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Functional equations in the spectral theory of random fields I.

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1. Introduction

Let H denote a separable Hilbert space and let G denote the group of motions in H, that is the family of one-to-one isometric transformations of H into itself.

By a one-dimensional random field defined on H we mean

$$X = \{X(t),\ t \in H\},$$

(i.e., X is a family of real random variables).

A random field X is called homogeneous isotropic in H if, for any $t_1 \in H$, $t_2 \in H$, $g \in G$,

$$M\, X(g(t_1))\, X(g(t_2)) = M\, X(t_1)\, X(t_2).$$

A random field X on H is called Markov if, for any sphere S in H and any two points t_1, t_2 separated by the sphere S, the random variables $X(t_1)$, and $X(t_2)$ are conditionally independent given $X(t)$ on S.

Professor M.I. Yadrenko (Kiew) proved the following

Theorem (see [3]). *The correlation function of a Gaussian homogeneous isotropic random field X of the Markov type in H is given by*

$$B(r) = \exp(-c\, r^2), \quad r \in \mathbb{R}, \tag{1}$$

where c is a nonnegative constant, $\mathbb{R}$ is the set of real numbers.

The PROOF of this theorem based on the fact that, in this case, the correlation function B satisfies the functional equation

$$B(R\sqrt{2})\, B(R_2) = B(R)\, B(\sqrt{R^2 + R_2^2}) \quad (R_2 > R > 0), \tag{2}$$

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and the correlation function of any homogeneous isotropic random fields on H is differentiable infinitely many times.

The purpose of this paper is to present the general solution of functional equation (2). From the general solution it follows that, under certain regularity conditions (e.g., B is differentiable, continuous or measurable), B has the from (1).

2. The general solution of functional equation (2).

Let $\mathbb{R}_+ = \{x \mid x \in \mathbb{R},\ x > 0\}$ and $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

Lemma 1. *If the function $B : \mathbb{R}_+ \to \mathbb{R}$ satisfies the functional equation* (2)*, then the function*

$$f : \mathbb{R}_+ \to \mathbb{R}, \qquad f(x) = B(\sqrt{x}), \tag{3}$$

satisfies the functional equation

$$f(x)\, f(y) = f\left(\frac{x}{2}\right)\, f\left(\frac{x}{2} + y\right), (x, y) \in D = \{(x, y) \mid y > \frac{x}{2} > 0\}. \tag{4}$$

Proof. By the transformation

$$R = \sqrt{\frac{x}{2}},\ R_2 = \sqrt{y}; \qquad (x, y) \in D, \tag{5}$$

it follows from (2) that

$$B(\sqrt{x})\, B(\sqrt{y}) = B\left(\sqrt{\frac{x}{2}}\right)\, B\left(\sqrt{\frac{x}{2} + y}\right), \qquad (x, y) \in D,$$

which, together with (3), implies the functional equation (4) for the function f.

Lemma 2. *If the function $f : \mathbb{R}_+ \in \mathbb{R}$ satisfies the functional equation* (4) *and there exists a subset $E \subset \mathbb{R}_+$ of positive Lebesgue-measure, such that $f(x) \neq 0$ for all $x \in E$, then $f(x) \neq 0$ for all $x \in \mathbb{R}_+$.*

Proof. Suppose $f(x) \neq 0$ for $x \in E$, where E has positive measure. Then there exists a compact subset $E_1 \subset E_0$ positive measure such that $E_1 \subset [a, b] \subset \mathbb{R}_+$ for some closed interval $[a, b]$. Further there exists a natural number n such that $2^n a > b$. It is obvious that one of the intervals $(a, 2a), \dots, (2^{n-2}a, 2^{n-1}a), (2^{n-1}a, b)$ contains a subset $E_2 \subset E_1$ of positive measure with the property $E_2 \times E_2 \subset D$ and then $f(x)\, f(y) \neq 0$ for $(x, y) \in E_2 \times E_2$.

Thus, by equation (4), $f(\frac{x}{2})\, f(\frac{x}{2} + y) \neq 0$, whenever $x, y \in E_2$. It follows that

$$f(u) \neq 0 \quad \text{if} \quad u \in \frac{E_2}{2} + E_2.$$

Since $\frac{E_2}{2}$ and E_2 have positive Lebesgue measure, by a theorem of STEINHAUS (see [2]), the set $\frac{E_2}{2} + E_2$ contains an interval $[c, d] \subset \mathbb{R}_+$ of positive length and thus

$$f(u) \neq 0, \quad u \in [c, d].$$

By the substitution $y = x = t \in [c, d]$, we get from (4) that $0 \neq f^2(t) = f(\frac{t}{2})\, f(\frac{3}{2}t)$, which implies $f(\frac{t}{2}) \neq 0$, $f(\frac{3}{2}t) \neq 0$ for all $t \in [c, d]$. Thus $f(t) \neq 0$ if $t \in [\frac{c}{2}, \frac{3}{2}d]$.

It follows by induction that $f(x) \neq 0$ for all $x \in \mathbb{R}_+$.

Lemma 3. *If the function f satisfies the functional equation (4) and $f(x) \neq 0$ for all $x \in \mathbb{R}_+$ then the function*

$$h : (-1, \infty) \to \mathbb{R},\ h(x) = \ln \frac{f(x+1)}{f(1)}, \tag{6}$$

satisfies the functional equation

$$h(x+y) = h(x) + h(y) \qquad (x \in (0,1),\, y > x - 1), \tag{7}$$

i.e., h is additive on the open connected domain $\{(x, y) \mid x \in (0,1),\ y > x - 1\}$

PROOF. Let us replace x by $\frac{x}{2}$ in (4), then we get the equation

$$f(2x)\, f(x) = f(x)\, f(x+y) \qquad (y > x > 0). \tag{8}$$

Since $f(x) \neq 0$, (8) shows that the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$g(x) = \frac{f(2x)}{f(x)} \qquad (x \in \mathbb{R}_+),$$

satisfies the Pexider type functional equation

$$f(x+y) = g(x)\, f(y) \qquad (y > x > 0). \tag{9}$$

Putting $y = 1$ in (9), we obtain that

$$g(x) = \frac{f(x+1)}{f(1)}, \qquad x \in (0,1),$$

which, together with (9), implies

$$f(x+y) = \frac{f(x+1)}{f(1)}\, f(y) \qquad (x \in (0,1),\, y > x).$$

Replacing $y + 1$ by y and dividing this equation by $f(1)$, we obtain

$$\frac{f(x+y+1)}{f(1)} = \frac{f(x+1)}{f(1)}\, \frac{f(y+1)}{f(1)} \ (x \in (0,1),\ y > x - 1). \tag{10}$$

Substitute here $x = y = \frac{t}{2}$, then we get

$$\frac{f(t+1)}{f(1)} = \frac{f^2(\frac{t}{2}+1)}{f^2(1)} > 0.$$

Thus (10) shows that the function h defined by (6) satisfies the functional equation (7).

Lemma 4. *If the function $h : (-1,\infty) \to \mathbb{R}$ satisfies the functional equation* (7)*, then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that*

$$h(x) = A(x), \qquad x \in (-1,\infty). \tag{11}$$

Proof. Using a theorem on the extension of additive functions due to of Daróczy and Losonczi (see [1]), it follows that there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and constants C_1, C_2, $C_3 \in \mathbb{R}$ such that

$$h(x) = \begin{cases} A(x) + C_1, & x \in D_x = (0,1), \\ A(x) + C_2, & x \in D_y = (-1,\infty), \\ A(x) + C_3, & x \in D_{x+y} = (-1,\infty). \end{cases}$$

Since $D_x \subset D_y = D_{x+y}$ and h satisfies (7), hence $C_1 = C_2 = C_3 = 0$ and h has the form (11).

Theorem. *If the function $B : \mathbb{R}_+ \to \mathbb{R}$ satisfies the functional equation* (2) *and there exists a set $E \subset \mathbb{R}_+$ of positive measure such that $B(R) \neq 0$ if $R \in E$, then*

$$B(x) = \alpha \exp[A(x^2)] \qquad (x \in \mathbb{R}_+), \tag{12}$$

where $\alpha \in \mathbb{R}_0$ is an arbitrary constant, $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$.

Proof. The condition of theorem and Lemma 1 imply that the function f defined by (3) satisfies the functional equation (4) and $f(x) \neq 0$ for $x \in E_1 = E^2 = \{x \in \mathbb{R}_+, \sqrt{x} \in E\}$, where E_1 has positive Lebesgue-measure, too. Then, by Lemma 2, it follows that $f(x) \neq 0$ for all $x \in \mathbb{R}_+$.

Thus the conditions of Lemma 3 and 4 are satisfied, too.

From (3), (6) and (11) we obtain the form

$$B(x) = f(1) \exp[A(x^2 - 1)] \qquad (x \in \mathbb{R}_+), \tag{13}$$

for the function B, where $f(1) \neq 0$ is an arbitrary real constant. Finally, we get from (13) the form (12) for B, where $\alpha = f(1) \exp[-A(1)] \in \mathbb{R}_0$ is an arbitrary constant.

3. The continuous and measurable solutions of (2)

Theorem 2. *A function* $B : \mathbb{R}_+ \to \mathbb{R}$ *is a continuous solution of equation* (2) *if and only if*

$$B(x) = \alpha \exp(ax^2) \qquad (x \in \mathbb{R}_+), \tag{14}$$

where $\alpha, a \in \mathbb{R}$ *are arbitrary constants.*

Proof. If B is continuous and there exists an $R_0 \in \mathbb{R}_+$, such that $B(R_0) \neq 0$, then because of (3) $f(x_0) \neq 0$ for $x_0 = R_0^2$ and so there exists an interval $I = [x_0 - \delta,\, x_0 + \delta] \subset \mathbb{R}_+$, such that $f(x) \neq 0$ for all $x \in I$ and $I \times I \subset D$. Thus the last part of the proof of Lemma 2 gives that $f(x) \neq 0$ for all $x \in \mathbb{R}_+$.

Using again the continuity of B, we get that the functions f, h in Lemma 3 and function A in Lemma 4 are continuous. But A is an additive function on $\mathbb{R}^2$, thus $A(x) = ax (x \in \mathbb{R})$ with some $a \in \mathbb{R}$. Formula (14) results from (12) for $\alpha \in \mathbb{R}_0$.

$B \equiv 0$ is also a solution of (2).

This completes the proof of Theorem 2.

Theorem 3. *If the measurable function* $B : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the functional equation* (2) *and there exists a set* $E \subset \mathbb{R}_+$ *of positive measure such that* $B(R) \neq 0$ *if* $R \in E$, *then* B *has the form* (14), *where* $\alpha \in \mathbb{R}_0$ *and* $a \in \mathbb{R}$ *arbitrary constants.*

Proof. The statements of Lemma 1,2,3 and 4 are true for measurable functions B, f, h, A, thus $A(x) = ax$ $(x \in \mathbb{R})$ with some $a \in \mathbb{R}$, and hence (14) follows from (12).

Remark. The characteristic function of the rationals satisfies the functional equation (4) in Lemma 1. Thus there is measurable solution of (2) which is almost everywhere zero, but not identically zero. This shows that there is a measurable solution of equation (2) which is not continuous.

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