency must tend to zero) can be satisfied by choosing the sequence $\ell(n)$ so that $\lim_{n \rightarrow \infty} (\ell(n)/n) = 0$. \square

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