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A note on Euler's φ-function

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This paper is dedicated to Kálmán Győry, János Pintz, Attila Pethő and András Sárközy on the occasion of their round birthdays

Abstract. For a large class of multiplicative arithmetic functions f a method is given to determine the value of the leading coefficient in the asymptotical formula for the number of integers $n \leq x$ with $f(n)$ prime to a given integer. This is applied to Euler's φ-function, generalizing the result of [6].

1. Introduction

For an integer-valued arithmetic function f and an integer $N \geq 2$ denote by $F_N(f;x)$ the number of integers $n \leq x$ satisfying $(f(n), N) = 1$. If N is a prime, then $F_N(f;x)$ counts the integers $n \leq x$ with $N \nmid f(n)$.

It has been shown in 1964 by E. J. Scourfield ([4]) that for the function $\varphi(n)$ and odd primes q one has

$$F_q(\varphi, x) = (c(q) + o(1)) \frac{x}{\log^{1/(q-1)} x} \tag{1}$$

with some positive constant $c(q)$. This has been later put in a more general context in [2], [3] (see also [5]). The value of the coefficient $c(q)$ has been determined by B. K. Spearman and K. S. Williams [6] in 2006. The purpose of this note is to show that the approach utilized in [2], [3], [5] can be used to obtain the

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value of the leading coefficient in a similar asymptotic formula for a large class of integer-valued multiplicative functions, without restricting the integer q to be a prime.

2. Notation

By χ we shall denote multiplicative characters mod N, $L(s,\chi)$ will be the corresponding Dirichlet L-function and χ_0 the principal character. The letter p in formulas will be reserved for prime numbers. For $(k,N)=1$ and $\Re s>1$ put

$$g(N,k;s)=\sum_{p\equiv k \bmod N}\frac{1}{p^s}-\frac{1}{\varphi(N)}\log\frac{1}{s-1}. \tag{2}$$

We shall need a simple formula for the value of

$$g(N,k;1)=\lim_{s\to 1+0} g(N,k;s).$$

It belongs to the folklore, but we sketch the proof for the convenience of the readers.

Lemma 1. *If* $(N,k)=1$, *then*

$$g(N,k;1)=\frac{1}{\varphi(N)}\sum_{\chi\neq\chi_0}\overline{\chi(k)}\log L(1,\chi)-\frac{\alpha(N)}{\varphi(N)}-\beta(N,k),$$

where

$$\alpha(N)=\log\frac{N}{\varphi(N)},$$

and

$$\beta(N,k)=\sum_{j=2}^{\infty}\frac{1}{j}\sum_{p^j\equiv k \bmod N}\frac{1}{p^j}.$$

Proof. For characters χ mod N and $\Re s>1$ one has

$$\log L(\chi,s)=\sum_p\frac{\chi(p)}{p^s}+\sum_{j=2}^{\infty}\frac{1}{j}\sum_p\frac{\chi^j(p)}{p^{js}},\qquad \sum_p\frac{\chi_0(p)}{p^s}=\sum_p\frac{1}{p^s}-\sum_{p|N}\frac{1}{p^s},$$

and

$$\log\frac{1}{s-1}+r(s)=\log\zeta(s)=\sum_p\frac{1}{p^s}+\sum_{j=2}^{\infty}\frac{1}{j}\sum_p\frac{1}{p^{js}},$$

where $r(s)$ is regular for $\Re s \geq 1$ and vanishes at $s = 1$.

Thus

$$\sum_p \frac{\chi_0(p)}{p^s} = \log \frac{1}{s-1} + r(s) - \sum_{p|N} \frac{1}{p^s} - \sum_{j=2}^{\infty} \frac{1}{j} \sum_p \frac{1}{p^{js}}.$$

Applying Dirichlet's formula

$$\varphi(N) \sum_{p \equiv k \bmod N} p^{-s} = \sum_\chi \overline{\chi(k)} \sum_p \frac{\chi(p)}{p^s}$$

one arrives at

$$\varphi(N) \sum_{p \equiv k \bmod N} p^{-s} = \log \frac{1}{s-1} + r(s) - \sum_{p|N} \frac{1}{p^s} - \sum_{j=2}^{\infty} \frac{1}{j} \sum_p \frac{1}{p^{js}}$$
$$+ \sum_{\chi \neq \chi_0} \overline{\chi(k)} \log L(s, \chi) - \sum_{j=2}^{\infty} \frac{1}{j} \sum_p \frac{1}{p^{js}} \sum_{\chi \neq \chi_0} \overline{\chi(k)} \chi(p^j).$$

In view of

$$\sum_{\chi \neq \chi_0} \chi(a) = \begin{cases} -1 & \text{if } a \not\equiv 1 \bmod N \text{ and } (a, N) = 1, \\ \varphi(N) - 1 & \text{if } a \equiv 1 \bmod N, \\ 0 & \text{if } (a, N) > 1, \end{cases}$$

we get

$$\varphi(N) \sum_{p \equiv k \bmod N} p^{-s} = \log \frac{1}{s-1} + r(s) - \sum_{p|N} \frac{1}{p^s} - \sum_{j=2}^{\infty} \frac{1}{j} \sum_p \frac{1}{p^{js}}$$
$$+ \sum_{\chi \neq \chi_0} \overline{\chi(k)} \log L(s, \chi) + \sum_{j=2}^{\infty} \frac{1}{j} \sum_{p \nmid N} \frac{1}{p^{js}} - \varphi(N) \sum_{j=2}^{\infty} \frac{1}{j} \sum_{p^j \equiv k \bmod N} \frac{1}{p^{js}},$$

and this leads to

$$\varphi(N) g(N, k; 1) = \sum_{\chi \neq \chi_0} \overline{\chi(k)} \log L(1, \chi) \left(\sum_{j=1}^{\infty} \frac{1}{j} \sum_{p|N} \frac{1}{p^j} + \varphi(N) \sum_{j=2}^{\infty} \frac{1}{j} \sum_{p^j \equiv k \bmod N} \frac{1}{p^j} \right).$$

It remains to observe that

$$\sum_{j=1}^{\infty} \frac{1}{j} \sum_{p|N} \frac{1}{p^j} = \sum_{p|N} \log \frac{1}{1 - 1/p} = \log \prod_{p|N} \frac{1}{1 - 1/p} = \log \frac{N}{\varphi(N)}. \qquad \square$$

We shall need also the following lemma:

Lemma 2. *If $m \geq 2$ and $(k,N)=1$, then for $\Re s > 1/m$ one has*

$$\lim_{s\to 1/m+0}\left(\sum_{p\,\equiv k \bmod N}\frac{1}{p^{sm}}-\frac{1}{\varphi(N)}\log\frac{1}{s-1/m}\right)=g(N,k;1)-\frac{\log m}{\varphi(N)}.$$

Proof. Follows from (2) and the equality

$$\log(ms-1)=\log(s-1/m)+\log m. \qquad \square$$

For shortness we shall write

$$T(y)=\sum_{j=2}^{\infty}\frac{(-1)^{j+1}}{j}y^{-j}=\log(1+1/y)-1/y,$$

and

$$V(y)=\sum_{j=2}^{\infty}\frac{1}{jy^j}=\log\frac{1}{1-1/y}-1/y.$$

Moreover we put

$$\Phi(n)=\prod_{p|n}\left(1-\frac{1}{p-1}\right),$$

and for $(N,k)=1$ and $\Re s>1$

$$Z(N,k;s)=\prod_{p\,\equiv k \bmod N}\frac{1}{1-p^{-s}}.$$

3. Multiplicative functions

We shall deal with integer-valued multiplicative functions $f(n)$ which are polynomial-like, i.e. have the property that for $i=1,2,\dots$ and every prime p one has $f(p^i)=V_i(p)$, where $V_1,V_2,\dots$ are polynomials with integral coefficients.

For a fixed integer $N\geq 3$ and $i=1,2,\dots$ put

$$R_i(f,N)=\{x \bmod N : (xV_i(x),N)=1\}$$

and denote by $m=m(f,N)$ the smallest integer i for which the set is nonempty, if such integer exists.

For $\Re s > 1$ put

$$F(f, N; s) = \sum_{n=1}^{\infty} \frac{\chi_0(f(n))}{n^s}.$$

Expanding $F(f, N; s)$ in an Euler product one gets

$$F(f, N; s) = g(s)G(s),$$

with

$$g(s) = \prod_{p \mid N} \left(1 + \sum_{j=1}^{\infty} \frac{\chi_0(f(p^j))}{p^{js}}\right) \tag{3}$$

being regular for $\Re s > 0$ and

$$G(s) = \prod_{p \nmid N} \left(1 + \sum_{j=m}^{\infty} \frac{\chi_0(f_j(p))}{p^{js}}\right).$$

Since

$$\left|1 + \frac{\chi_0(f(p^m))}{p^{ms}}\right| \geq \frac{1}{2},$$

the product

$$H(s) = \prod_{p \nmid N} \frac{1 + \sum_{j=m}^{\infty} \chi_0(f(p^j))p^{-js}}{1 + \chi_0(f(p^m))p^{-ms}} \tag{4}$$

is regular for $\Re s \geq 1/m$.

Moreover

$$\prod_{p \nmid N} \left(1 + \frac{\chi_0(f(p^m))}{p^{ms}}\right) = \prod_{p \nmid N} \left(1 + \frac{\chi_0(V_m(p))}{p^{ms}}\right)$$

$$= \exp\left(\sum_{p \nmid N} \log\left(1 + \frac{\chi_0(V_m(p))}{p^{ms}}\right)\right) = \exp\left(\sum_{p \nmid N} \frac{\chi_0(V_m(p))}{p^{ms}} + h(s)\right),$$

where

$$h(s) = \sum_{p \nmid N} \chi_0(V_m(p)) \sum_{r=2}^{\infty} \frac{(-1)^{r+1}}{r} \frac{1}{p^{mrs}} \tag{5}$$

is regular for $\Re s \geq 1/m$.

This implies the equality

$$G(s) = H(s) \exp\left(\sum_{p \nmid N} \frac{\chi_0(V_m(p))}{p^{ms}} + h(s)\right), \tag{6}$$

valid for $\Re s > 1/m$.

Now write $R_m(f,N) = \{r_1 < r_2 < \cdots < r_t\}$, observe that

$$\sum_{p\nmid N} \frac{\chi_0(V_m(p))}{p^{ms}} = \sum_{j=1}^{t} \sum_{p\equiv r_j \bmod N} \frac{1}{p^{ms}},$$

and apply Lemma 2 to arrive at

$$\sum_{p\nmid N} \frac{\chi_0(V_m(p))}{p^{ms}} = \frac{t}{\varphi(N)} \log \frac{1}{s-1/m} + \sum_{j=1}^{t} g(N,r_j;1) - \frac{t\log m}{\varphi(N)} + \gamma(s),$$

where $\gamma(s)$ is regular for $\Re s \geq 1/m$ and $\gamma(1/m) = 0$.

The last equality and (6) give now

$$F(f,N;s) = A(s)/(s-1/m)^B,$$

where

$$A(s) = g(s)H(s)\exp\left(h(s) + \sum_{j=1}^{t} g(N,r_j;1) - \frac{t\log m}{\varphi(N)} + \gamma(s)\right) \tag{7}$$

and

$$B = t/\varphi(N). \tag{8}$$

Observe also that $A(s)$ cannot vanish at $s = 1/m$. Indeed, otherwise one would have either $1 \leq g(1/m) = 0$, or $H(1/m) = 0$, thus for some prime $p \nmid N$ one must have

$$\sum_{j=m}^{\infty} \frac{\chi_0(V_j(p))}{p^{j/m}} = -1,$$

which is obviously not possible.

Applying the Tauberian theorem of Delange ([1], [7], p. 275) to the function $F(f,N;s)$ one gets the following assertion:

Theorem 1. *Let f be a polynomial-like integral-valued multiplicative function and let $N \geq 2$ be an integer. Then*

$$\#\{n \leq x : (f(n),N) = 1\} = \left(\frac{mA}{\Gamma(t/\varphi(N))} + o(1)\right) \frac{x^{1/m}}{\log^{1-t/\varphi(N)} x},$$

with

$$A = g(1/m)H(1/m)\exp\left(h(1/m) + \sum_{j=1}^{t} g(N,r_j;1) - \frac{t\log m}{\varphi(N)}\right), \tag{9}$$

the functions $g(N,k;s)$, $g(s)$, $H(s)$, $h(s)$ being defined in (2), (3), (4) *and* (5), *respectively, and $m = m(f,N)$.*

4. Euler's function

We apply now Theorem 1 to Euler's function $\varphi(n)$.

Theorem 2. *If $N \geq 3$ is an odd integer, then*

$$F_N(\varphi, x) = (c(N) + o(1))\frac{x}{\log^a x},$$

where

$$a = 1 - \Phi(N)$$

and

$$c(N) = \frac{1}{\Gamma(\Phi(N))}\left(\frac{\varphi(N)}{N}\right)^{\Phi(N)} \prod_{p|N,(p-1,N)=1}\left(1+\frac{1}{p}\right)$$
$$\cdot \prod_{\chi\neq\chi_0} L(1,\chi)^{-\Phi(f_\chi)}\left(\prod_{1<d|N}\prod_{1<\delta|\varphi(d)}\prod_{k\in T_d(\delta)} Z(N,k;\delta)^{\mu(d)/\delta}\right)^{-1},$$

where f_χ is the conductor of χ and $T_d(\delta)$ denotes the set of residue classes mod N *having order δ* mod d.

Proof. Let $N = \prod_{j=1}^s p_j^{\alpha_j} \geq 3$ be an odd integer, let R_1 denote the set of residue classes k mod N satisfying $(k(k-1), N) = 1$ and R let be their union. In our case we have $2 \in R_1(\varphi, N) = R_1 \neq \emptyset$ hence $m = m(\varphi, N) = 1$, and since the Chinese remainder theorem implies

$$\#R_1(\varphi, N) = \prod_{j=1}^{s} \#R_1(\varphi, p_j^{\alpha_j}),$$

hence

$$t = \#R_1 = \prod_{j=1}^{s}(p_j - 2)p_j^{\alpha_j - 1} = N\prod_{p|N}\left(1-\frac{2}{p}\right) \neq 0.$$

Since $\varphi(p^j) = (p-1)p^{j-1}$, the condition $(\varphi(p^j), N) = 1$ is equivalent to either $p \nmid N$ and $(p-1, N) = 1$, or $p|N$, $(p-1, N) = 1$ and $j = 1$. Therefore

$$g(1) = \prod_{p|N,(N,p-1)=1}\left(1+\frac{1}{p}\right), \qquad H(1) = \prod_{p\in R}\frac{1}{1-1/p^2}, \tag{10}$$

and

$$h(1) = \sum_{p\in R} T(p).$$

Moreover, by Lemma 1 we have

$$\sum_{k\in R} g(N,k;1) = \frac{1}{\varphi(N)} \sum_{\chi\neq\chi_0} \sum_{k\in R} \overline{\chi(k)} \log L(1,\chi) - \frac{t}{\varphi(N)} \log\frac{N}{\varphi(N)} - S \tag{11}$$

with

$$S = \sum_{k\in R} \beta(N,k). \tag{12}$$

Write P_d for the set of primes $p\nmid N$, congruent to 1 mod d. We can write S in the form

$$\begin{aligned}
S &= \sum_{j\geq 2}\frac{1}{j}\sum_{p^j\in R}\frac{1}{p^j} = \sum_{j\geq 2}\frac{1}{j}\sum_{p\nmid N}\frac{1}{p^j} - \sum_{j\geq 2}\frac{1}{j}\sum_{p\nmid N,\, p^j\notin R}\frac{1}{p^j}\\
&= \sum_{p\nmid N} V(p) + \sum_{d|N,\, d>1}\mu(d)\sum_{j\geq 2}\sum_{\substack{p\nmid N\\ p^j\equiv 1 \bmod d}}\frac{1}{jp^j}\\
&= \sum_{p\nmid N} V(p) + \sum_{d|N,\, d>1}\mu(d)\sum_{j\geq 2}\Bigg(\sum_{p^j\equiv 1 \bmod}d\frac{1}{jp^j} - \sum_{\substack{p|N\\ p^j\equiv 1 \bmod d}}\frac{1}{jp^j}\Bigg)\\
&= \sum_{p\nmid N} V(p) + \sum_{d|N,\, d>1}\mu(d)\sum_{j\geq 2}\frac{1}{j}\Bigg(\sum_{p\in P_d}\frac{1}{p^j} + \sum_{\substack{p\nmid N,\, p\notin P_d\\ p^j\equiv 1 \bmod d}}\frac{1}{p^j}\Bigg)\\
&= \sum_{p\nmid N} V(p) + \sum_{d|N,\, d>1}\mu(d)\sum_{p\in P_d} V(p)\\
&\quad + \sum_{d|N,\, d>1}\mu(d)\sum_{j\geq 2}\frac{1}{j}\sum_{\substack{p\nmid N,\, p\notin P_d\\ p^j\equiv 1 \bmod d}}\frac{1}{p^j} = A+B+C,
\end{aligned}$$

say. Since

$$B = \sum_{p\nmid N} V(p)\Bigg(\sum_{d|(N,p-1)}\mu(d) - 1\Bigg) = -\sum_{p\nmid N,\,(N,p-1)>1} V(p),$$

we get

$$A+B = \sum_{p\in R} V(p).$$

Now observe that if $o_d(p)$ denote the multiplicative order of p mod d, then $p^j \equiv 1 \bmod d$ holds if and only if $o_d(p)$ divides j. Therefore

$$\begin{aligned} C &= \sum_{1<d|N} \mu(d) \sum_{1<\delta|\varphi(d)} \sum_{\substack{p\nmid N \\ o_d(p)=\delta}} \frac{1}{\delta} \sum_{m=1}^{\infty} \frac{1}{mp^{m\delta}} \\ &= \sum_{1<d|N} \sum_{1<\delta|\varphi(d)} \frac{1}{\delta} \sum_{\substack{p\nmid N \\ o_d(p)=\delta}} \left(V(p^\delta) + \frac{1}{p^\delta} \right) \\ &= \sum_{1<d|N} \mu(d) \sum_{1<\delta|\varphi(d)} \frac{1}{\delta} \sum_{p, o_d(p)=\delta} \log \frac{1}{1-p^{-\delta}}, \end{aligned}$$

and finally we arrive at

$$S = A + B + C = \sum_{p\in R} V(p) + \log \left(\prod_{1<d|N} \prod_{1<\delta|\varphi(d)} \prod_{k\in T_d(\delta)} Z(N,k;\delta)^{\mu(d)/\delta} \right), \quad (13)$$

where $T_d(\delta)$ denotes the set of residue classes mod N having order δ mod d.

Moreover for $\chi \neq \chi_0$ we have

$$\sum_{k\in R_1} \overline{\chi(k)} = \sum_{k=1}^{N-1} \overline{\chi(k)} - \sum_{k\notin R_1} \overline{\chi(k)} = -\sum_{d|N} \mu(d) \sum_{k\equiv 1 \bmod d, (k,N)=1} \overline{\chi(k)}.$$

Let H_d be the subgroup of the multiplicative group mod N consisting of residue classes congruent to 1 mod d. In view of

$$\sum_{k\equiv 1 \bmod d, (k,N)=1} \overline{\chi(k)} = \begin{cases} \varphi(N)/\varphi(d) & \text{if } d|f_\chi, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\sum_{k\in R_1} \overline{\chi(k)} = -\varphi(N) \sum_{d|f_\chi} \frac{\mu(d)}{\varphi(d)} = -\varphi(N)\Phi(f_\chi),$$

hence

$$\exp\left(\frac{1}{\varphi(N)} \sum_{\chi\neq\chi_0} \sum_{k\in R_1} \overline{\chi(k)} \log L(1,\chi) \right) = \prod_{\chi\neq\chi_0} L(1,\chi)^{-\Phi(f_\chi)}. \quad (14)$$

Observe finally that one has

$$H(1) \exp\left(h(1) - \sum_{p\in R} V(p) \right) = 1. \quad (15)$$

The assertion follows now from (10), (11), (12), (13), (14), (15) and Theorem 1. □

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