Publ. Math. Debrecen
79/3-4 (2011), 497–505
DOI: 10.5486/PMD.2011.5148

A correspondence theorem for L-functions and partial differential operators

By JERZY KACZOROWSKI (Poznań) and ALBERTO PERELLI (Genova)

Dedicated to Professors K. Győry, A. Pethő, J. Pintz and A. Sárközy with admiration to their works in number theory

Abstract. Given an L-function $F(s)$ from the extended Selberg class, we associate a function $\Phi_F(x,y)$. We show that the functions $\Phi_F(x,y)$ are, in the general case, the analogs of the modular forms associated with the GL_2 L-functions. Indeed, we prove that every $\Phi_F(x,y)$ is eigenfunction of a certain partial differential operator. Moreover, we prove a general correspondence theorem for such L-functions involving the functions $\Phi_F(x,y)$.

Let $F(s)$ be a function in the extended Selberg class $\mathcal{S}^\sharp$. This means that $(s-1)^m F(s)$ is entire of finite order for some non-negative integer m, $F(s)$ is representable for $\sigma > 1$ as an absolutely convergent Dirichlet series with coefficients $a(n)$ and satisfies the functional equation

$$\gamma(s)F(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s)$$

with $|\omega| = 1$ and

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j),$$

Mathematics Subject Classification: 11M41, 11F66, 11F25.
Key words and phrases: L-functions, Selberg class, correspondence theorems, modular forms, partial differential operators.

where $Q > 0$, $\Re\mu_j \geq 0$ and $\lambda_j > 0$. Here $\bar{f}(s) = \overline{f(\bar{s})}$. We also write $d_F = 2\sum_{j=1}^r \lambda_j$ for the degree of $F(s)$ and

$$\mu = \frac{1}{2} + \sum_{j=1}^{r} \left(\mu_j - \frac{1}{2}\right).$$

Moreover, for $x, y \in \mathbb{R}$ with $y > 0$, we let

$$\Phi_F(x,y) = y^{-\mu} \sum_{n=1}^{\infty} a(n)\tilde{\gamma}(n\sqrt{q}y)e(nx)$$

where $q = q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$ is the conductor of $F(s)$, $e(x) = e^{2\pi i x}$ and $\tilde{\gamma}(\xi)$ is the inverse Mellin transform of the gamma-factor $\gamma(s)$, i.e. for $\xi > 0$

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s)\xi^{-s}\mathrm{d}s.$$

Since $\tilde{\gamma}(\xi) \ll \xi^{-A}$ for every $A > 0$, the series defining $\Phi_F(x,y)$ has good convergence properties.

Examples of $\Phi_F(x,y)$. The function $\Phi_F(x,y)$ becomes a familiar object when $F(s)$ is a classical L-function of degree 2.

1. *Holomorphic cusp forms.* Let $f(z)$ be a holomorphic cusp form of weight k and level N

$$f(z) = \sum_{n=1}^{\infty} \alpha(n)e(nz), \quad z = x + iy;$$

see Ch. 7 of Iwaniec [3]. Writing $a(n) = \alpha(n)n^{-(k-1)/2}$, the normalized L-function associated with $f(z)$ is

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

hence $F(s) = L(s,f)$ is an entire function of degree 2 in $\mathcal{S}^\sharp$ with

$$\gamma(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right), \quad \mu = \frac{k-1}{2}, \quad q = N. \tag{1}$$

We therefore have

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s+\mu) \left(\frac{2\pi\xi}{\sqrt{q}}\right)^{-s} \mathrm{d}s = \left(\frac{\sqrt{q}}{2\pi\xi}\right)^{-\mu} e^{-2\pi\xi/\sqrt{q}}$$

and hence

$$\Phi_F(x,y) = (2\pi)^{\mu} \sum_{n=1}^{\infty} a(n) n^{\mu} e^{-2\pi ny + 2\pi inx} = (2\pi)^{\mu} f(z). \tag{2}$$

2. *Maass forms.* Let $f(z)$ be a Maass form of level N and given parity

$$f(z) = \sqrt{y} \sum_{n\neq 0} a(n) n^{\varepsilon} K_{i\kappa}(2\pi|n|y) e(nx), \quad z = x+iy$$

where $1/4+\kappa^2$ is the eigenvalue of $f(z)$, $K_{i\kappa}(z)$ is the Bessel K-function and $\varepsilon = 0$ if $f(z)$ is even, $\varepsilon = 1$ otherwise; see Ch. 3 of Terras [12]. The L-function associated with $f(z)$ is

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

hence $F(s) = L(s,f)$ is an entire function of degree 2 in $\mathcal{S}^{\sharp}$ with

$$\gamma(s) = \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s+\varepsilon+i\kappa}{2}\right) \Gamma\left(\frac{s+\varepsilon-i\kappa}{2}\right), \quad \mu = \varepsilon - \frac{1}{2},\ q = N.$$

In this case we have

$$\tilde{\gamma}(\xi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma\left(\frac{s+\varepsilon+i\kappa}{2}\right) \Gamma\left(\frac{s+\varepsilon-i\kappa}{2}\right) \left(\frac{\pi\xi}{\sqrt{q}}\right)^{-s} \mathrm{d}s.$$

Making the substitution $s+\varepsilon+i\kappa = 2w$ and using the following integral representation (obtained by the inverse Mellin transform of formula 11.1 of p. 115 of Oberhettinger [10], choosing $a = 2$)

$$K_{\nu}(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w)\Gamma(w-\nu) \left(\frac{z}{2}\right)^{-2w} \mathrm{d}w$$

with $c > \max(0, \Re\nu)$, we obtain

$$\tilde{\gamma}(\xi) = \left(\frac{\pi\xi}{\sqrt{q}}\right)^{\varepsilon} K_{i\kappa}\left(\frac{2\pi\xi}{\sqrt{q}}\right)$$

and hence

$$\Phi_F(x,y) = \pi^{\varepsilon}\sqrt{y} \sum_{n=1}^{\infty} a(n) n^{\varepsilon} K_{i\kappa}(2\pi ny) e(nx) = \frac{\pi^{\varepsilon}}{2} f(z). \tag{3}$$

Therefore, in both cases $\Phi_F(x,y)$ reduces (essentially) to the modular form to which $F(s)$ is associated. $\square$

In Theorem 1 below we assume that $\lambda_j \in \mathbb{Q}$ for every j. Hence, without loss of generality, we may assume that the λ_j are all equal and of the form

$$\lambda_j = \frac{1}{k} \quad k \in \mathbb{N},\ j = 1, \dots, r \tag{4}$$

(this can be seen by means of the multiplication formula for the Γ function, see [6]). In particular, the degree d_F is a rational number; note that it is expected that $d_F \in \mathbb{N}$ for every $F \in \mathcal{S}^\sharp$. For $F \in \mathcal{S}^\sharp$ satisfying (4) we consider the partial differential operator

$$\mathcal{D} = \prod_{j=1}^{r} \left(-\frac{1}{k} y \frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k} \right) - \frac{(2\pi)^{r-k}}{k^r i^k} y^k \frac{\partial^k}{\partial x^k},$$

where multiplication means composition of differential operators. Note that $\Phi_F(x,y)$ depends strongly on $F(s)$, while $\mathcal{D}$ depends only on the functional equation satisfied by $F(s)$. However, the operator $\mathcal{D}$ is not invariant, in the sense that it depends on the shape of the functional equation of $F(s)$, which may be changed by applications of the multiplication formula for the Γ function. We have

Theorem 1. *Let $F \in \mathcal{S}^\sharp$ satisfy* (4). *Then*

$$\mathcal{D}\Phi_F(x,y) = 0.$$

Remark. Let $\mathcal{D}_0$ be a partial differential operator of the form

$$\mathcal{D}_0 = \prod_{j=1}^{r} \left(-\lambda y \frac{\partial}{\partial y} + \nu_j \right).$$

It is not difficult to detect the structure of such operators. Indeed, it can be proved that $\mathcal{D}_0$ can be written as

$$\mathcal{D}_0 = \sum_{j=0}^{r} W_{j,r}(\lambda, \nu_j) y^j \frac{\partial^j}{\partial y^j}$$

where the $W_{j,r}$ are polynomials satisfying

$$W_{j,r+1} = (\nu_{r+1} - j\lambda) W_{j,r} - \lambda W_{j-1,r}, \quad W_{0,r} = \prod_{j=1}^{r} \nu_j, \quad W_{r,r} = (-1)^r \lambda^r.$$

Moreover, the ring generated by such operators is the polynomial ring $\mathbb{C}\left[y\frac{\partial}{\partial y}\right]$. $\square$

Proof of Theorem 1. From the definition of $\tilde{\gamma}(\xi)$ and $\Phi_F(x,y)$ we have

$$\frac{1}{k}y\frac{\partial}{\partial y}\Phi_F(x,y) = -\frac{\mu}{k}\Phi_F(x,y) - y^{-\mu}\sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(s)(n\sqrt{q}y)^{-s}\frac{s}{k}\mathrm{d}s\, e(nx)$$

$$= \left(\mu_j - \frac{\mu}{k}\right)\Phi_F(x,y) - y^{-\mu}\sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(s)\left(\frac{s}{k}+\mu_j\right)(n\sqrt{q}y)^{-s}\mathrm{d}s\, e(nx),$$

hence

$$\left(-\frac{1}{k}y\frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k}\right)\Phi_F(x,y)$$
$$= y^{-\mu}\sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(s)\left(\frac{s}{k}+\mu_j\right)(n\sqrt{q}y)^{-s}\mathrm{d}s\, e(nx).$$

Therefore

$$\prod_{j=1}^{r}\left(-\frac{1}{k}y\frac{\partial}{\partial y} + \mu_j - \frac{\mu}{k}\right)\Phi_F(x,y)$$
$$= y^{-\mu}\sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(s)\prod_{j=1}^{r}\left(\frac{s}{k}+\mu_j\right)(n\sqrt{q}y)^{-s}\mathrm{d}s\, e(nx). \quad (5)$$

By (4) and the factorial formula for the Γ function we have

$$\gamma(s)\prod_{j=1}^{r}\left(\frac{s}{k}+\mu_j\right) = Q^s\prod_{j=1}^{r}\Gamma\left(\frac{s}{k}+\mu_j+1\right) = Q^{-k}\gamma(s+k),$$

hence, by the substitution $s+k=w$ and using Cauchy's theorem, the right hand side of (5) becomes

$$Q^{-k}y^{-\mu}\sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(s+k)(n\sqrt{q}y)^{-s}\mathrm{d}s\, e(nx)$$
$$= Q^{-k}y^{-\mu}(\sqrt{q}y)^k\sum_{n=1}^{\infty} a(n)n^k\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \gamma(w)(n\sqrt{q}y)^{-w}\mathrm{d}w\, e(nx)$$
$$= Q^{-k}y^{-\mu}(\sqrt{q}y)^k\sum_{n=1}^{\infty} a(n)n^k\tilde{\gamma}(n\sqrt{q}y)e(nx)$$
$$= \left(\frac{\sqrt{q}}{2\pi i Q}\right)^k y^k\frac{\partial^k}{\partial x^k}\Phi_F(x,y). \quad (6)$$

The result follows now from (5), (6) and the definition of q. □

Special cases of Theorem 1. Let us examine again two classical cases of Theorem 1.

1. *Holomorphic cusp forms.* In this case by (1) we have

$$\mathcal{D} = -y\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} = iy\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right),$$

hence $\mathcal{D}\Phi_F(x,y) = 0$ means

$$\frac{\partial \Phi_F(x.y)}{\partial x} = -i\frac{\partial \Phi_F(x.y)}{\partial y}$$

i.e. the Cauchy–Riemann equations. But, see (2), $\Phi_F(x,y) = (2\pi)^\mu f(z)$, hence $\mathcal{D}\Phi_F(x,y) = 0$ is equivalent to the fact that $f(z)$ is holomorphic.

2. *Maass forms.* In this case we have

$$\begin{aligned}\mathcal{D} &= \left(-\frac{y}{2}\frac{\partial}{\partial y} + \frac{\varepsilon + i\kappa}{2} - \frac{\varepsilon - 1/2}{2}\right)\left(-\frac{y}{2}\frac{\partial}{\partial y} + \frac{\varepsilon - i\kappa}{2} - \frac{\varepsilon - 1/2}{2}\right) + \left(\frac{y}{2}\right)^2 \frac{\partial^2}{\partial x^2} \\ &= \frac{y^2}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{4}\left(\frac{1}{4} + \kappa^2\right).\end{aligned}$$

Hence $\mathcal{D}\Phi_F(x,y) = 0$ becomes

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_F(x,y) = \left(\frac{1}{4} + \kappa^2\right)\Phi_F(x,y)$$

i.e. $\Phi_F(x,y)$ is an eigenfunction of the hyperbolic laplacian, with eigenvalue $\frac{1}{4}+\kappa^2$, as expected by (3). □

The classification of the degree 2 L-functions in the Selberg class (i.e., roughly, the Dirichlet series with functional equation of degree 2 and Euler product) is a well known open problem, see Selberg [11], Conrey–Ghosh [2] and our survey papers [5], [4], [7], [8], [9]. Roughly speaking, it is expected that such functions are the L-functions associated with the holomorphic and non-holomorphic modular forms. The classification of the degree 2 functions in the extended Selberg class (where no Euler product is assumed) is more difficult, and as far as we know there isn't even a clear expectation about it. The above special cases suggest that the functions $\Phi_F(x,y)$ are, in the general case, the analogs of the classical modular forms. Such analogy is supported also by the following extension of the Hecke correspondence theorem; we refer e.g. to Berndt–Knopp [1] for an account of Hecke's correspondence theorem. Suppose that the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

is absolutely convergent for $\sigma>1$, and let $\mu\in\mathbb{C}$, $q>0$, $Q>0$, $\lambda_j>0$ and $\Re\mu_j\geq 0$. We write $\gamma(s)$, $\tilde{\gamma}(\xi)$ and $\Phi_F(x,y)$ as above, and

$$\tilde{\gamma}^*(\xi)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\bar{\gamma}(s)\xi^{-s}\mathrm{d}s \qquad \Phi_F^*(x,y)=y^{-\bar{\mu}}\sum_{n=1}^{\infty}\overline{a(n)}\tilde{\gamma}^*(n\sqrt{q}y)e(nx).$$

With this notation our general correspondence theorem is

Theorem 2. *Let* $|\omega|=1$. *With the above notation the following statements are equivalent.*

(i) $F(s)$ *extends to an entire function of finite order and satisfies the functional equation*

$$\gamma(s)F(s)=\omega\bar{\gamma}(1-s)\bar{F}(1-s).$$

(ii) *For* $y>0$ *we have*

$$\Phi_F(0,y)=\omega q^{-\bar{\mu}-1/2}y^{-\bar{\mu}-\mu-1}\Phi_F^*\left(0,\frac{1}{qy}\right).$$

Remarks. **1.** Results of type of Theorem 2 already exist in the literature, mainly due to S. Bochner and his collaborators and followers. We refer to Chapters 7 and 8 of Berndt–Knopp [1] and to the literature quoted there for several variants of the principle underlying Theorem 2.

2. The condition that $F(s)$ is entire is not crucial in Theorem 2. Indeed, the same argument proves a version of Theorem 2 where $F(s)$ in (i) has a pole at $s=1$ and the modular relation in (ii) is slightly modified by adding terms corresponding to the residue of $F(s)$ at $s=1$. □

Proof of Theorem 2. We first show that (i) implies (ii). Since $F(s)$ has polynomial growth, the integrals below have good convergence properties, justifying our formal manipulations. Note first that, thanks to the functional equation, $F(s)\gamma(s)$ is an entire function since $F(s)$ is entire and $\gamma(s)$ is holomorphic for $\sigma>0$. Hence by Cauchy's theorem we have

$$\Phi_F(0,y)=y^{-\mu}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\gamma(s)F(s)(\sqrt{q}y)^{-s}\mathrm{d}s$$

$$= \omega y^{-\mu} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(1-s)\bar{F}(1-s)(\sqrt{q}y)^{-s} \mathrm{d}s$$
$$= \omega y^{-\mu} \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \bar{\gamma}(w)\bar{F}(w)(\sqrt{q}y)^{w-1} \mathrm{d}w$$
$$= \omega y^{-\mu} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(w)\bar{F}(w)(\sqrt{q}y)^{w-1} \mathrm{d}w$$
$$= \omega q^{-1/2} y^{-\mu-1} \sum_{n=1}^{\infty} \overline{a(n)} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \bar{\gamma}(w) \left(\frac{n\sqrt{q}}{qy}\right)^{-w} \mathrm{d}w,$$

hence

$$\Phi_F(0,y) = \omega q^{-\bar{\mu}-1/2} y^{-\bar{\mu}-\mu-1} \Phi_F^*\left(0, \frac{1}{qy}\right)$$

and the first statement follows. Now we prove that *(ii)* implies *(i)*. We have

$$\left(\frac{y}{\sqrt{q}}\right)^{\mu} \Phi_F\left(0, \frac{y}{\sqrt{q}}\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \gamma(s)F(s)y^{-s} \mathrm{d}s,$$

hence by the inversion formula of Mellin's transform we get

$$\gamma(s)F(s) = q^{-\mu/2} \int_0^{\infty} \Phi_F\left(0, \frac{y}{\sqrt{q}}\right) y^{s+\mu-1} \mathrm{d}y.$$

Now we apply the well known trick of splitting the integration over $(0,\infty)$ into $(0,1)\cup(1,\infty)$ and then transforming the integral over $(0,1)$. We get

$$q^{-\mu/2} \int_0^1 \Phi_F\left(0, \frac{y}{\sqrt{q}}\right) y^{s+\mu-1} \mathrm{d}y = \omega q^{-\bar{\mu}/2} \int_0^1 \Phi_F^*\left(0, \frac{1}{\sqrt{q}y}\right) y^{s-\bar{\mu}-2} \mathrm{d}y$$
$$= \omega q^{-\bar{\mu}/2} \int_1^{\infty} \Phi_F^*\left(0, \frac{y}{\sqrt{q}}\right) y^{-s+\bar{\mu}} \mathrm{d}y.$$

Therefore

$$\gamma(s)F(s) = q^{-\mu/2} \int_1^{\infty} \Phi_F\left(0, \frac{y}{\sqrt{q}}\right) y^{s+\mu-1} \mathrm{d}y + \omega q^{-\bar{\mu}/2} \int_1^{\infty} \Phi_F^*\left(0, \frac{y}{\sqrt{q}}\right) y^{-s+\bar{\mu}} \mathrm{d}y. \tag{7}$$

Thanks to the decay properties of $\tilde{\gamma}(\xi)$ and $\tilde{\gamma}^*(\xi)$ as $\xi \to \infty$, the right hand side of (7) provides the analytic continuation of $\gamma(s)F(s)$ to $\mathbb{C}$, as well as the functional equation. The other properties in *(i)* follow from this in a standard way. □

Acknowledgments. This research was partially supported by the Istituto Nazionale di Alta Matematica, by a grant PRIN2008 and by grant N N201 605940 of the National Science Centre.

References

[1] B. C. Berndt and M. I. Knopp, Hecke's Theory of Modular Forms and Dirichlet Series, *World Scientific*, 2008.

[2] J. B.Conrey and A. Ghosh, On the Selberg class of Dirichlet series: small degrees, *Duke Math. J.* **72** (1993), 673–693.

[3] H. Iwaniec, Topics in classical automorphic forms, *Graduate Studies in Mathematics* **17**, *American Mathematical Society, Providence, RI*, 1997.

[4] J. Kaczorowski, Axiomatic theory of L-functions: the Selberg class, In: Analytic Number Theory, C.I.M.E. Summer School, Cetraro (Italy) 2002, (A. Perelli and C. Viola, eds.), *Springer L.N. 1891*, 2006.

[5] J. Kaczorowski and A. Perelli, The Selberg class: a survey, In: Number Theory in Progress, Proc. Conf. in Honor of A. Schinzel, (K.Győry *et al.*, eds.), *de Gruyter*, 1999.

[6] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, II: invariants and conjectures, *J. Reine Angew. Math.* **524** (2000), 73–96.

[7] A. Perelli, A survey of the Selberg class of L-functions, part I, *Milan J. Math.* **73** (2005), 19–52.

[8] A. Perelli, A survey of the Selberg class of L-functions, part II, *Riv. Mat. Univ. Parma* (7) **3*** (2004), 83–118.

[9] A. Perelli, Non-linear twists of L-functions: a survey, *Milan J. Math.* **78** (2010), 117–134.

[10] F. Oberhettinger, Tables of Mellin Transforms, *Springer Verlag*, 1974.

[11] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, In: Proc. Amalfi Conf. Analytic Number Theory, (E. Bombieri *et al.*, eds.), *Università di Salerno*, 1992, 367–385, *Collected Papers*, vol. II, 47–63, *Springer Verlag*, 1991.

[12] A. Terras, Harmonic Analysis on Symmetric Spaces and Applications I, *Springer Verlag*, 1985.

JERZY KACZOROWSKI
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
A. MICKIEWICZ UNIVERSITY
61-614 POZNAŃ
POLAND

AND

INSTITUTE OF MATHEMATICS
OF THE POLISH ACADEMY OF SCIENCES
00-956 WARSAW
POLAND

E-mail: kjerzy@amu.edu.pl

ALBERTO PERELLI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI GENOVA
VIA DODECANESO 35
16146 GENOVA
ITALY

E-mail: perelli@dima.unige.it

(Received February 14, 2011; revised September 7, 2011)