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Conjecture of Pomerance for some even integers and odd primorials

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Dedicated to Professors K. Győry, A. Pethő, J. Pintz and A. Sárközy

Abstract. We solve some cases of a conjecture of Pomerance concerning reduced residue systems modulo k consisting of the first $\varphi(k)$ primes not dividing k when k is even or when k is an odd primorial, thus extending a recent result of Hajdu and Saradha.

1. Introduction

Let $k > 1$ be an integer. We denote by $\varphi(k)$, Euler's totient function and by $\omega(k)$, the number of distinct prime divisors of k. We say that k is a P–integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k. In 1980, Pomerance [5] proved the finiteness of P–integers and conjectured that

if k is a P–integer, then $k \leq 30$.

This conjecture is still *open*. It is easy to check that the only P–integers less than or equal to 30 are $2, 4, 6, 12, 18, 30$. In fact, it has been verified by Hajdu and Saradha [3] that

there are no other P–integers up to 5.5×10^5.

Further it was shown that

$$\text{the only prime } P\text{– integer is } 2. \tag{1}$$

This follows from the following general result proved in [3]. Let $\ell(k)$ denote the least prime divisor of k and we put $\ell(1) = 1$.

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If k is an integer with $\ell(k) > \log(k)$, then k is a P–integer if and only if $k \in \{2, 4, 6\}$.

This result depends on explicit computations done by Hagedorn [2] on the values of the Jacobsthal function. Note that this result does not include even integers > 8 since $2 < \log 8$. In this note we extend the above result as follows. Let $\alpha \geq 0$ be an integer. We write $k = 2^\alpha k_1$ with $2 \nmid k_1$.

Theorem 1.1. *Let $k = 2^\alpha k_1 > 1$ with $k_1 = 1$ or $\ell(k_1) > (.88)\log(k)$. Then k is a P–integer if and only if $k \in \{2, 4, 6, 12, 18, 30\}$.*

The following corollary is immediate and it extends (1).

Corollary 1.1. *Let q be an odd prime.*

(i) *The only P–integers which are powers of 2 are 2 and 4.*

(ii) *Any integer of the form q^β with $\beta < 1.136\frac{q}{\log q}$ is not a P–integer. In particular, none of the integers of the form q, q^2 or q^3 is a P–integer.*

(iii) *The only P–integers of the form $2q$, 2^2q, 2^3q, $2q^2$ are $6, 12, 18$.*

Let $N_h = p_1 \dots p_h$ i.e., product of the first h primes. These are called *primorials.* In Theorem 3 of [3], it was shown that all primorials are not P–integers except $2, 6$ and 30. Here we consider *odd primorials* i.e.,

$$N'_h = p_2 \dots p_h.$$

We show that

Theorem 1.2. *All odd primorials are not P–integers.*

2. Lemmas

We record some lemmas required for the proofs of Theorems 1.1 and 1.2. As the proofs are similar to the proof of Theorem 2 of [3], we will be brief at many places and give details only where the arguments are different. Let $2 = p_1 < p_2 < \dots$ denote the sequence of all primes. For any positive real x, let $\log_1 x = \log(x)$ and for $t \geq 2, \log_t(x) = \log(\log_{t-1}(x))$. We denote by $P(k)$ the maximum of the least primes in the reduced residue classes mod k. For any integer $n > 1$, let $g(n)$ denote the Jacobsthal function i.e., the least integer such that in any sequence of $g(n)$ consecutive integers there is an integer coprime to n. For the properties of $g(n)$, we refer to [1], [3] and [4] and the references mentioned therein. We begin with some properties of $g(n)$ that we need in this article.

Lemma 2.1. *For any integer $n > 1$, let $N(n)$ denote its radical. Then $g(n) = g(N(n))$. For any prime p, we have $g(p^\alpha) = 2$. If n is an odd integer, then $g(2n) = 2g(n)$. Further if $\ell(n) > \omega(n) + 1$, then $g(n) = \omega(n) + 1$.*

The first two assertions follow from the definition of $g(n)$. For the proof of the third assertion we refer to Lemma 2.2 of [4] or the argument in Proposition 2.8 of [2]. The last assertion was an observation of Jacobsthal, see Erdős [1]. The next lemma is due to Stevens [7] in which an explicit upper bound for $g(k)$ is given.

Lemma 2.2. *We have $g(k) \leq 2\omega(k)^{2+2e\log(\omega(k))}$ for all $k > 1$.*

The next lemma gives estimates from Prime Number Theory due to Rosser and Schoenfeld [6].

Lemma 2.3. *Let p_n denote the n-th prime. Then*

(i) $p_n > n\big(\log(n) + \log_2(n) - \frac{3}{2}\big)$ *for* $n > 1$;

(ii) $p_n < n(\log(n) + \log_2(n))$ *for* $n \geq 6$;

(iii) *For* $x \geq 2$ *write* $\vartheta(x) = \sum_{p \leq x} \log(p)$. *For any* $x \geq 563$ *we have*

$$x\left(1 - \frac{1}{2\log(x)}\right) < \vartheta(x) < x\left(1 + \frac{1}{2\log(x)}\right).$$

It is well known that the normal order of $\omega(n)$ is $\log_2(n)$. For the purpose of this article we use the following explicit estimate for $\omega(k)$. Let $k = 2^\alpha k_1$ with $k_1 = 1$ or $\ell(k_1) > (.88)\log(k)$. Suppose $k > 5.5 \times 10^5$. Then for $k_1 \neq 1$, we see that

$$\omega(k) = \omega(k_1) + 1 < \frac{\log(k)}{\log_2(k) - (.12)} + 1 < \frac{1.25\log(k)}{\log_2(k)} < (.49)\log(k) < \ell(k_1). \quad (2)$$

From the definition of P–integers and a result of Pomerance [5], we get the following estimates for $P(k)$.

Lemma 2.4. *Let k be given. Suppose m is an integer such that* $\gcd(m,k)=1$ *and $1 < m \leq \frac{k}{1+g(k)}$. Then k is a P–integer if and only if*

$$(g(m) - 1)k < P(k) \leq p_{\varphi(k)+\omega(k)}.$$

Let

$$\delta_1 = \begin{cases} 0 & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \end{cases} \quad \text{and} \quad \delta_2 = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Suppose it is possible to choose m in Lemma 2.4 as the product of the first h primes if k is odd and first $h-1$ odd primes if k is even i.e.,

$$m = 2^{\delta_1} p_2 p_3 \dots p_h. \tag{3}$$

Then by Proposition 1.1 of Hagedorn [2] we get

$$g(m) \geq 2p_{h-1} \quad \text{if } \delta_1 = 1.$$

Hence by Lemma 2.1, if $\delta_1 = 0$ i.e., when m is odd, we get

$$g(m) = \frac{1}{2} g(2m) \geq p_{h-1}.$$

Thus for the choice of m as in (3), we have

$$g(m) \geq 2^{\delta_1} p_{h-1}.$$

Now by Lemmas 2.4 and 2.3, we have

$$p_{\varphi(k)+\omega(k)} > (2^{\delta_1} p_{h-1} - 1)k > 2^{\delta_1}(h \log(h))k \quad \text{for } h \geq 8. \tag{4}$$

When $k = 2^{\alpha} k_1$ with $\ell(k_1) > (.88)\log(k)$ and $k > 5.5 \times 10^5$ we observe by (2) and Lemma 2.1 that

$$g(k) = g(N(k)) = g(2^{\delta_2} N(k_1)) = 2^{\delta_2}(\omega(k_1) + 1).$$

Also if $k_1 = 1$, then $g(k) = g(2^{\alpha}) = 2$. Hence we have

$$g(k) = 2^{\delta_2}(\omega(k_1) + 1) = 2^{\delta_2}\omega(k) \quad \text{for } k > 5.5 \times 10^5. \tag{5}$$

Further $\varphi(k) < \frac{k}{2^{\delta_2}}$. Hence using $\varphi(k) + \omega(k) \leq k$, the upper estimate for p_n from Lemma 2.3 (ii) and (2), we get

$$p_{\varphi(k)+\omega(k)} \leq p_{\frac{k}{2^{\delta_2}}+1+\frac{1.25\log(k)}{\log_2(k)}} \leq \left(\frac{k}{2^{\delta_2}} + 1 + \frac{1.25\log(k)}{\log_2(k)}\right)(\log(k) + \log_2(k)). \tag{6}$$

Thus we have

$$p_{\varphi(k)+\omega(k)} \leq \frac{1.026}{2^{\delta_2}} k \log(k) \quad \text{for } k \geq 10^{90}. \tag{7}$$

Applying (4) and (7) in Lemma 2.4, we obtain the following lemma.

Lemma 2.5. *Let $k \geq 10^{90}, k = 2^\alpha k_1$ with $k_1 = 1$ or $\ell(k_1) > (.88)\log(k)$. Then k is not a P–integer.*

PROOF. Let $k \geq 10^{90}$ and $m = 2^{\delta_1} p_2 \dots p_h$ with

$$h = \left[\frac{.85\log(k)}{\log_2(k)}\right] + 1.$$

Then

$$\frac{.85\log(k)}{\log_2(k)} < h < \frac{.88\log(k)}{\log_2(k)}.$$

Hence

$$p_h < .88\log(k) < \ell(k_1)$$

showing that $\gcd(m,k) = 1$ and also using (2) and (5) we get

$$m < e^{.88\log(k)} < \frac{k\log_2(k)}{2.5\log(k) + \log_2(k)} \leq \frac{k}{1 + 2\omega(k)} \leq \frac{k}{1 + g(k)}.$$

On the other hand, using (4) and (7) in Lemma 2.4, we get

$$\begin{aligned}\log(k) &> 2^{\delta_1+\delta_2}(.974)h\log(h) > 1.948h\log(h)\\ &> \frac{1.65\log(k)}{\log_2(k)}\{\log_2 k - .17 - \log_3 k\} > 1.07\log(k),\end{aligned}$$

a contradiction. □

3. Proofs of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. We take $k = 2^\alpha k_1$ with $k_1 = 1$ or $\ell(k_1) > .88\log(k)$. By Lemma 2.5 and the computations made in [3] we may assume that

$$5.5 \times 10^5 < k < 10^{90}. \tag{8}$$

As in [3] we use "boot-strapping" technique and the explicit values of $g(m)$ given by the work of HAGEDORN [2] to cover this range.

First, we take k odd. Then $k = k_1 > 1$ and by (2), we have $\ell(k) = \ell(k_1) > \omega(k_1) + 1 = \omega(k) + 1$. Hence by Lemma 2.1 we have $g(k) = \omega(k) + 1 < \log(k) + 1$. Now we follow the argument exactly as in [3] (see pages 22-23) to show that no odd value of k in (8) is a P–integer. Next we take k even in the range given by (8). Then by (5) and (2) we have,

$$g(k) \leq 2\omega(k) \leq \frac{2.5\log(k)}{\log_2(k)}.$$

Suppose $\beta_1 < k \leq \beta_2$. Let $m = p_2 \dots p_h$ with a suitable h such that

$$p_h < .88\log(\beta_1) \tag{9}$$

and

$$1 < m < \frac{\beta_1\log_2(\beta_1)}{2.5\log(\beta_1) + \log_2(\beta_1)}. \tag{10}$$

Then $\gcd(m, k) = 1$ since $\ell(k) > .88\log(k) > .88\log\beta_1 > p_h$ and we also have

$$m < \frac{k\log_2 k}{2.5\log(k) + \log_2(k)} \leq \frac{k}{1 + g(k)}.$$

Then by Lemma 2.4 and (6), we find that k is a P–integer only if

$$g(m) - 1 < \log(\beta_2)\left(\frac{1}{2} + \frac{1}{\beta_2} + \frac{1.25\log(\beta_2)}{\beta_2\log_2(\beta_2)}\right)\left(1 + \frac{\log_2(\beta_2)}{\log(\beta_2)}\right). \tag{11}$$

Thus when (11) is contradicted, then no value of k in $(\beta_1, \beta_2]$ is a P–integer. We begin with $\beta_1 = 5.5 \times 10^5$ and $\beta_2 = 10^7$. Then $\omega(k) \leq \frac{1.25\log(\beta_2)}{\log_2(\beta_2)} \leq 7.3$ giving $g(k) \leq 8$. We choose $h = 6$. Then $m = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and hence $m < 5.5 \times 10^5/9$ and $g(m) = \frac{1}{2} \times 22 = 11$ so that the left hand side of (11) equals 10. On the other hand, the right hand side of (11) does not exceed 9.5 giving the necessary contradiction. Proceeding successively from $10^{\alpha_1} = 10^7$, we give in Table 1, the value $\alpha_i = \alpha$ for $i > 1$ such that k is taken in the range $(10^{\alpha_{i-1}}, 10^{\alpha_i}]$, the value of h such that $m = p_2 \dots p_h$ satisfies (10) and the exact value of $g(m) = \frac{1}{2}g(2m)$ as provided by HAGEDORN (see Table 1 of [2]). One checks that (11) is contradicted in each of the range specified, thereby proving the assertion of the theorem. □

h	7	8	10	13	20	23
$g(m)$	13	17	23	37	87	108
α	9	12	17	29	72	90

Table 1

Proof of Theorem 1.2. We follow the argument as in the proof of Theorem 3 of [3]. First we take $p_h > 1000$ and we choose $m = 2p_{h+1}\dots p_{h+\theta}$ with $p_{h+\theta} \le 1.777p_h$ and such that

$$\text{(i)} \quad m \le \frac{k}{1+g(k)} \qquad \text{(ii)} \quad (g(m)-1)k \ge p_{\varphi(k)+\omega(k)}.$$

This would imply that k is not a P–integer. Condition (i) requires that

$$g(k)+1 \le \frac{1}{2}\left(\exp(2\vartheta(p_h)) - \exp(\vartheta(p_{h+\theta}))\right).$$

Using the upper bound for $g(k)$ from Lemma 2.2, this amounts to checking

$$2 + 4h^{2+2e\log(h)} \le \exp(2\vartheta(p_h)) - \exp(\vartheta(p_{h+\theta})).$$

As in [3] this inequality is verified by using approximate values of $\vartheta(x)$ given by Lemma 2.3(iii) for $x = p_h \ge 12000$ and exact values of $\vartheta(p_h)$ for $1000 < p_h < 12000$. The second condition (ii) leads to showing

$$\begin{aligned} g(m)-1 \ge \omega(m) \ge \theta \ge \pi(1.777p_h) - h \\ \ge \Big(\prod_{i=1}^{h}\left(1-\frac{1}{p_i}\right) + \frac{h}{k}\Big)\left(\vartheta(p_h) + \log(\vartheta(p_h))\right). \end{aligned}$$

This is checked to be valid for $p_h > 1000$. Thus no odd primorial with $p_h > 1000$ is a P–integer. Now we assume that $p_h < 1000$. In order to check all those $k = p_2\dots p_h$ with $p_h < 1000$, we proceed as follows. For each such k we find a power of 2, say $2^q < k$ and $0 \le i < j$ such that $ik + 2^q$ and $jk + 2^q$ are both primes and

$$jk + 2^q < (\varphi(k) + h - 1)\log(\varphi(k) + h - 1). \tag{12}$$

This implies that both the primes $ik + 2^q$ and $jk + 2^q$ belong to the set of first $\varphi(k)$ primes coprime to k, but they belong to the same residue class $2^q \pmod{k}$. Hence k is not a P–integer. We give two examples to illustrate the above procedure. Let $k = 3\cdot 5\dots 29$. Then $k+2$ is a prime and it is one of the first $\varphi(k)$ primes coprime to k, but it falls in the residue class $2 \pmod{k}$. Hence by the above procedure with $i = 0, j = 1$ and $q = 1$, we conclude that k is not a P–integer. Let $k = 3\cdot 5\dots p_{39}$. Then $3k + 2^{32}$ and $5k + 2^{32}$ are primes, $2^{32} < k$ and (12) is satisfied by taking $i = 3, j = 5$ and $q = 32$. Hence we conclude that k is not a P–integer. □

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