*Publ. Math. Debrecen*
**79/3-4** (2011), 663–673
DOI: 10.5486/PMD.2011.5105

# Integer solutions to decomposable and semi-decomposable form inequalities

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*Dedicated to Professor K. Győry on the occasion of his 70th birthday*

**Abstract.** In this paper, we give a survey on the recent development in the study of the finiteness of the number of integer solutions to decomposable and semi-decomposable form inequalities.

## 1. Introduction

Let $F(\mathbf{X}) = F(X_0, \dots, X_m) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form, i.e. a homogeneous polynomial which factorizes into linear forms over $\bar{\mathbb{Q}}$, the algebraic closure of the field of reational numbers $\mathbb{Q}$. Assume that $q = \deg F > 2m$, and consider the decomposable form inequality

$$0 < |F(\mathbf{x})| < c|\mathbf{x}|^{\lambda} \text{ in } \mathbf{x} = (x_0, \dots, x_m) \in \mathbb{Z}^{m+1}, \tag{1.1}$$

where $|\mathbf{x}| = \max_{0 \le i \le m} |x_i|$, $0 \le \lambda < q - 2m$ and $c > 0$ is a fixed constant. For $m = 1$, it follows from Roth's approximation theorem that if the linear factors of $F$ are pairwise non-proportional, then (1) has only finitely many solutions. Note that when $m = 1$, every homogeneous polynomial is decomposable. Using

*Mathematics Subject Classification:* Primary: 11J68; Secondary: 11J25.
*Key words and phrases:* Integral points, decomposable inequalities, semi-decomposable form inequalities, diophantine approximation.
The author is supported in part by NSA grant H98230-09-1-0004, and H98230-11-1-0201 and grant number 100376.

his subspace theorem, W. M. SCHMIDT ([Sch1], [Sch2]) generalized this for arbitrary $m$, under the assumptions that (i) any $m+1$ of the linear factors of $F$ are linearly independent over $\bar{\mathbb{Q}}$, and that (ii) $F$ is not divisible by a form with rational coefficients of degree less than $m+1$. Later H. P. SCHLICKEWEI [Schl] extended this theorem to the case when the ground ring is an arbitrary finitely generated subring of $\mathbb{Q}$. These results have obvious applications to decomposable form equations of the form

$$F(\mathbf{x}) = G(\mathbf{x}) \text{ in } \mathbf{x} \in \mathbb{Z}^{m+1}, \tag{1.2}$$

where $G \in \mathbb{Z}[\mathbf{X}]$ is a non-zero polynomial of degree $< \deg F - 2m$. For the case when $G$ is a constant, it is then reduced to the decomposable form equation $F(x_0, \dots, x_m) = b$ with $b \in \mathbb{Q}^*$. In this case, one can use the well-known unit-lemma to deal with it. In fact, EVERTSE and GYŐRY [EG1] obtained a necessary and sufficient condition for the equation $F(x_0, \dots, x_m) = b$ with $b \in \mathbb{Q}^*$ to have finitely many integer solutions.

This paper intends to give a partial survey of the recent results along this direction. In Section 2, we recall the result of EVERTSE and GYŐRY [EG1] in the study of decomposable form equations, and its generalization given by CHEN–RU (see [CR]) to decomposable form inequalities. Section 3 reviews the sharp result of GYŐRY–RU (see [GR]) about the integer solutions to decomposable form inequalities. The final section reviews the recent result of CHEN–RU–YAN ([CRY]) on the integral solutions to semi-decomposable form inequalities.

## 2. Integer solutions to decomposable form equations

Let $k$ be a finitely generated (but not necessarily algebraic) extension field of $\mathbb{Q}$. Let $F(X_0, \dots, X_m)$ be a form (homogeneous polynomial) in $m \geq 1$ variables with coefficients in $k$ and suppose that $F$ is *decomposable*, i.e. it factorizes into linear factors over some finite extension of $k$. Let $b \in k^*$, where $k^*$ is the set of non-zero elements of $k$, and consider the decomposable form equation

$$F(x_0, \dots, x_m) = b \quad \text{in} \quad (x_0, \dots, x_m) \in R^{m+1} \tag{2.1}$$

where $R$ is a subring of $k$ finitely generated over $\mathbb{Z}$.

When $m = 1$, such equations are called *Thue equations.* The Thue equations are named after A. THUE [Th] who proved, in the case $k = \mathbb{Q}$, $R = \mathbb{Z}, m = 1$, that if $F$ is a binary form having at least three pairwise linearly independent linear factors in its factorization over the field of algebraic numbers, then (3) has only finitely many solutions. Later, Lang extended Thue's result to the general case when $k$ is a finitely generated extension field of $\mathbb{Q}$ and $R$ is a subring of $k$ finitely generated over $\mathbb{Z}$. For the case $m \geq 2$, after the works of Schmidt, Schlickewei, Laurent and others, EVERTSE and GYŐRY [EG1] finally obtained a necessary and sufficient condition for (2.1) to have finitely many solutions, independently of the choice of $b$ and $R$. In §3 of [EG1], EVERTSE and GYŐRY gave an equivalent form of this condition in the case where $F$ factors into a product of linear forms over $k$. Given a field $k$ and a set of linear forms $\mathcal{M} \subset k[X_0, \ldots, X_m]$, we denote by $(\mathcal{M})_k$ the $k$-linear subspace of $k[X_0, \ldots, X_m]$ generated by $\mathcal{M}$. The following is the statement of their result.

**Theorem 2.1** (Evertse and Győry)**.** *Let $k$ be a finitely generated extension field of $\mathbb{Q}$. Let $F(X_0, \ldots, X_m)$ be a decomposable form in $m+1$ variables with coefficients $k$. Assume that it factors into a product of linear forms over $k$. Denote by $\mathcal{L}$ a maximal set of linear factors of $F$ in $k[X_0, \ldots, X_m]$ which are pairwise linearly independent. Then the following two statements are equivalent:*

(i) *For every $b \in k^*$, the equation*

$$F(x_0, \ldots, x_m) = b, \quad \textit{in} \quad (x_0, \ldots, x_m) \in R^{m+1}$$

*has only finitely many solutions for every subring $R$ of $k$ which is finitely generated over $\mathbb{Z}$.*

(ii) *The subspace $(\mathcal{L})_k$ of $k[X_0, \ldots, X_m]$ generated by $\mathcal{L}$ has dimension $m+1$ and for each proper, non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, the intersection $(\mathcal{L}_1)_k \cap (\mathcal{L} \backslash \mathcal{L}_1)_k$ contains an element of $\mathcal{L}$.*

Note that the condition (ii) is independent of the choice of $\mathcal{L}$.

We now focus ourself on the case when $k$ is a number field. We first introduce some notations. Let $k$ be a number field. Denote by $\bar{k}$ the algebraic closure of $k$. Denote by $\mathbf{M}(k)$ the set of places of $k$ and write $\mathbf{M}_\infty(\mathbf{k})$ for the set of archimedean places of $k$. For $v \in \mathbf{M}(k)$ denote by $|\ |_v$ the associated absolute value, normalized such that $|\ |_v = |\ |$ (standard absolute value) on $\mathbf{Q}$ if $v$ is archimedean, whereas for $v$ non-archimedean $|p|_v = p^{-1}$ if $v$ lies above the rational prime p. Denote by $k_v$ the completion of $k$ with respect to $v$ and by $d_v = [k_v : \mathbf{Q}_v]$ the local degree. We put $||\ ||_v = |\ |_v^{d_v/d}$, where $d$ is the degree of $k$.

For $\mathbf{x} = (x_0, \ldots, x_m) \in k^{m+1}$, we put $\|\mathbf{x}\|_v = \max_{0 \leq i \leq m} \|x_i\|_v$, we denote by $H(\mathbf{x}) = \prod_{v \in \mathbf{M}(k)} \|\mathbf{x}\|_v$ and

$$h(\mathbf{x}) = \log H(\mathbf{x}) = \sum_{v \in \mathbf{M}(k)} \log \|\mathbf{x}\|_v$$

the absolute logarithmic height of $\mathbf{x}$. Given a polynomial $P$ with coefficient in $\mathbf{K}$, we define $\|P\|_\upsilon$ and $h(P)$ as the $\| \, \|_\upsilon$-value and absolute logarithmic height, respectively, of the point whose coordinates are the coefficients of $P$. As is known, $h(\mathbf{x})$ and $h(P)$ are independent of the choice of the field $k$. Further, $h(\lambda\mathbf{x}) = h(\mathbf{x})$ and $h(\lambda P) = h(P)$ for all $\lambda \in \bar{\mathbf{Q}}^*$.

Let S be a finite subset of $\mathbf{M}(k)$ containing $\mathbf{M}_\infty(k)$. An element $x \in k$ is said to be S-integer if $||x||_\upsilon \leq 1$ for each $\upsilon \in \mathbf{M}(k) - S$. Denote by $\mathcal{O}_S$ the set of S-integers. The units of $\mathcal{O}_S$ are called $S$-units. They form a multiplicative group which is denoted by $\mathcal{O}_S^*$. For $\mathbf{x} = (x_0, \ldots, x_m) \in k^{m+1}$, define the $S$-height as $H_S(\mathbf{x}) = \prod_{\upsilon\in S} \|\mathbf{x}\|_\upsilon$. If $\mathbf{x} \in \mathcal{O}_S^{m+1} - \{0\}$, then $H_S(\mathbf{x}) \geq 1$ and $H_S(\alpha\mathbf{x}) = H_S(\mathbf{x})$ for $\alpha \in \mathcal{O}_S^*$. Let $h_S = \log H_S$. For a polynomial $P$ with coefficients in $k$, let $H_S(P)$ denote the $S$-height of that point whose coordinates are the coefficients of $P$.

Let

$$F(\mathbf{X}) = F(X_0, \ldots, X_m) \in \mathcal{O}_S[\mathbf{X}]$$

be a homogeneous polynomial of $m+1$ variables. $F$ is said to be *decomposable* if $F$ factorizes into a product of linear forms over $\bar{k}$ with at least $m+1$ factors. For given real numbers $c, \lambda$ with $c > 0$, consider the solutions of the inequality

$$0 < \prod_{\upsilon\in S} \|F(\mathbf{x})\|_\upsilon \leq cH_S(\mathbf{x})^\lambda \text{ in } \mathbf{x} \in \mathcal{O}_S^{m+1}. \tag{2.2}$$

If $\mathbf{x}$ is a solution of (4), then so is $\mathbf{x}' = \eta\mathbf{x}$ for every $\eta \in O_S^*$. Such solution $\mathbf{x}, \mathbf{x}'$ are called $\mathbf{O}_S^*$-proportional.

Motivated by (ii) of Theorem 2.1, we introduce the following definition.

*Definition 2.1.* Let $k$ be a number field and let $F(X_0, \ldots, X_m)$ be a decomposable form in $m+1$ variables with coefficients in $k$. We say that $F$ is non-degenerate if it satisfies the following conditions: there exists a finite algebraic extension $k'$ of $k$ such that $F$ factors into a product of linear forms over $k'$ and if we denote by $\mathcal{L}$ a maximal set of linear factors of $F$ which are pairwise linearly independent, then the subspace $(\mathcal{L})_{k'}$ of $k'[X_0, \ldots, X_m]$ generated by $\mathcal{L}$ over $k'$ has dimension $m+1$ and for each proper, non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, the intersection $(\mathcal{L}_1)_{k'} \cap (\mathcal{L}\backslash\mathcal{L}_1)_{k'}$ contains an element of $\mathcal{L}$.

Note that the above definition is independent of the choice of $\mathcal{L}$. Chen–Ru (see [CR]) extended the result of Evertse and Győry to the following:

**Theorem 2.2** (see [CR] Theorem 1.1)**.** *Let $k$ be a number field and let $F(X_0, \ldots, X_m)$ be a non-degenerate decomposable form with coefficients in $k$.*

*Then, for every finite set of places $S$ of $k$ containing the archimedean places of $k$, for each real number $\lambda < \frac{1}{m-1}$ and for each constant $c > 0$, the inequality*

$$0 < \prod_{v \in S} \|F(x_0, \ldots, x_m)\|_v \le c H_S^\lambda(x_0, \ldots, x_m) \text{ in } (x_1, \ldots, x_m) \in \mathcal{O}_S^m$$

*has only finitely many $\mathcal{O}_S^*$-non-proportional solutions.*

Important examples of non-degenerate decomposable forms are those $F(\mathbf{X})$ such that $\deg F > 2m$ and that any $m+1$ linear factors of $F$ over $\overline{\mathbb{Q}}$ are linearly independent. In this case, Győry and Ru actually obtained a a stronger result. This leads to the discussion of the next section.

## 3. Integer solutions to decomposable form inequalities

By applying Ru–Wong's degenerate Schmidt's subspace theorem (see [RW]), K. Győry and Min Ru (see [GR]) dropped the assumption (ii) in Schmidt's result mentioned in the introduction. Furthermore, they obtained a more general result: under the (weak) assumption that $\lambda < q - 2m + l - 1$, where $l > 0$ is an integer, then the set of integer solutions is contained in a finite union of subspaces of dimension at most $l$.

**Theorem 3.1** (see [GR] Theorem 7)**.** *Let $k$ be a number field and let $F(X_0, \ldots, X_m)$ be a decomposable form of degree $q$ which factorizes into linear factors over $\bar{k}$. Suppose that $\lambda < q - 2m + l - 1$ and that the linear factors of $F$ over $\bar{\mathbf{Q}}$ are in general position. Then the set of solutions of* (2.2) *is contained in a finite union of linear subspaces of $k^{m+1}$ of dimension at most $l$. In particular, if $\lambda < q - 2m$ then* (2.2) *has only finitely many $\mathcal{O}_S^*$-non-proportional solutions.*

The above theorem was derived from the following result due to Ru–Wong (see [RW]).

**Theorem 3.2** (see [RW] Theorem 4.1)**.** *Given linear forms $L_1, \ldots, L_q \in k[X_0, \ldots, X_m]$ in general position. Then for any $\epsilon > 0$, the set of points $\mathbf{x} \in k^{m+1}$ such that $L_j(\mathbf{x}) \neq 0$ for $j = 1, \ldots, q$ and*

$$\sum_{v \in S} \sum_{j=1}^{q} \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \ge (2m - l + 1 + \epsilon) h(\mathbf{x})$$

*is contained in a finite union of linear subspaces of $k^{m+1}$ of dimension at most $l$.*

## 4. Integer solutions to semi-decomposable form inequalities

In this section, we review the recent result of CHEN–RU–YAN (see [CRY]) on the integer Solutions to Semi-Decomposable Form Inequalities. Let

$$F(\mathbf{X}) = F(X_0, \ldots, X_m) \in \mathcal{O}_S[\mathbf{X}]$$

be a homogeneous polynomial of $m+1$ variables. $F$ is said to be *semi-decomposable* if $F$ factorizes into a product of irreducible homogeneous polynomials over $\bar{\mathbf{Q}}$ with at least $m+1$ factors.

Again, we study the inequality,

$$0 < \prod_{v \in S} \|F(\mathbf{x})\|_v \leq cH_S(\mathbf{x})^{\lambda} \text{ in } \mathbf{x} \in \mathcal{O}_S^{m+1}. \tag{4.1}$$

except in this case, $F$ is only assumed to be "semi-decomposable". We call such inequality the *semi-decomposable form inequality.*

We first establish a Schmidt's subspace type theorem. To do so, we recall the following result from [CZ] (See Addendum, 128(2006)).

**Theorem 4.1** (Corvaja and Zannier)**.** *Let $k$ be a number field and let $S$ be a finite set of places of $k$. Let $V \subset \mathbb{P}^N(k)$ be a (irreducible) projective variety with $\dim V = n$. For each $v \in S$, let $Q_v \in \bar{k}[X_0, \ldots, X_N]$ be a homogeneous polynomial of degree $d$. Then, for every $\epsilon > 0$, there are only finitely many points $\mathbf{x} \in V(\mathcal{O}_S)$ such that*

$$0 < \prod_{v \in S} \|Q_v(\mathbf{x})\|_v < H(\mathbf{x})^{-dn-\epsilon}.$$

*Definition 4.1.* Let $V \subset \mathbb{P}^N(k)$ be a projective subvariety with $\dim V = n$, and $D_1, \ldots, D_q, q > n$, be given hypersurfaces in $\mathbb{P}^N(k)$. We say they are *located in general position with respect to* $V$ if for any distinct $j_1, \ldots, j_{n+1}$, $\bigcap_{i=1}^{n+1} \mathrm{supp} D_{j_i} \cap V(\bar{k}) = \emptyset$.

*Definition 4.2.* Let $V \subset \mathbb{P}^N(k)$ be a projective subvariety with $n = \dim V$, and $D_1, \ldots, D_q$ be given hypersurfaces in $\mathbb{P}^N(k)$. Given a positive integer $m \geq n$, we say they are *located in $m$-subgeneral position with respect to $V$* if $q > m$ and for any distinct $j_1, \ldots, j_{m+1}$, $\bigcap_{i=1}^{m+1} \mathrm{supp} D_{j_i} \cap V(\bar{k}) = \emptyset$.

*Remark 4.1.* Obviously, if $W \subset V$ is a subvariety of $V$ and if $D_1, \ldots, D_q$, are in general position with respect to $V$, they are in $n$-subgeneral position with respect to $W$, where $n = \dim V$.

We prove the following theorem.

**Theorem 4.2.** *Let $k$ be a number field and let $S$ be a finite set of places of $k$. Let $V \subset \mathbb{P}^N(k)$ be a (irreducible) projective variety. Let $Q_1, \ldots, Q_q \in \bar{k}[X_0, \ldots, X_N]$ be homogeneous polynomials of degree $d_1, \ldots, d_q$ respectively, and assume that for some $m \geq \dim V$ they are located in $m$-subgeneral position with respect to $V$. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \sum_{v \in S} \log \left( \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \right)^{1/d_j} \leq (m(\dim V + 1) + \epsilon) h(\mathbf{x})$$

*holds for all $\mathbf{x} \in V(k)$, outside a finite union of proper subvarieties of $V$.*

**Corollary 4.1.** *Let $k$ be a number field and let $S$ be a finite set of places of $k$. Let $V \subset \mathbb{P}^N(k)$ be a (irreducible) projective variety with $\dim V = n$. Let $Q_1, \ldots, Q_q \in \bar{k}[X_0, \ldots, X_N]$ be homogeneous polynomials of $d_1, \ldots, d_q$ respectively, which are located in general position with respect to $V$. Then, for every $\epsilon > 0$, the set of points $\mathbf{x} \in V(k) \setminus \bigcup_{j=1}^{q} \{Q_j = 0\}$ with*

$$\sum_{j=1}^{q} \sum_{v \in S} \log \left( \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \right)^{1/d_j} \geq (n(n+1) + \epsilon) h(\mathbf{x})$$

*is a finite set.*

Proof of Corollary 4.1. Since $Q_1, \ldots, Q_q$ are in general position with respect to $V$, by applying Theorem 4.2 with $m = n$ we conclude that the set of $\mathbf{x} \in V(k)$ with

$$\sum_{j=1}^{q} \sum_{v \in S} \log \left( \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \right)^{1/d_j} \geq (n(n+1) + \epsilon) h(\mathbf{x})$$

is contained a finite union of proper subvariaties of $V(k)$. Say $W$ is one of them. From Remark 4.1, we know that $Q_1, \ldots, Q_q$ are in $n$-subgeneral position with respect to $W$. Applying Theorem 4.2 to $W$ with $m = n$ and noticing that $n(n+1) \geq n(\dim V + 1)$, we get that the set of $\mathbf{x} \in V(k) \setminus \bigcup_{j=1}^{q} \{Q_j = 0\}$ with

$$\sum_{j=1}^{q} \sum_{v \in S} \log \left( \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \right)^{1/d_j} \geq (n(n+1) + \epsilon) h(\mathbf{x})$$

is contained a finite union of proper subvarieties of $W(k)$. Eventually, the set will be finite. This proves the Corollary. □

Proof of Theorem 4.2. Assume that $\dim V = n$. Let $Q_j, 1 \leq j \leq q$, be the given homogeneous polynomials in $\bar{k}[x_0, \dots, x_n]$ of degree $d_j$. Replacing $Q_j$ by $Q_j^{d/d_j}$ if necessary, where $d$ is the l.c.m. of $d_j's$, we can assume that $Q_1, \dots, Q_q$ have the same degree of $d$. Denote by $P_1, \dots, P_r$ the generators of $(I_V)_d$, where $I_V$ is the prime ideal defining $V$ and $(I_V)_d$ is the subset of $I_V$, which consists only of the homogeneous polynomials with degree $d$. For every fixed $\mathbf{b} = [b_0 : \cdots : b_N] \in V(k)$, and every $v \in S$, take a renumbering $\{l_1, \dots, l_q\}$ (which depends on $v$ and $\mathbf{b}$) of the indices $\{1, \dots, q\}$ such that

$$\|Q_{l_1}(\mathbf{b})\|_v \leq \cdots \leq \|Q_{l_q}(\mathbf{b})\|_v. \tag{4.2}$$

Then the assumption that $Q_1, \dots, Q_q$ are in $m$-subgeneral position with respect to $V$ implies that $P_1, \dots, P_r, Q_{l_1}, \dots, Q_{l_{m+1}}$ have no common zeros in $\mathbb{P}^N(\bar{k})$. By Hilbert's Nullstellensatz, for any integer $t, 0 \leq t \leq N$, there is an integer $m_i \geq d$ such that

$$x_t^{m_t} = \sum_{j=1}^{m+1} \alpha_{jt} Q_{l_j} + \sum_{i=1}^{r} \beta_{it} P_i,$$

where $\alpha_{jt}, 1 \leq j \leq m+1$, and $\beta_{it}, 1 \leq i \leq r$, are the homogeneous polynomials of degree $m_t - d$. So, for $0 \leq t \leq N$,

$$\|x_t\|_v^{m_t} \leq c_{1,v} \|\mathbf{x}\|_v^{m_t - d} \max\{\|Q_{l_1}(\mathbf{x})\|_v, \dots, \|Q_{l_{m+1}}(\mathbf{x})\|_v\}$$

for all $\mathbf{x} \in V(k)$, where $c_{1,v}$ is a positive constant. That is

$$\|\mathbf{x}\|_v^d \leq c_{1,v} \max\{\|Q_{l_1}(\mathbf{x})\|_v, \dots, \|Q_{l_{m+1}}(\mathbf{x})\|_v\} \tag{4.3}$$

for all $\mathbf{x} \in V(k)$. Combining (4.2) and (4.3), we get

$$\begin{aligned}\sum_{j=1}^{q} \log \frac{\|\mathbf{b}\|_v^d \cdot \|Q_j\|_v}{\|Q_j(\mathbf{b})\|_v} &\leq \sum_{i=1}^{m} \log \left( \frac{\|\mathbf{b}\|_v^d \cdot \|Q_{l_i}\|_v}{\|Q_{l_i}(\mathbf{b})\|_v} \right) + C_v \\ &\leq m \log \left( \frac{\|\mathbf{b}\|_v^d \cdot \|Q_{l_1}\|_v}{\|Q_{l_1}(\mathbf{b})\|_v} \right) + C_v.\end{aligned}$$

Theorem 4.1 then implies that

$$\sum_{j=1}^{q} \sum_{v \in S} \log \left( \frac{\|\mathbf{x}\|_v^d \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \right)^{1/d} \quad \leq \quad m(n+1+\epsilon) h(\mathbf{x}).$$

This proves Theorem 4.2. $\square$

**Theorem 4.3** (Finiteness result)**.** *Let $k$ be a number field and let $F(\mathbf{X})$ be a semi-decomposable form in $m+1$ variables with coefficients in $k$. Write $F = Q_1 \dots Q_q$ over $\bar{k}$. Assume that $Q_1, \dots, Q_q$ are in general position with $\deg Q_j = d_j$. Let $d = \max_{1 \le j \le q} d_j$. Assume that $\deg F > dm(m+1)$. Then, for every finite set $S$ of places of $k$ containing the archimedean places of $k$, for each positive number $\lambda < \deg F - dm(m+1)$, (4.1) has only finitely many $\mathcal{O}_S^*$-non-proportional solutions.*

Proof. We shall prove Theorem 4.3 by using Corollary 4.1. Write $F = Q_1 \dots Q_q$ over $k'$, where $k'$ is a finite algebraic extension of $k$. Let $S' \subset \mathbf{M}(k')$ consist of the extension of the places of $S$ to $k'$, then every $S$-integer in $k$ is also an $S'$-integer in $k'$. Moreover, we have $H_S(x_0, \dots, x_m) = H_{S'}(x_0, \dots, x_m)$ and

$$\prod_{v \in S} \|F(x_0, \dots, x_m)\|_v = \prod_{w \in S'} \|F(x_0, \dots, x_m)\|_w \quad \text{for } (x_0, \dots, x_m) \in \mathcal{O}_S^{m+1}.$$

So (4.1) is preserved when we work on $k'$. Therefore, for simplicity, we assume that $k' = k$. By enlarging $S$ if necessary, we may assume that the coefficients of $Q_j, 1 \le j \le q$, are in $\mathcal{O}_S$. Hence, by Corollary 4.1, for all $\mathbf{x} = (x_0, \dots, x_m) \in \mathcal{O}_S^{m+1}$, except for finitely many, with $F(\mathbf{x}) \neq 0$, we have

$$\sum_{j=1}^{q} \sum_{v \in S} \frac{1}{d_j} \log \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \le \big(m(m+1) + \epsilon\big) h(\mathbf{x}).$$

This gives, for $d = \max_{1 \le j \le q} d_j$,

$$\sum_{j=1}^{q} \sum_{v \in S} \log \frac{\|\mathbf{x}\|_v^{d_j} \cdot \|Q_j\|_v}{\|Q_j(\mathbf{x})\|_v} \le \big(dm(m+1) + \epsilon\big) h(\mathbf{x}).$$

Hence

$$(d_1 + \dots + d_q) h_S(\mathbf{x}) \le \big(dm(m+1) + \epsilon\big) h(\mathbf{x}) + \log \prod_{v \in S} \|F(\mathbf{x})\|_v + O(1).$$

Using (4.1), the above becomes

$$(d_1 + \dots + d_q) h_S(\mathbf{x}) \le \big(dm(m+1) + \epsilon\big) h(\mathbf{x}) + \lambda h_S(\mathbf{x}) + O(1).$$

Since $\deg F = d_1 + \dots + d_q$, it yields

$$(\deg F) h_S(\mathbf{x}) \le \big(dm(m+1) + \epsilon\big) h(\mathbf{x}) + \lambda h_S(\mathbf{x}) + O(1). \tag{4.4}$$

On the other hand, for $\mathbf{x} \in \mathcal{O}_S^m$, we have

$$h(\mathbf{x}) \leq h_S(\mathbf{x}). \tag{4.5}$$

(4.4) and (4.5) then yield

$$\big(\deg F - dm(m+1) - \lambda - \epsilon\big) h_S(\mathbf{x}) \leq C,$$

for some positive constant $C$. Choose an $\epsilon > 0$ such that $\deg F - \epsilon - dm(m+1) - \lambda > 0$. Then it gives that $H_S(\mathbf{x})$ is bounded. By the Dirichlet–Chevalley–Weil $S$-unit Theorem, there is an $S$-unit $u$ such that $\|u\mathbf{x}\|_\upsilon \leq D_\upsilon H_S(\mathbf{x})^{1/\#S}$ for $\upsilon \in S$, where the $D_\upsilon$ are constants depending only on $k$, $S$. Thus $\mathbf{x}$ is $\mathcal{O}_S^*$-proportional to $\mathbf{x}' := u \cdot \mathbf{x}$, and $\|\mathbf{x}'\|_\upsilon$ is bounded for every $\upsilon \in \mathbf{M}(k)$. This implies that there are only finitely many possibilities for $\mathbf{x}'$. Hence up to $\mathcal{O}_S^*$-proportionality, (4.1) has only finitely many solutions $\mathbf{x} \in \mathcal{O}_S^m$. This finishes the proof of Theorem 4.3 $\square$

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*(Received January 26, 2011; revised March 24, 2011)*