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Characterization of additive functions with values in the circle group II.

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1. This paper is a continuation of [1], consequently we shall use the terminology and notations that were used there. By a topological group we mean a T_0 group that guarantees the existence of a translation invariant metric.

Our aim is to determine all $\varphi \in \mathcal{A}_G^* (D[2,-1])$, where G is a metrically compact Abelian topological group.

Assume that $\varphi \in \mathcal{A}_G^* (D[2,-1])$.

Let X_0^+ (resp. X_0^-) denote the set of limit points of $\{\varphi(4n+1) | n \in \mathbb{N}\}$ (resp. $\{\varphi(4n-1) | n \in \mathbb{N}\}$). Since the natural numbers $m \equiv 1 \pmod 4$ form a semigroup, therefore $\{\varphi(4n+1) | n \in \mathbb{N}\}$ is a semigroup as well, therefore X_0^+ has to be a closed semigroup, and by a known theorem (see[2]), X_0^+ is a compact subgroup in G. Hence we get immediately that $\varphi(n) \in X_0^+$ if $n \equiv 1 \pmod 4$. Let m_ν be an arbitrary sequence of positive integers from the residue class $\equiv -1 \pmod 4$, such that $\varphi(m_\nu) \to g$. Then $g \in X_0^-$. Then for (d, 2)=1 the limit $\varphi(dm_\nu) \to g + \varphi(d)$ exists as well, furthermore $g + \varphi(d) \in X_0^+$ if $d \equiv -1 \pmod 4$ and $g + \varphi(d) \in X_0^-$ if $d \equiv 1 \pmod 4$. This implies immediately that $X_0^- + X_0^- \subseteq X_0^+,\ X_0^- + X_0^+ \subseteq X_0^-$, whence we get that $X_0^- = X_0^+ + \varphi(3)$.

Furthermore, it is clear that $L[X_0] = X_0^+,\ S[X_0] = X_0^-$. Since $0 \in X_0^+$, therefore there exists a $\gamma \in X_0$ such that $L(\gamma) = 0$. But from (2.14) ([1]) stating that $S(\tau - L(\tau)) = 0\ \forall \tau \in X$, it follows that $S(\gamma) = 0$, i.e. $0 \in X_0^-$. Then $X_0^- = X_0^+ = X_0$.

By using the relation $S(\tau - L(\tau)) = 0\ \forall \tau \in X$ again, we have immediately

Lemma 1. *If $n_\nu \to \infty$ is a sequence of odd integers for which $\varphi(n_\nu) \to 0$, then $\varphi(n_\nu + 2) \to 0$, and consequently $\varphi(n_\nu + 2s) \to 0$ for each $s \in \mathbb{N}$.*

Let k be an odd integer. Then $\varphi(k) \in X_0$, and X_0 being a group, $-\varphi(k) \in X_0$, consequently there exists a suitable sequence of odd integers

$n_\nu \to \infty$, such that $\varphi(n_\nu) \to -\varphi(k)$. Let n_ν be such a sequence. Then $\varphi(kn_\nu) \to 0$, and so by Lemma 1, $\varphi(kn_\nu + 2s) \to 0$, in particular $\varphi(kn_\nu + 2k) \to 0$, $\varphi(n_\nu + 2) \to -\varphi(k)$. So we have proved

Lemma 2. *Let $k \in \mathbb{N}$ be an arbitrary odd integer. If $n_\nu \to \infty$ is such a sequence of odd integers for which $\varphi(n_\nu) \to -\varphi(k)$, then $\varphi(n_\nu + 2s) \to -\varphi(k)$ for each $s \in \mathbb{N}$.*

As an obvious consequence of Lemma 2, we get that $\varphi(2n_\nu + 4s - 1) \to L(-\varphi(k)), \varphi(2n_\nu + 4s + 1) \to S(-\varphi(k))$.

In other words, if $\varphi(n_\nu) \to -\varphi(k)$, k odd, then

$$\varphi(2n_\nu + m) \to \begin{cases} L(-\varphi(k)) & \text{for } m \equiv -1 \pmod 4 \\ S(-\varphi(k)) & \text{for } m \equiv 1 \pmod 4 \end{cases} \tag{1.1}$$

Let now $p, k \in \mathbb{N}$ be odd integers, and let N_ν be such a sequence of odd integers, for wich $\varphi(N_\nu) \to -\varphi(pk)$, i.e. $\varphi(pN_\nu) \to -\varphi(k)$.

Assume first that $p \equiv 1 \pmod 4$. Then $pm \equiv m \pmod 4$. Apply (1.1) first with $n_\nu = N_\nu$ and with an arbitrary odd $m \in \mathbb{N}$, then with $n_\nu = pN_\nu$ and with pm instead of m. From (1.1) we get that

$$\varphi(p) + S(-\varphi(p) - \varphi(k)) = S(-\varphi(k)) \tag{1.2}$$

$$\varphi(p) + L(-\varphi(p) - \varphi(k)) = L(-\varphi(k)) \tag{1.3}$$

In the case $p \equiv -1 \pmod 4$ we get similary that

$$\varphi(p) + S(-\varphi(p) - \varphi(k)) = L(-\varphi(k)), \tag{1.4}$$

$$\varphi(p) + L(-\varphi(p) - \varphi(k)) = S(-\varphi(k)). \tag{1.5}$$

Let us fix p, and let k run over all the odd integers. Taking into account that $\{-\varphi(k) | (k, 2) = 1, k \in \mathbb{N}\}$ is everywhere dense in X_0, furthermore that L, S are continuous, we get that

$$S(g) = S(g - \varphi(p)) + \varphi(p),\ L(g) = L(g - \varphi(p)) + \varphi(p) \tag{1.6}$$

if $p \equiv 1 \pmod 4$, and that

$$S(g) = L(g - \varphi(p)) + \varphi(p),\ L(g) = S(g - \varphi(p)) + \varphi(p) \tag{1.7}$$

if $p \equiv -1 \pmod 4$.

Since $\{\varphi(p) | p \equiv 1 \pmod 4\}$ is everywhere dense in $X_0^+ = X_0$, from (1.6) we have:

$$S(g) = S(g + h) - h,\ L(g) = L(g + h) - h,\ \forall g, h \in X_0 \tag{1.8}$$

Similarly, $\{\varphi(p) | p \equiv -1 \pmod 4\}$ is everywhere dense in $X_0^- = X_0$, consequently from (1.7) we have

$$S(g) = L(g + h) - H,\ L(g) = S(g + h) - h,\ \forall g, h \in X_0. \tag{1.9}$$

Hence we deduce immediately

Lemma 3. *If $g \in X_0$, then $L(g) = S(g)$, $L(g) = g + L(0)$, $S(g) = g + L(0)$.*

In [1] (Lemma 4) it was proved that

$$L(g) = S(g)\ \forall g \in X_0 \Rightarrow L(g) = S(g)\ \forall g \in X. \tag{1.10}$$

From (1.10), by using an argument based upon a suitable rarefaction of sequences, we get immediately

Lemma 4. *We have $\varphi(2n-1) - \varphi(2n+1) \to 0$ as $n \to \infty$.*

With some unimportant modification in the proof of Wirsing's theorem and in that of ours (see[3]) we have

Lemma 5. *There exists a continuous homomorphism $\Psi : \mathbb{R}_x \to G$, such that $\Psi(n) = \varphi(n)\ \forall n \in \mathbb{N}$, $(n, 2) = 1$.*

Let $\varphi \in \mathcal{A}_G^*(D[2,-1])$ and let Ψ be the continuous homomorphism corresponding to it. Let $u(n) := \varphi(n) - \Psi(n)$. Then $u \in \mathcal{A}_G^*$, $u(n) = 0$ for each odd n. Let Γ be the closed group generated by $\{ku(2) | k \in \mathbb{N}\}$. If $u(2) = 0$, then $\Gamma = \{0\}$ is the trivial group.

Lemma 6. *We have $\Gamma \cap X_0 = \{0\}$.*

Proof. Since Γ, X_0 are subgroups in G, therefore $0 \in \Gamma \cap X_0$. Let us assume that $\sigma \neq 0$, $\sigma \in \Gamma \cap X_0$. Let k_ℓ be such a sequence of positive integers for which $u(2^{k_\ell}) = k_\ell u(2) \to \sigma$. Let us choose an arbitrary $\lambda \in X_0$ and two sequences $\{m_\nu\}$; $\{N_\nu\}$ of odd integers so that $\varphi(m_\nu) \to \lambda$, $\varphi(N_\nu) \to \lambda + \sigma$. Then we have $\varphi(2^{k_\nu} m_\nu) \to \lambda + \sigma$, and so for the composed sequence

$$\{M_1, M_2, ...\} = \{2^{k_1} m_1, N_1, 2^{k_2} m_2, N_2, ...\}$$

we have that $\varphi(M_\ell) \to \lambda + \sigma$, and that $u(M_\ell)$ does not have a limit. From our assumption, $\varphi(2M_\ell - 1) \to L(\lambda + \sigma)$. Since $\varphi = \Psi$ for odd integers, therefore $\varphi(2M_\ell - 1) = \Psi(2M_\ell - 1)$, consequently $\Psi(2M_\ell - 1) \to L(\lambda + \sigma)$. Since Ψ is a continuous homomorphism, therefore $\Psi(M_\ell) \to L(\lambda + \sigma) - -\Psi(2)$, wich by $\varphi(M_\ell) \to \lambda + \sigma$ implies that $u(M_\ell)$ has a limit. This is a contradiction.

From the definition of X_0, Γ, X it is clear that $X = \Gamma + X_0$. Consequently each $\xi \in X$ can be written as $\gamma + \eta$, $\gamma \in \Gamma$, $\eta \in X_0$. Furthermore, this representation is unique, since from $\gamma_1 + \eta_1 = \gamma_2 + \eta_2$, $\gamma_1 \neq \gamma_2$ it would follow that $\gamma_1 - \gamma_2 = \eta_2 - \eta_1 \in \Gamma \cap X_0$.

Let now $u(2)$ be so chosen that $\Gamma \cap X_0 = \{0\}$. Then a sequence $\xi_n = \gamma_n + \eta_n \in X$, $\gamma_n \in \Gamma$, $\eta_n \in X_0$. has a limit if and only if the sequences γ_n and η_n are convergent.

Let Ψ be a continuous homomorphism, $\Psi : \mathbb{R}_x \to G$, and let $u(2)$ be so chosen that $\Gamma \cap X_0 = \{0\}$. Let $\varphi(n) := \Psi(n) + u(n) \in \mathcal{A}_G^*$.

Assume that for some subsequence of integers n_ν the sequence $\varphi(n_\nu)$ converges. Then, with the notation $\xi_\nu = \varphi(n_\nu) = u(n_\nu) + \Psi(n_\nu) = \gamma_\nu + \eta_\nu$, we have that the sequences γ_ν, η_ν are convergent, consequently $\lim \varphi(2n_\nu - 1) = \lim \Psi(2n_\nu - 1) = \Psi(2) + \lim \eta_\nu$ exists as well. So we have that $\varphi \in \mathcal{A}_G^*(D[2,-1])$.

2. We have proved the following

Theorem. *Let $\varphi \in \mathcal{A}_G^*(D[2,-1])$, where G is a metrically compact Abelian group. Then there exists a continuous homomorphism $\Psi{:}\mathbb{R}_x \to G$, such that $\Psi(n) = \varphi(n) \forall n \in \mathbb{N}$, $(n,2) = 1$. Let $u(2) = \varphi(2) - \Psi(2)$, and let Γ be the smallest closed group generated by $u(2)$. Let $u(n) := \varphi(n) - \Psi(n)$ be extended as a completely additive function.*

Then $\Gamma \cap X_0 = \{0\}$.

Conversely, let Ψ be an arbitrary continuous homomorphism, $\Psi : \mathbb{R}_x \to G$, and let X_0 be the smallest compact subgroup generated by $\Psi(\mathbb{N})$. Let $\alpha \in G$ be so chosen that the smallest closed group Γ generated by α has the property $\Gamma \cap X_0 = \{0\}$. Let $u \in \mathcal{A}_G^*$ be so that $u(2) = \alpha$ and $u(n) = 0\ \forall n$, $(n,2) = 1$. Then the function $\varphi(n) = \Psi(n) + u(n)$ belongs to $\mathcal{A}_G^*(D[2,-1])$.

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