Publ. Math. Debrecen
79/3-4 (2011), 531–537
DOI: 10.5486/PMD.2011.5120

On Kleinbock's Diophantine result

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Abstract. We give an elementary proof of a metrical Diophantine result by D. Kleinbock related to badly approximable vectors in affine subspaces.

1. Definitions and notation

Let $\mathbb{R}^d$ be a Euclidean space with the coordinates $(x_1, \dots, x_d)$, let $\mathbb{R}^{d+1}$ be a Euclidean space with the coordinates $(x_0, x_1, \dots, x_d)$. For $\mathbf{x} \in \mathbb{R}^d$ or $\mathbf{x} \in \mathbb{R}^{d+1}$ we denote by $|\mathbf{x}|$ its sup-norm:

$$|\mathbf{x}| = \max_{1 \le j \le d} |x_j| \quad \text{or} \quad |\mathbf{x}| = \max_{0 \le j \le d} |x_j|.$$

Consider an affine subspace $A \subset \mathbb{R}^d$ and define the affine subspace $\mathcal{A} \subset \mathbb{R}^{d+1}$ in the following way:

$$\mathcal{A} = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in A\}.$$

Let B be another affine subspace such that $B \subset A$. Define

$$\mathcal{B} = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in B\}, \quad \mathcal{B} \subset \mathcal{A}.$$

Put

$$a = \dim A = \dim \mathcal{A}, \quad b = \dim B = \dim \mathcal{B}.$$

We define *linear* subspaces

$$\mathfrak{A} = \operatorname{span} \mathcal{A}, \quad \mathfrak{B} = \operatorname{span} \mathcal{B}$$

Mathematics Subject Classification: 11J13.
Key words and phrases: badly approximable vectors, affine subspaces.
This paper is supported by grants NSh-8784.2010.1 and RFBR 09-01-00371.

as the smallest linear subspaces in $\mathbb{R}^{d+1}$ containing $\mathcal{A}$ or $\mathcal{B}$ respectively. So

$$\dim \mathfrak{A} = a + 1, \quad \dim \mathfrak{B} = b + 1.$$

Let $\psi(T)$, $T \geq 1$ be a positive valued function decreasing to zero as $T \to +\infty$. We define an affine subspace B to be *ψ-badly approximable* if

$$\inf_{\mathbf{x} \in \mathbb{Z}^{d+1} \setminus \{\mathbf{0}\}} \left(\frac{1}{\psi(|\mathbf{x}|)} \inf_{\mathbf{y} \in \mathfrak{B}} |\mathbf{x} - \mathbf{y}| \right) > 0. \tag{1}$$

This definition is very convenient for our exposition.

Here we would like to note that in the case when the affine subspace B has zero dimension (and hence $B = \{\mathbf{w}\}$ consists of just one nonzero vector $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d$) the definition (1) gives

$$\inf_{\mathbf{x} \in \mathbb{Z}^{d+1} \setminus \{\mathbf{0}\}} \left(\frac{1}{\psi(|\mathbf{x}|)} \inf_{t \in \mathbb{R}} |\mathbf{x} - t\mathbf{w}^*| \right) > 0 \tag{2}$$

with $\mathbf{w}^* = (1, w_1, \ldots, w_d)$. Under mild restrictions on the function ψ the inequality (2) is equivalent to the condition that there exists a positive constant $\gamma = \gamma(\mathbf{w})$ such that

$$\max_{1 \leq j \leq d} \|w_j q\| \geq \gamma \cdot \psi(|q|), \quad \forall\, q \in \mathbb{Z} \setminus \{0\} \tag{3}$$

(here $\|\cdot\|$ denotes the distance to the nearest integer). We can consider the inequality

$$\inf_{T \geq 1} \frac{\psi(\kappa T)}{\psi(T)} > 0, \quad \forall\, \kappa \geq 1 \tag{4}$$

as an example of such a condition on the function ψ.

The condition (3) is a usual condition of ψ-badly simultaneously approximability of the vector $\mathbf{w}$. This explains our definition (1).

2. Badly approximable vectors in affine subspaces

We consider a special case $b = 0$ and $\psi_d(T) = T^{-1/d}$. Then ψ_d-badly approximable vectors $\mathbf{w}$ are known as simultaneously badly approximable vectors (see [Schm], Chapter 2). Take

$$\varphi_{d,\Delta}(T) = \psi_d(T) \cdot (\log T)^{-\Delta} = T^{-\frac{1}{d}} (\log T)^{-\Delta}. \tag{5}$$

Theorem 1. *Consider an affine subspace $A \subset \mathbb{R}^d$. Suppose that there exists a badly approximable vector $\mathbf{w} \in A$. Suppose that $\Delta > \frac{1}{a}$. Then almost all vectors from the subspace A are $\varphi_{d,\Delta}$-badly approximable vectors, where $\varphi_{d,\Delta}(T)$ is defined in* (5).

We should note that a similar result dealing with 'not very well approximable' vectors (this is a condition weaker than 'badly approximabe' vectors) was obtained by Dmitry Kleinbock (see [Kle1], Theorem 4.2) by means of theory of dynamics on homogeneous spaces. Some generalizations are due to Dmitry Kleinbock [Kle2] and Yuqing Zhang [Zha]. We also refer to a relevant paper [Kle3]. We would like to note that the main subject of the papers [Kle1], [Kle2], [Zha] is dynamical approach to Diophantine approximations on submanifolds. A weaker result was announced by Mikhail B. Sevryuk [Sevr].

Consider the function $\psi_a(T) = T^{-\frac{1}{a}}$. It can be shown that for a given affine subspace $A \subset \mathbb{R}^d$ of dimension $\dim A = a \geq 1$ there exists a ψ_a-badly approximable vector $\mathbf{w}$ such that $\mathbf{w} \in A$. Moreover in this case ψ_a-badly approximable vectors in A form a winning set (see [Mosh], where even a little bit stronger result is proved).

Obviously we have the following corollary to Theorem 1.

Corollary 1. *For any affine subspace $A \subset \mathbb{R}^d$ almost all vectors from A are $\varphi_{a,\Delta}$-badly approximable vectors, where $\varphi_{a,\Delta}(T)$ is defined in* (5).

3. General result

In the sequel we consider two positive decreasing to zero functions $\psi(T)$, $\varphi(T)$ under the condition $\varphi(T) \leq \psi(T)$. We consider the following condition on function $\varphi(T)$:

$$0 < \inf_{T \geq 1} \frac{\varphi(\kappa T)}{\varphi(T)} \leq \sup_{T \geq 1} \frac{\varphi(\kappa T)}{\varphi(T)} < +\infty, \quad \forall \kappa \geq 1. \tag{6}$$

For these functions we define quantities

$$\mu_T = \left(\frac{T}{\psi(T)}\right)^{a-b}, \quad \lambda_T = \left(\frac{\varphi(T)}{T}\right)^a - \left(\frac{\varphi(T+1)}{T+1}\right)^a.$$

Now we are ready to formulate the main result of this paper.

Theorem 2. *Consider function $\psi(T)$ decreasing to zero as $T \to +\infty$. Suppose that for all $T \geq 1$ one has $0 < \varphi(T) \leq \psi(T)$ and the series*

$$\sum_{T=1}^{+\infty} \mu_T \lambda_T \tag{7}$$

converges. Suppose that ψ satisfies (4) *and φ satisfies* (6)*. Suppose that $B \subset A$ are affine subspaces and $0 \leq b = \dim B < a = \dim A \leq d$. Let B be a ψ-badly approximable affine subspace.*

Then almost all (in the sense of Lebesgue measure) vectors $\mathbf{w} \in A$ are φ-badly approximable vectors.

We should note that Theorem 1 immediately follows from Theorem 2 (one should take $b = 0$) as in the case $\Delta > \frac{1}{a}$ the series (7) converges.

We believe that the condition that (7) converges in Theorem 1 is optimal. Here we would like to refer to related papers [BBDD], [Gho1], [Gho2]; in particular these papers study metric Diophantine approximations with respect to convergence/divergence condition.

4. Proof of Theorem 2

Take $R \geq 1$. In the sequel we do not pay attention to the constants. All constants in the symbols $\ll$, $\asymp$ may depend on d, subspaces A, B and R.

For a set $\mathfrak{C} \subset \mathbb{R}^{d+1}$ and a point $\mathbf{x} \in \mathbb{R}^{d+1}$ we define the distance $|\mathbf{x}|_{\mathfrak{C}}$ from $\mathbf{x}$ to $\mathfrak{C}$ by

$$|\mathbf{x}|_{\mathfrak{C}} = \inf_{\mathbf{y} \in \mathfrak{C}} |\mathbf{x} - \mathbf{y}|.$$

Consider the set

$$\Omega_T = \{\mathbf{z} = (z_0, z_1, \ldots, z_d) \in \mathbb{R}^{d+1} : 0 \leq z_0 \leq T, \ \max_{1 \leq j \leq d} |z_j| \leq RT, \\ |\mathbf{z}|_{\mathfrak{B}} \leq \gamma \cdot \psi(RT)\}.$$

Here we suppose that $\gamma > 0$ is strictly less than the infimum in (1). As B is a ψ-badly approximable subspace we see that

$$\Omega_T \cap \mathbb{Z}^{d+1} = \{\mathbf{0}\}$$

(we use the definition (1)). Now we observe that any translation of the 1/2-contracted set

$$\frac{1}{2} \cdot \Omega_T + \mathbf{c}, \quad \mathbf{c} \in \mathbb{R}^{d+1} \tag{8}$$

consists of not more than one integer point. Indeed if two different integer points $\mathbf{x}$, $\mathbf{y}$ belong to the same set of the form (8) then $\mathbf{0} \neq \mathbf{x} - \mathbf{y} \in \Omega_T$. This is not possible.

Consider the set

$$\Pi_T = \{\mathbf{z} \in \mathbb{R}^{d+1} : 0 \le z_0 \le T,\ \max_{1 \le j \le d} |z_j| \le RT,\ |\mathbf{z}|_{\mathfrak{A}} \le \varphi(RT)\}.$$

Let ν_T be the minimal number of points $\mathbf{c}_j, 1 \le j \le \nu_T$ such that

$$\Pi_T \subset \bigcup_{j=1}^{\nu_T} \left(\frac{1}{2} \cdot \Omega_T + \mathbf{c}_j \right).$$

As $\varphi(T) \le \psi(T)$ and ψ satisfies (4), we see that $\nu_T \ll \mu_T$. That is why the set Π_T can be covered by not more than $C \cdot \mu_T$ different sets of the form (8) (here C is a positive constant). So we deduce a conclusion about an upper bound for the number of integer points in Π_T:

$$\# \left(\Pi_T \cap \mathbb{Z}^{d+1} \right) \ll \mu_T. \tag{9}$$

For an integer $T \ge 1$ we consider the set

$$\mathbb{Z}_T = \{\mathbf{z} = (z_0, z_1, \ldots, z_d) \in \mathbb{Z}^{d+1} : z_0 = T,\ \max_{1 \le j \le d} |z_j| \le RT,\ |\mathbf{z}|_{\mathfrak{A}} \le \varphi(RT)\}$$

of the cardinality

$$\zeta_T^{(R)} = \# \mathbb{Z}_T, \quad 0 \le \zeta_T^{(R)} \ll T^a. \tag{10}$$

For $\rho > 0$ and $\mathbf{z} \in \mathbb{R}^{d+1}$ put

$$\mathfrak{U}_\rho(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^{d+1} : |\mathbf{z} - \mathbf{y}| \le \rho\}.$$

Consider the set

$$\mathfrak{U}_T = \bigcup_{\mathbf{z}} \mathfrak{U}_{\varphi(RT)}(\mathbf{z})$$

where the union is taken over all integer points $\mathbf{z}$ such that

$$z_0 = T, \quad \max_{1 \le j \le d} |z_j| \le RT, \quad \mathfrak{U}_{\varphi(RT)}(\mathbf{z}) \cap \mathfrak{A} \neq \varnothing.$$

Clearly

$$\mathfrak{U}_T = \bigcup_{\mathbf{z} \in \mathbb{Z}_T} \mathfrak{U}_{\varphi(RT)}(\mathbf{z})$$

Consider the cone

$$\mathfrak{G}_T = \{\mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{x} = t \cdot \mathbf{y},\ t \in \mathbb{R}, \mathbf{y} \in \mathfrak{U}_T\}$$

and the projection

$$\mathcal{U}_T = \mathcal{A} \cap \mathfrak{G}_T.$$

Now we suppose that $\mathbf{w} \in A \cap \{|\mathbf{z}| \le R\}$ is a φ-badly approximable vector. Then from (3) it follows that for any positive γ there exist infinitely many positive integers q such that

$$\max_{1 \le j \le d} \|w_j q\| < \gamma\varphi(q).$$

Define integers z_j from the condition

$$\|w_j q\| = |w_j q - z_j|, \quad 1 \le j \le d.$$

So we consider integer vector $\mathbf{z} = (q, z_1, \ldots, z_d) \in \mathbb{Z}^{d+1}$. We suppose γ to be small enough. Then by the left inequality from (6) for the corresponding vector $\mathbf{w}^* \in \mathcal{A}$ one has

$$q\mathbf{w}^* \in \mathfrak{U}_{\gamma\varphi(q)}(\mathbf{z}) \subset \mathfrak{U}_{\varphi(Rq)}(\mathbf{z}).$$

As $\mathbf{w} \in \{|\mathbf{z}| \le R\}$ and $\mathbf{w}^* \in \mathcal{A}$ we have $q\mathbf{w}^* \in \mathfrak{U}_T$ and $\mathbf{w}^* \in \mathcal{U}_T$. So if $\mathbf{w} \in A \cap \{|\mathbf{z}| \le R\}$ is a φ-badly approximable vector then there exist infinitely many T such that

$$\mathbf{w}^* \in \mathcal{U}_T.$$

Consider the series

$$\sum_{T=1}^{+\infty} \mathrm{mes}_a \mathcal{U}_T \tag{11}$$

where mes_a stands for the a-dimensional Lebesgue measure. Suppose that the series (11) converges. Then by the Borel–Cantelli lemma argument we see that the set of non-φ-badly approximable vectors $\mathbf{w}^* \in \mathcal{A} \cap \{|\mathbf{z}| \le R\}$ is a set of zero measure. As R is arbitrary we see that Theorem 2 follows from the convergence of the series (11).

Now we show that the series (11) converges. Note that the set $\mathfrak{U}_T$ is a union of not more than $\zeta_T^{(R)}$ balls (in sup-norm) of radius $\varphi(RT)$. Hence the set $\mathcal{U}_T \subset \mathcal{A}$ can be covered by not more than $\zeta_T^{(R)}$ balls of radius $\varphi(RT)/T$. So in order to prove the convergence of the series (11) one can establish the convergence of the series

$$\sum_{T=1}^{+\infty} \zeta_T^{(R)} \cdot \left(\frac{\varphi(T)}{T}\right)^a \tag{12}$$

(we take the right inequality form (6) into account).

It follows from (9) and monotonicity of ψ that

$$\sum_{j=1}^{T}\zeta_j^{(R)} \ll \sum_{\nu\leq\log T}\mu_{T/2^\nu} = \sum_{\nu\leq\log T}\left(\frac{T/2^\nu}{\psi(T/2^\nu)}\right)^{a-b} \ll \left(\frac{T}{\psi(T)}\right)^{a-b} = \mu_T.$$

Now we see that the convergence of the series (12) follows from the convergence of the series (7) by partial summation as from (10) we see that

$$\zeta_T^{(R)}\cdot\left(\frac{\varphi(T+1)}{T+1}\right)^a \ll (\varphi(T))^a \to 0, \quad T\to+\infty.$$

Theorem 2 is proved.

Acknowledgement. The author thanks anonymous referees for important suggestions.

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(Received February 1, 2011; revised October 13, 2011)