Publ. Math. Debrecen
80/1-2 (2012), 1–23
DOI: 10.5486/PMD.2012.4561

Self-stabilization in certain infinite-dimensional matrix algebras

By GYULA LAKOS (Budapest)

Abstract. Analytical tools to K-theory; namely, self-stabilization of rapidly decreasing matrices, linearization of cyclic loops, and the contractibility of the pointed stable Toeplitz algebra are discussed in terms of concrete formulas. Adaptation to the $*$-algebra and finite perturbation categories is also considered. The finite linearizability of algebraically finite cyclic loops is demonstrated.

1. Introduction

Learning K-theory, one likely encounters stabilization of matrices, linearization of cyclic loops, and the contractibility of the pointed stable Toeplitz algebra. Stabilization of matrices is a fundamental feature of K-theory; linearization of cyclic loops is an important method to prove complex Bott periodicity; the Toeplitz algebra can also be used for the same purpose, but it is also a tool to construct classifying spaces. Although considered simple, these basic constructions are often treated in quite awkward manners. The purpose of this paper is to show that these topics can be discussed in a unified and simple way. Our statements are formulated primarily in the setting of locally convex algebras. This is not just for the sake of extreme generality but to demonstrate that concrete formulas and maps can be very successful, without using approximations. The main statements of this paper are as follows:

Statement 1.1 (Self-stabilization)**.** *Assume that* $\mathfrak{A} = \mathcal{K}_{\mathbb{Z}}(\mathfrak{S})$, *i.e. the locally convex algebra of rapidly decreasing* $\mathbb{Z} \times \mathbb{Z}$ *matrices over an other locally convex*

Mathematics Subject Classification: Primary: 19K99; Secondary: 47B35.
Key words and phrases: linearization of loops, stabilization, Toeplitz algebra, Bott periodicity, locally convex algebras.

algebra $\mathfrak{S}$. Let $r:\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})\to\mathfrak{A}$ be an isomorphism which comes from relabeling $\mathcal{K}_{\mathbb{Z}\times\mathbb{Z}}(\mathfrak{S})$ into $\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})$. Then there is a smooth homotopy

$$E:\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})\times[0,\pi/2]\to\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})$$

such that it yields a family of endomorphisms of $\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})$, which are isomorphisms for $\theta\in[0,\pi/2)$, and a closed injective endomorphism for $\theta=\pi/2$, with

$$E(A,0)=A,\qquad E(A,\pi/2)=\mathrm{Diag}(\dots,0,0\mid r(A),0,0,\dots);$$

cf. (1) *for the diagonal notation. This statement extends to unit groups, showing that $\mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A}))$ can be pushed down by a homotopy into $\mathtt{U}(\mathfrak{A}\mathbf{e}_{00})$.*

(The continuous map $\phi:\mathfrak{A}\times[0,\pi/2]_\theta\to\mathfrak{B}$ is smooth in the variable θ if the higher partial derivatives $\partial_\theta^n\phi:\mathfrak{A}\times[0,\pi/2]_\theta\to\mathfrak{B}$ are still continuous functions.)

Let $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ be the algebra of formal Laurent series with rapidly decreasing coefficients.

Statement 1.2 (Linearization of cyclic loops)**.** *There is a smooth homotopy*

$$K:\mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])\times[0,\pi/2]\to\mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{z}^{-1},\mathsf{z}]),$$

such that

$$K(a(\mathsf{z}),\pi/2)=\mathrm{Diag}(\dots 1,1\mid a(\mathsf{z})a(1)^{-1},1,1,\dots),$$

but

$$K(a(\mathsf{z}),0)=\mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)^{-1}\Lambda(\mathsf{z},\mathsf{Q})^{-1};$$

where $\Lambda(\mathsf{z},\mathsf{Q})=\mathrm{Diag}(\dots,\mathsf{z},\mathsf{z}\mid 1,1,\dots)$ is the linear loop generated by the Hilbert transform, cf. (1)*, and $\mathsf{U}(a)$ is the matrix of multiplication by $a(\mathsf{z})$, cf.* (2).

Statement 1.3 (Toeplitz contractibility)**.** *Let $\mathfrak{A}=\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})$. Then the unit group of the pointed Toeplitz algebra over $\mathfrak{A}$, i.e. $\mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathfrak{A})^{\mathrm{po}})$, is contractible.*

These statements were formulated in the smooth category. However, it is often useful to work in slightly different categories. One case is when $\mathfrak{A}$ is a $*$-algebra. In those cases, instead of the general unit group $\mathtt{U}(\mathfrak{A})$ of invertible elements, one should work with the group $\mathtt{U}^*(\mathfrak{A})$ of unitary elements. Another type of restriction occurs in the finite perturbation category, when the algebra $\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})$ of rapidly decreasing matrices is replaced by the algebra $\mathcal{K}^{\mathrm{f}}_{\mathbb{Z}}(\mathfrak{S})$ of matrices with finitely many nonzero entries, and the algebra $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ of rapidly decreasing Laurent series is replaced by the algebra $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{f}}$ of finite Laurent series. (Here one should be careful, because for smooth loops being finite and invertible does not generally imply that the inverse is finite.)

Statement 1.4. *Statements 1.1–1.3 restrict to the $*$-algebra and/or finite perturbation categories.*

The setting of finite perturbations, may, however, be too restrictive. Let us call an element $a(\mathsf{z}) \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])$ algebraically finite if $a = a_s \dots a_1$, where for each s either a_s or $(a_s)^{-1}$ has finite Laurent series form. The algebraically finite elements of $\mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])$ fall into various finiteness classes F depending on the length of the elements a_s or $(a_s)^{-1}$. Let $\mathfrak{A}_F$ be the set of decompositions $\{a_j\}_{1\leq j\leq s}$ compatible with F. Then Statement 1.2 can be augmented as follows:

Statement 1.5. *For any finiteness class F, there is a smooth homotopy*

$$K_F^{\mathrm{e}} : \mathfrak{A}_F \times [0,1] \times [0,\pi/2] \to \mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{z}^{-1},\mathsf{z}]),$$

such that

(i) $K_F^{\mathrm{e}}(\tilde{a},0,\theta) = K(a,\theta)$;

(ii) $K_F^{\mathrm{e}}(\tilde{a},1,\theta)$ *differs from* $1_{\mathbb{Z}}$ *in finitely many places (depending on F);*

(iii) $K_F^{\mathrm{e}}(\tilde{a},h,\pi/2)$ *is constant in* h*;*

(iv) $K_F^{\mathrm{e}}(\tilde{a},h,0) = \mathsf{U}_F(\tilde{a},h)\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}_F(\tilde{a},h)^{-1}\Lambda(\mathsf{z},\mathsf{Q})^{-1}$.

Here $\mathsf{U}_F(\tilde{a},h)$ differs from $\mathsf{U}(a)$ in a rapidly decreasing matrix.

In particular, $K_F^{\mathrm{e}}(\tilde{a},1,0)$ yields a finite linearization of $a(\mathsf{z})a(1)^{-1}$.

These statements are known, but in lesser generality, in various ways: Statement 1.1, as stated here in the smooth category (however, see 1.4), follows from Cuntz, [3], Section 2. Statement 1.2 is a quantitative version of the well-known linearization technique of Atiyah and Bott, [1]; but much resembling to the formulas of Pressley and Segal, [8], Ch. 6, who work with Hilbert–Schmidt matrices, instead of rapidly decreasing ones. Statement 1.3 comes from the original Toeplitz argument of Cuntz, [2], originally stated in the context of C^*-algebras, but subsequently adapted to the smooth case, cf. also [3]. One can also find some explicit homotopies in [4]. Statement 1.4 is useful, because $*$-algebras are prominent in operator algebraic discussions; and the finite perturbation category is the technically easiest setting to provide large contractible spaces for the purposes of algebraic topology. Statement 1.5 amounts to an explicit computation in the less functorial but more concrete setting of [1].

The constructions presented here are improved versions of some constructions which can be found in the author's thesis [5]. The author indebted to Prof. Richard B. Melrose, his advisor, for helpful discussions. In fact, much of this content was motivated by the geometric idea of Melrose, Rochon [7]. The author would also like to thank Prof. Joachim Cuntz, who called his attention to some related papers, and Prof. Balázs Csikós, for some useful advices.

2. A general framework for computations

If $\mathfrak{A}$ is a not necessarily unital algebra, then one can consider the semigroup $1+\mathfrak{A}$, with elements of form $1+a$, $(a\in\mathfrak{A})$, which multiply as $(1+a)(1+b) = 1+(a+b+ab)$. If $\mathfrak{A}$ is unital, then it is customary to identify $\mathfrak{A}$ and $1+\mathfrak{A}$ by the recipe $a\in\mathfrak{A} \leftrightarrow 1-(1_{\mathfrak{A}}-a)\in 1+\mathfrak{A}$. This is also the situation if there is a natural identity element which can be associated to $\mathfrak{A}$, like the identity matrix in the case of matrix algebras. The unit group $\mathtt{U}(\mathfrak{A})$ of $\mathfrak{A}$ is the unit group of the semigroup $1+\mathfrak{A}$, i.e., it is the group of pairs $(1+a,1+b)\in(1+\mathfrak{A})\times(1+\mathfrak{A})$ such that $(1+a)(1+b)=(1+b)(1+a)=1$; they multiply as $(1+a_1,1+b_1)$ $(1+a_2,1+b_2)=((1+a_1)(1+a_2),(1+b_2)(1+b_1))$. If $\mathfrak{A}$ is a topological ring, then the natural topology on $\mathtt{U}(\mathfrak{A})$ comes from the product topology of $(1+\mathfrak{A})\times(1+\mathfrak{A})$ by restriction. As $1+a$ determines $1+b$, we write "$1+a$" instead of "$(1+a,1+b)$". If $\phi:\mathfrak{A}\to\mathfrak{B}$ is a homomorphism, then it induces a homomorphism $\mathtt{U}\,\phi:\mathtt{U}(\mathfrak{A})\to\mathtt{U}(\mathfrak{B})$ defined by $1+a\mapsto 1+\phi(a)$. We will write ϕ instead of $\mathtt{U}\,\phi$.

In what follows, a "locally convex vector space $\mathfrak{A}$" means a sequentially complete, Hausdorff, locally convex vector space $\mathfrak{A}$. The completeness is essential for analytic purposes. If the topology of $\mathfrak{A}$ is induced by a set $\Pi_{\mathfrak{A}}$ of seminorms, then we assume that any positive integral combination of these seminorms also belongs to the generating seminorm set. A locally convex algebra $\mathfrak{A}$ is a locally convex vector space with continuous bilinear multiplication. So, for each seminorm $p\in\Pi_{\mathfrak{A}}$ there is an other seminorm $\tilde{p}\in\Pi_{\mathfrak{A}}$ such that for all $X_1,X_2\in\mathfrak{A}$ the inequality $p(X_1X_2)\le\tilde{p}(X_1)\tilde{p}(X_2)$ holds. An inductive locally convex vector space $\mathfrak{A}$ is an indexed family of locally convex vector spaces $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ such that the following holds: Λ is an upward directed partially ordered set, i.e. for all $\lambda,\mu\in\Lambda$ there is an element $\nu\ge\lambda,\mu$. For all $\mu\ge\lambda$ there exist continuous inclusions $T^\lambda_\mu:\mathfrak{A}_\lambda\to\mathfrak{A}_\mu$; and for $\nu\ge\mu\ge\lambda$ one has $T^\mu_\nu\circ T^\lambda_\mu=T^\lambda_\nu$. Now, $\mathfrak{A}$ is an inductive locally convex algebra if for each $\lambda,\mu\in\Lambda$ there is an element $\mathrm{prod}(\lambda,\mu)\in\Lambda$, and for $\nu\ge\mathrm{prod}(\lambda,\mu)$, bilinear products $M^\nu_{\lambda,\mu}:\mathfrak{A}_\lambda\times\mathfrak{A}_\mu\to\mathfrak{A}_\nu$ compatible with the inclusions and the usual algebraic prescriptions are given. An element of $\mathfrak{A}$ is an element of $\bigcup_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ making identifications along the inclusion maps. Then $\mathfrak{A}$ will be an algebra endowed with an "inductive" topology coming from the filtration $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$, such that the vector space structure respects the filtration but the algebra structure does not. If the spaces $\mathfrak{A}$ and $\mathfrak{B}$ have inductive topologies with filtrations $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ and $\{\mathfrak{B}_\mu\}_{\mu\in M}$, then a map $\phi:\mathfrak{A}\to\mathfrak{B}$ is continuous if for each $\lambda\in\Lambda$ there is an element $\mu\in M$ such that there is a continuous map $\phi_\lambda:\mathfrak{A}_\lambda\to\mathfrak{B}_\mu$ which is set-theoretically a restriction of ϕ.

Suppose that Θ_1,Θ_2 are sets and $\mathfrak{V}$ is a vector space. Then a $\mathfrak{V}$-valued

Θ_1 times Θ_2 matrix is just a formal sum $s = \sum_{a\in\Theta_1,b\in\Theta_2} s_{a,b}\mathbf{e}_{a,b} \in \mathcal{M}_{\Theta_1,\Theta_2}(\mathfrak{V})$ with coefficients $s_{a,b}$ from $\mathfrak{V}$. We write $\mathcal{M}_\Theta(\mathfrak{V})$ instead of $\mathcal{M}_{\Theta,\Theta}(\mathfrak{V})$ and use similar notation for other spaces as well. For column and row matrices, we use the notation $\mathbf{e}_a = \mathbf{e}_{a,*}$ and $\mathbf{e}_b^\top = \mathbf{e}_{*,b}$ respectively, and we make the formal identification $\mathbf{e}_{a,b} = \mathbf{e}_a \otimes \mathbf{e}_b^\top$. For column spaces, we use the notation $\mathcal{S}(\Theta;\mathfrak{V}) = \mathcal{M}_{\Theta,\{*\}}(\mathfrak{V})$. We use the notation $1_\Theta = \sum_{\theta\in\Theta} \mathbf{e}_{\theta,\theta}$, and in general circumstances we consider the identity matrix 1_Θ as the adjoint unit in any non-unital Θ times Θ matrix algebra. If $s_i \in \mathfrak{V}$, $i \in \mathbb{Z}$ are given then

$$\mathrm{Diag}(\dots s_{-2}, s_{-1}|s_0, s_1, s_2, \dots) = \sum_{i\in\mathbb{Z}} s_i\mathbf{e}_{i,i} \in \mathcal{M}_\mathbb{Z}(\mathfrak{V}) \tag{1}$$

is the corresponding diagonal matrix; $\mathrm{Diag}(s_0, s_1, s_2, \dots) \in \mathcal{M}_\mathbb{N}(\mathfrak{V})$, similarly. For $a \in \mathfrak{A}$, we define the matrices $\mathsf{E}_\mathbb{N}(a) = a\mathbf{e}_{00} \in \mathcal{K}_\mathbb{N}(\mathfrak{A})$ and $\mathsf{E}_\mathbb{Z}(a) = a\mathbf{e}_{00} \in \mathcal{K}_\mathbb{Z}(\mathfrak{A})$. Then, as usual, for $\tilde{a} = 1+a \in 1+\mathfrak{A}$, we extend these maps as $\mathsf{E}_\mathbb{N}(\tilde{a}) = 1_\mathbb{N}+\mathsf{E}_\mathbb{N}(a)$ and $\mathsf{E}_\mathbb{Z}(\tilde{a}) = 1_\mathbb{Z}+\mathsf{E}_\mathbb{Z}(a)$; i.e., for $\tilde{a} \in 1+\mathfrak{A}$, it yields $\mathsf{E}_\mathbb{N}(\tilde{a}) = \mathrm{Diag}(\tilde{a}, 1, 1, \dots)$, and $\mathsf{E}_\mathbb{Z}(\tilde{a}) = \mathrm{Diag}(\dots, 1, 1 \mid \tilde{a}, 1, 1, \dots)$.

On the set $\mathbb{N}$ of natural numbers, there is the natural space $\mathcal{S}^\infty(\mathbb{N};\mathbb{R})^\star$, i.e. the space of multiplicatively invertible polynomially growing functions. A countable set Θ is called a set of polynomial growth if it is endowed with a set of functions $\mathcal{S}^\infty(\Theta;\mathbb{R})^\star$ from Θ to $\mathbb{R}$ such that there is a bijection $\omega : \Theta \to \mathbb{N}$ so that $\omega^*\mathcal{S}^\infty(\mathbb{N};\mathbb{R})^\star = \mathcal{S}^\infty(\Theta;\mathbb{R})^\star$. It is notable that $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}\,\dot\cup\,\mathbb{N}$ are sets of polynomial growth naturally; and that way we can define the direct product $\Theta_1\times\Theta_2$ and direct sums $\Theta_1\,\dot\cup\,\Theta_2$ of sets of polynomial growth Θ_1 and Θ_2. In what follows, the sets of polynomial growth we use will be like $\mathbb{N}, \mathbb{Z}$, or $\{1,\dots,n\}\times\mathbb{Z}$, where the description of the relevant function spaces is evident, so it will not be detailed. The point is that a set Θ of polynomial growth is just like $\mathbb{N}$ for practical purposes. If Θ_1, Θ_2 are sets of polynomial growth, and $\mathfrak{V}$ is a locally convex vector space, then we can define some matrix spaces as follows:

(a) With functions $F : \Pi_\mathfrak{V} \to \mathcal{S}^\infty(\Theta_1;\mathbb{R})^\star \times \mathcal{S}^\infty(\Theta_2;\mathbb{R})^\star$, the filtering spaces

$$\mathcal{M}^{\infty,\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})_F = \Bigg\{ s \in \mathcal{M}_{\Theta_1,\Theta_2}(\mathfrak{V}) : \forall p \in \Pi_\mathfrak{V} \quad |s|_{\frac{1}{F_1(p)},p,\frac{1}{F_2(p)}} = \sum_{(a,b)\in\Theta_1\times\Theta_2} \left|\frac{1}{F_1(p)(a)}\right| p(s_{a,b}) \left|\frac{1}{F_2(p)(b)}\right| < +\infty \Bigg\}$$

form the inductive locally convex space $\mathcal{M}^{\infty,\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})$.

(b) With functions $F : \Pi_\mathfrak{V} \times \mathcal{S}^\infty(\Theta_2;\mathbb{R})^\star \to \mathcal{S}^\infty(\Theta_1;\mathbb{R})^\star$, the filtering spaces

$$\mathcal{M}^{\infty,-\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})_F = \Bigg\{ s \in \mathcal{M}_{\Theta_1,\Theta_2}(\mathfrak{V}) : \forall p \in \Pi_{\mathfrak{V}} \forall g \in \mathcal{S}^\infty(\Theta_2;\mathbb{R})^\star \quad |s|_{\frac{1}{F(p,g)},p,g} = \sum_{(a,b)\in\Theta_1\times\Theta_2} \left| \frac{1}{F(p,g)(a)} \right| p(s_{a,b})|g(b)| < +\infty \Bigg\}$$

form the inductive locally convex space $\mathcal{M}^{\infty,-\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})$. We can define the space $\mathcal{M}^{-\infty,\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})$ similarly.

(c) We define

$$\mathcal{M}^{-\infty,-\infty}(\Theta_1,\Theta_2;\mathfrak{V}) = \Bigg\{ s \in \mathcal{M}(\Theta_1,\Theta_2;\mathfrak{V}) : \forall p \in \Pi_{\mathfrak{V}} \forall f \in \mathcal{S}^\infty(\Theta_1;\mathbb{R})^\star \quad \forall g \in \mathcal{S}^\infty(\Theta_2;\mathbb{R})^\star \qquad |s|_{f,p,g} = \sum_{(a,b)\in\Theta_1\times\Theta_2} |f(a)|\, p(s_{a,b})|g(b)| < +\infty \Bigg\}.$$

(d) It is natural to define $\Psi_{\Theta_1,\Theta_2}(\mathfrak{V}) = \mathcal{M}^{-\infty,\infty}_{\Theta_1,\Theta_2}(\mathfrak{V}) \cap \mathcal{M}^{\infty,-\infty}_{\Theta_1,\Theta_2}(\mathfrak{V})$, the space of matrices of "pseudodifferential size".

If $\mathfrak{A}\times\mathfrak{B}\to\mathfrak{C}$ is a continuous bilinear pairing between locally convex spaces, then we have induced continuous pairings $\mathcal{M}^{\mathsf{X},\infty}_{\Theta_1,\Theta_2}(\mathfrak{A}) \times \mathcal{M}^{-\infty,\mathsf{Y}}_{\Theta_2,\Theta_3}(\mathfrak{B}) \to \mathcal{M}^{\mathsf{X},\mathsf{Y}}_{\Theta_1,\Theta_3}(\mathfrak{C})$, $\Psi_{\Theta_1,\Theta_2}(\mathfrak{A}) \times \Psi_{\Theta_2,\Theta_3}(\mathfrak{B}) \to \Psi_{\Theta_1,\Theta_3}(\mathfrak{C})$, etc. So come the algebra and module structures associated to matrices. Instead of $\mathcal{M}^{-\infty,-\infty}_{\Theta_1,\Theta_2}(\mathfrak{A})$ we will use the shorter notations $\mathcal{K}_{\Theta_1,\Theta_2}(\mathfrak{A})$. Instead of $\mathcal{M}^{\pm\infty,\mathsf{X}}_{\Theta,\{*\}}(\mathfrak{A})$ it is reasonable to use $\mathcal{S}^{\pm\infty}(\Theta;\mathfrak{A})$, which is a consistent extension of our earlier notation. There are natural isomorphisms like $\mathcal{K}_{\Theta_1\times\Theta_1',\Theta_2\times\Theta_2'}(\mathfrak{A}) \simeq \mathcal{K}_{\Theta_1,\Theta_2}(\mathcal{K}_{\Theta_1',\Theta_2'}(\mathfrak{A}))$, etc. One often uses is relabeling of matrices, which is as follows: Suppose that $\omega:\Omega\to\Omega'$ is a map between sets of polynomial growth, such that $\omega^*\mathcal{S}^\infty(\Omega',\mathbb{R})^\star = \mathcal{S}^\infty(\Omega,\mathbb{R})^\star$. This includes the case when ω is an isomorphism of sets of polynomial growth, and also the natural inclusions $\iota:\Omega\to\Omega'\dot\cup\Omega''$, where Ω'' is finite or an other set of polynomial growth. Let us now consider the matrix $R_\omega = \sum_{\alpha\in\Omega}\mathbf{e}_{\alpha,\omega(\alpha)} \in \Psi_{\Omega,\Omega'}(\mathbb{R})$. Then, for a matrix $A\in\mathcal{M}^{\mathsf{X},\mathsf{Y}}_{\Theta}(\mathfrak{A})$ or $\Psi_\Theta(\mathfrak{A})$, we can take the matrix $r_\omega(A) = R_\omega^\top A R_\omega$, which is a matrix of the same kind as A but Ω is replaced by Ω'. This relabeling r_ω is a continuous, smooth operation, which is an isomorphism if ω is an isomorphism.

The advantage of the spaces $\mathcal{M}^{\mathsf{X},\mathsf{Y}}_{\Theta_1,\Theta_2}(\mathfrak{V})$ is that they are sufficiently large for the purposes of arithmetic calculations. In what follows, only the algebras $\mathcal{K}$ will be used explicitly. On the other hand, all calculations, except in Section 7 will be governed by the principle every matrix expression will be understood as an element of $\Psi_{\Omega_1,\Omega_2}(\mathfrak{A})$, where Ω_i are sets of polynomial growth, and $\mathfrak{A}$ is a locally convex algebra; but we always hope that our expressions will yield results which turn out to be continuous in stronger topologies.

3. The environment of cyclic and Toeplitz algebras

Cyclic and Toeplitz algebras. In what follows, let $\overline{\mathbb{N}} = \mathbb{Z} \setminus \mathbb{N}$, so $\mathbb{Z} = \overline{\mathbb{N}} \dot{\cup} \mathbb{N}$. We make a canonical correspondence between $\mathbb{N}$ and $\overline{\mathbb{N}}$ by relabeling every n to $-1-n$. We can consider every $\mathbb{Z}\times\mathbb{Z}$ matrix U as a 2×2 matrix of $\mathbb{N}\times\mathbb{N}$ matrices:

$$U = \left[\begin{array}{c|c} U|_{\overline{\mathbb{N}}\times\overline{\mathbb{N}}} & U|_{\overline{\mathbb{N}}\times\mathbb{N}} \\ \hline U|_{\mathbb{N}\times\overline{\mathbb{N}}} & U|_{\mathbb{N}\times\mathbb{N}} \end{array}\right] \simeq \begin{bmatrix} U^{--} & U^{-+} \\ U^{+-} & U^{++} \end{bmatrix},$$

such that the matrix entries on the right side are $\mathbb{N}\times\mathbb{N}$ matrices obtained by the correspondence explained above.

An element $a = \sum_{i\in\mathbb{Z}} a_i \mathbf{e}_i \in \mathcal{S}^{-\infty}(\mathbb{Z};\mathfrak{A})$ can and will, in general, be identified with the Laurent series $\sum_{i\in\mathbb{Z}} a_i \mathsf{z}^i \in \mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ with rapidly decreasing coefficients. We call this algebra the algebra of cyclic loops, in contrast to the algebra of proper loops $\mathcal{C}^\infty(\mathrm{S}^1;\mathfrak{A})$. Elements $a = \sum_{i\in\mathbb{Z}} a_i \mathsf{z}^i \in \mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ can be represented by $\mathbb{Z}\times\mathbb{Z}$ matrices

$$\mathsf{U}(a) = \sum_{n,m\in\mathbb{Z}} a_{n-m}\mathbf{e}_{n,m} = \left[\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & & a_{-2} & a_{-3} & \ddots \\ \ddots & a_1 & a_0 & & a_{-1} & a_{-2} & \ddots \\ \hline \ddots & a_2 & a_1 & & a_0 & a_{-1} & \ddots \\ \ddots & a_3 & a_2 & & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots \end{array}\right]. \tag{2}$$

If $\mathcal{W}_\mathbb{Z}(\mathfrak{A})$ is the image set of $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ under U, then it is a subset of $\Psi(\mathbb{Z};\mathfrak{V})$ algebraically, but we put the topology of $\mathcal{S}^{-\infty}(\mathbb{Z};\mathfrak{V})$ to it. If $\mathfrak{A}$ is a locally convex algebra, then $\mathsf{U} : \mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}] \to \mathcal{W}_\mathbb{Z}(\mathfrak{A})$ is an isomorphism of algebras. When it comes to the 2×2 decomposition as explained above, in order to simplify the notation, we will just write $\mathsf{W}(a)$ instead of $\mathsf{U}(a)^{++}$, and $\mathsf{Y}(a)$ instead of $\mathsf{U}(a)^{+-}$. If $a = a(\mathsf{z})$ then, with some abuse of notation, we also write $a^\top = a(\mathsf{z}^{-1})$. Then

$$\mathsf{U}(a) = \begin{bmatrix} \mathsf{W}(a^\top) & \mathsf{Y}(a^\top) \\ \mathsf{Y}(a) & \mathsf{W}(a) \end{bmatrix}.$$

So $\mathsf{W}(a)$ is the infinite Toeplitz matrix associated to a, and $\mathsf{Y}(a)$ is the infinite Hankel matrix associated to ("the positive part" of) a.

As far as the linear structure is concerned, we could have just used the matrices $\mathsf{W}(a)$ to represent the elements a, and use the set $\mathcal{W}_\mathbb{N}(\mathfrak{A})$ of restricted matrices. The difference is that in terms of matrix multiplication $\mathsf{W}(a)\mathsf{W}(b) = \mathsf{W}(ab) - \mathsf{Y}(a)\mathsf{Y}(b^\top)$, so there is an "anomalous" term $-\mathsf{Y}(a)\mathsf{Y}(b^\top) \in \mathcal{K}_{\mathbb{N}}(\mathfrak{A})$. One can see that algebraically $\mathcal{W}_{\mathbb{N}}(\mathfrak{A}) \cap \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) = 0$. Hence it is reasonable to define the Toeplitz algebra

$$\mathcal{T}_{\mathbb{N}}(\mathfrak{A}) = \mathcal{W}_{\mathbb{N}}(\mathfrak{A}) + \mathcal{K}_{\mathbb{N}}(\mathfrak{A}),$$

which is topologically just $\mathcal{W}_{\mathbb{N}}(\mathfrak{A}) \oplus \mathcal{K}_{\mathbb{N}}(\mathfrak{A})$ but with the algebraic product rule $(\mathsf{W}(a)+p)((\mathsf{W}(b)+q)) = \mathsf{W}(ab) + (-\mathsf{Y}(a)\mathsf{Y}(b^\top) + \mathsf{W}(a)q + p\mathsf{W}(b) + pq)$, induced from the matrix structure. Algebraically, $\mathcal{T}_{\mathbb{N}}(\mathfrak{A})$ is just a subset of $\Psi(\mathbb{N};\mathfrak{A})$ but a locally convex algebra. So, one can see that there is a short exact sequence of algebras $0 \to \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \xrightarrow{\iota} \mathcal{T}_{\mathbb{N}}(\mathfrak{A}) \xrightarrow{\sigma} \mathcal{W}_{\mathbb{Z}}(\mathfrak{A}) \to 0$. The map ι is the inclusion of the ideal of rapidly decreasing matrices into the Toeplitz algebra, while σ is the symbol map. In what follows, we rather consider the value of the symbol map as an element of $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$, so we have the symbol homomorphism

$$\sigma : \mathcal{T}_{\mathbb{N}}(\mathfrak{A}) \to \mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}].$$

We can naturally extend this symbol map to unit groups as we have seen.

For technical reasons, we define the algebra

$$\mathcal{T}_{\mathbb{Z}}(\mathfrak{A}) = \begin{bmatrix} \mathcal{T}_{\mathbb{N}}(\mathfrak{A}) & \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \\ \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) & \mathcal{T}_{\mathbb{N}}(\mathfrak{A}) \end{bmatrix},$$

which is also naturally a locally convex algebra. Then $\mathcal{W}_{\mathbb{Z}}(\mathfrak{A}) \subset \mathcal{W}_{\mathbb{Z}}(\mathfrak{A}) + \mathcal{K}_{\mathbb{Z}}(\mathfrak{A}) \subset \mathcal{T}_{\mathbb{Z}}(\mathfrak{A})$. For the sake of notational convenience, we define the block matrix

$$\widehat{\mathsf{U}}(a) = \left[\begin{array}{c|c} \mathsf{W}(a^\top) & -\mathsf{Y}(a^\top) \\ \hline & 0 \\ -\mathsf{Y}(a) & \mathsf{W}(a) \end{array}\right] \in \mathcal{T}_{\mathbb{Z}}(\mathfrak{A}).$$

We remark that for $\tilde{a} \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])$ an "1" appears in the place of "0". Elements of $\mathcal{T}_{\mathbb{Z}}(\mathfrak{A})$ have two symbols; one belonging to the lower right quadrant, and one belonging to the upper left quadrant. It is a small but important observation regarding $\mathsf{U}(a) \in \mathcal{T}_{\mathbb{Z}}(\mathfrak{A})$ that the Toeplitz element in the lower right quadrant has symbol $a = a(\mathsf{z})$, but the Toeplitz element in the upper left quadrant has symbol $a^\top = a(\mathsf{z}^{-1})$. One can also see that there are natural isomorphisms like $\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\Omega}(\mathfrak{A})) \simeq \mathcal{K}_{\Omega}(\mathcal{T}_{\mathbb{N}}(\mathfrak{A}))$, etc. In fact, all of our matrix space constructions considered as functors are naturally "commutative".

Let $\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{po}}$ be the set of pointed loops, i.e., where $a(1) = 0$. Then the elements $\tilde{a} \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{po}})$ are those for which $\tilde{a}(1) = 1$. These pointed spaces are closed subspaces of the unpointed spaces. We can define the pointed Toeplitz algebra $\mathcal{T}_{\mathbb{N}}(\mathfrak{A})^{\mathrm{po}}$ similarly, the symbols are pointed there.

The Bott involution map. In what follows, we use the abbreviation $\Lambda(a,b) = \frac{1}{2}(1+a+b-ab)$. Let $\mathsf{Q} = \begin{bmatrix} -1_{\mathbb{N}} & \\ & 1_{\mathbb{N}} \end{bmatrix}$. Then $\Lambda(\mathsf{z},\mathsf{Q}) = \begin{bmatrix} \mathsf{z}1_{\mathbb{N}} & \\ & 1_{\mathbb{N}} \end{bmatrix}$. We use the delta function $\delta_{n,m}$, which is 1 if $n=m$, and it is 0 otherwise.

If $a \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])$, then we define the "Bott" involution

$$\mathsf{B}(a) = \mathsf{U}(a)\mathsf{Q}\mathsf{U}(a)^{-1} \in \mathsf{Q} + \mathcal{K}_{\mathbb{Z}}(\mathfrak{A})$$

(cf. the symbols).

"Shifting rotations". Our natural deformation parameter variable, in general, will be $\theta \in [0,\pi/2]$, or, more generally, $\theta \in \mathrm{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. In order to save space, we often use $t = \sin\theta$ and $s = \cos\theta$ instead. It is useful to keep in mind that $s^2 = 1 - t^2$. For $\theta \in \mathrm{S}^1$, we define the matrices

$$\mathsf{C}(\theta) = \begin{bmatrix} s & ts & t^2s & t^3s & \cdots \\ -t & s^2 & ts^2 & t^2s^2 & \ddots \\ & -t & s^2 & ts^2 & \ddots \\ & & -t & s^2 & \ddots \\ & & & -t & \ddots \\ & & & & \ddots \end{bmatrix} \in \mathcal{T}_{\mathbb{N}}(\mathbb{R}).$$

Lemma 3.1. *Let $\mathsf{C}(\theta)^\dagger$ denote the transpose of $\mathsf{C}(\theta)$. Then*

(a)
$$\mathsf{C}(\theta)^\dagger\mathsf{C}(\theta) = 1_{\mathbb{N}}.$$

(b)
$$\mathsf{C}(\theta)\mathsf{C}(\theta)^\dagger = -\delta_{t,1}\mathbf{e}_{0,0} - \delta_{t,-1}\mathbf{e}_{0,0} + 1_{\mathbb{N}}.$$

(c)
$$\mathsf{C}(\theta)\mathbf{e}_{n,m}\mathsf{C}(\theta)^\dagger = \begin{bmatrix} t^{n+m}s^2 & t^{n+m-1}s^3 & \cdots & t^ns^3 & -t^{n+1}s & & \\ t^{n+m-1}s^3 & t^{n+m-2}s^4 & \cdots & t^{n-1}s^4 & -t^ns^2 & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ t^ms^3 & t^{m-1}s^4 & \cdots & s^4 & -ts^2 & & \\ -t^{m+1}s & -t^ms^2 & \cdots & -ts^2 & t^2 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{bmatrix}.$$

(d) *For $n > 0$,*

$$\mathsf{C}(\theta)\mathsf{W}(\mathsf{z}^n)\mathsf{C}(\theta)^\dagger = -\delta_{t,1}\mathbf{e}_{0,0} - (-1)^n\delta_{t,-1}\mathbf{e}_{0,0} + \begin{bmatrix} t^n & & & \\ t^{n-1}s & & & \\ \vdots & & & \\ ts & & & \\ s & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix},$$

and $\mathsf{C}(\theta)\mathsf{W}(\mathsf{z}^{-n})\mathsf{C}(\theta)^\dagger$ is the transpose of the matrix above.

Proof. Direct computation. □

Remark 3.2. The presence of the terms $\delta_{t,1}$, $\delta_{t,-1}$ might be surprising at first sight. It reflects the phenomenon that in a topological algebra one cannot simultaneously topologize the families $\mathsf{C}(\theta)$ and $\mathsf{C}(\theta)^\dagger$ correctly. In fact, the $\mathsf{C}(\theta)$'s are isometries for $-1 < t < 1$, but they are just partial isometries for $t = \pm 1$.

4. Stabilizing homotopies

Proposition 4.1. *The continuous map*

$$T_{\mathcal{K}} : \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \times \mathrm{S}^1 \to \mathcal{K}_{\mathbb{N}}(\mathfrak{A})$$

given by

$$A, \theta \mapsto T_{\mathcal{K}}(A,\theta) = \mathsf{C}(\theta) A \mathsf{C}(\theta)^\dagger$$

is smooth in θ. It yields a family of endomorphisms of $\mathcal{K}_{\mathbb{N}}(\mathfrak{A})$ when θ is fixed. These are isomorphisms for $-1 < t < 1$, and closed injective endomorphisms for $t = \pm 1$. In particular, for $\theta = 0$ ($t = 0$),

$$T_{\mathcal{K}}(A,0) = A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix};$$

but for $\theta = \pm\pi/2$ ($t = \pm 1$),

$$T_{\mathcal{K}}(A,\pm\pi/2) = \begin{bmatrix} 0 & & & \\ & a_{11} & a_{12} & \cdots \\ & a_{21} & a_{22} & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. Well-definedness and smoothness follows from Lemma 3.1.c. Lemma 3.1.a implies that we have a family of endomorphisms. Furthermore, it also shows that $A = \mathsf{C}(\theta)^\dagger T_{\mathcal{K}}(A,\theta)\mathsf{C}(\theta)$; from which the statement about the nature of the endomorphisms follows easily. □

Hence, taking $\theta \in [0,\pi/2]$, we see that the deformation $T_{\mathcal{K}}$ does indeed realize a stabilizing homotopy, even if only with one "extra dimension". Nevertheless, after this, stabilization becomes a matter of standard tricks:

Corollary 4.2 (⇒ Statement 1.1)**.** *Let Ω_1 and Ω_2 be sets of polynomial growth; $\Omega = \Omega_1 \dot\cup \Omega_2$, and let $\omega : \Omega \to \Omega_1 \subset \Omega$ be the composition of an isomorphism $\Omega \simeq \Omega_1$ and the natural inclusion $\Omega_1 \to \Omega_1 \dot\cup \Omega_2$. Then we claim:*

There is a smooth map $\widehat{T}_{\mathcal{K}} : \mathcal{K}_\Omega(\mathfrak{S}) \times [0,\pi/2] \to \mathcal{K}_\Omega(\mathfrak{S})$ such that it yields a family of endomorphisms of $\mathcal{K}_\Omega(\mathfrak{S})$, which are isomorphisms for $\theta \in [0,\pi/2)$, and a closed injective endomorphism for $\theta = \pi/2$, such that $\widehat{T}_{\mathcal{K}}(A,0) = \mathrm{id}_{\mathcal{K}_\Omega(\mathfrak{S})}$ and $\widehat{T}_{\mathcal{K}}(A,\pi/2) = r_\omega$. The map $\widehat{T}_{\mathcal{K}}$ extends to unit groups naturally.

Proof. Take $\mathfrak{A} = \mathcal{K}_{\mathbb{N}}(\mathfrak{S})$ in the previous statement. It yields our statement with $\Omega_1 = (\mathbb{N} \setminus \{0\}) \times \mathbb{N}$, $\Omega_2 = \{0\} \times \mathbb{N}$, $\omega((n,m)) = (n+1,m)$. Now, using an appropriate relabeling r_η of Ω we obtain the general statement. □

Remark 4.3. Another way to achieve stabilization by many dimensions is to "quantize" $\mathsf{C}(\theta)$, see [6].

Due to the multiplicative structure, the concatenation of group valued homotopies is particularly simple: If $f,g : Y \times [0,1] \to G$ yield homotopies $f_0 \simeq f_1$, $g_0 \simeq g_1$ where $f_1 = g_0$, then $h(y,t) = f(y,t)f(y,1)^{-1}g(y,t)$ yields a homotopy between f_0 and g_1. Then polynomial/smooth homotopies yield polynomial/smooth homotopies, and the operation is associative; in contrast to concatenation by reparametrization. Using this observation and the stabilizing homotopies above, one can easily prove

Corollary 4.4. *Let Ω_1, Ω_2 be sets of polynomial growth, and let $\iota_1 : \Omega_1 \to \Omega_1 \dot\cup \Omega_2$ be the natural inclusion. Assume that $H : X \times [0,1] \to \mathtt{U}(\mathcal{K}_{\Omega_1 \dot\cup \Omega_2}(\mathfrak{S}))$ is a smooth homotopy with maps $f_0, f_1 : X \to \mathtt{U}(\mathcal{K}_{\Omega_1}(\mathfrak{S}))$ such that $H_0 = r_{\iota_1}(f_0)$ and $H_1 = r_{\iota_1}(f_1)$. Then we claim that there is a smooth homotopy $f : X \times [0,1] \to \mathtt{U}(\mathcal{K}_{\Omega_1}(\mathfrak{S}))$ between f_0 and f_1. This f can be chosen so that there is a smooth homotopy between H and $r_{\iota_1}(f)$ relative to endpoints. In other words: "In stable algebras stable homotopies can be reduced to ordinary homotopies."* □

Using the same techniques, the statement extends to stable homotopies of (stable) involutions.

5. Linearization of cyclic loops

5.1. Let v be a cyclic formal variable, and take $\mathsf{V}=\sum_{n\in\mathbb{N}}\mathsf{v}^n\mathbf{e}_{n,n}$. Furthermore, take $\mathsf{G}(\theta,\mathsf{v})=\begin{bmatrix}1_{\mathbb{N}}&\\&\mathsf{V}^{-1}\mathsf{C}(\theta)\mathsf{V}\end{bmatrix}$ and $\mathsf{G}^\dagger(\theta,\mathsf{v})=\begin{bmatrix}1_{\mathbb{N}}&\\&\mathsf{V}^{-1}\mathsf{C}(\theta)^\dagger\mathsf{V}\end{bmatrix}$. For $a\in\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$, we define

$$U(a,\theta,\mathsf{v})=\delta_{t,1}a(\mathsf{v})\mathbf{e}_{00}+\delta_{t,-1}a(-\mathsf{v})\mathbf{e}_{00}+\mathsf{G}(\theta,\mathsf{v})\mathsf{U}(a)\mathsf{G}^\dagger(\theta,\mathsf{v}).$$

5.2. For $n>0$, this definition yields

$$U(\mathsf{z}^n,\theta,\mathsf{v})$$
$$=\left[\begin{array}{ccccccc|ccc}
\ddots &&&&&&&&&\\
& 1 &&&&&&&&\\
\hline
&& s & ts\mathsf{v} & t^2s\mathsf{v}^2 & \cdots & t^{n-1}s\mathsf{v}^{n-1} & t^n\mathsf{v}^n &&\\
&& -t\mathsf{v}^{-1} & s^2 & ts^2\mathsf{v} & \cdots & t^{n-2}s^2\mathsf{v}^{n-2} & t^{n-1}s\mathsf{v}^{n-1} &&\\
&&& \ddots & \ddots & \ddots & \vdots & \vdots &&\\
&&&& \ddots & \ddots & ts^2\mathsf{v} & t^2s\mathsf{v}^2 &&\\
&&&&& \ddots & s^2 & ts\mathsf{v} &&\\
&&&&&& -t\mathsf{v}^{-1} & s &&\\
&&&&&&&& 1 &\\
&&&&&&&&& \ddots
\end{array}\right],$$

$U(1,\theta,\mathsf{v})=1_{\mathbb{Z}}$, and $U(\mathsf{z}^{-n},\theta,\mathsf{v})=U(\mathsf{z}^n,\theta,\mathsf{v}^{-1})^\top$; i.e. $U(\mathsf{z}^n,\theta,\mathsf{v})$ is just a rather nice perturbation / deformation of $\mathsf{U}(\mathsf{z}^n)$.

Lemma 5.3. *The continuous map*

$$U:\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]\times\mathrm{S}^1\to\mathcal{W}_{\mathbb{Z}}(\mathfrak{A})+\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{v}^{-1},\mathsf{v}]$$

defined by

$$a,\theta\mapsto U(a,\theta,\mathsf{v})$$

is smooth in θ. It yields a family of homomorphisms with fixed θ. The symbols remain constant. For $\theta=0$,

$$U(a,0,\mathsf{v})=\begin{bmatrix}\mathsf{W}(a^\top)&\mathsf{Y}(a^\top)\\\mathsf{Y}(a)&\mathsf{W}(a)\end{bmatrix}=\mathsf{U}(a);$$

while for $\theta = \pi/2$,

$$U(a,\pi/2,\mathsf{v}) = \left[\begin{array}{c|c} \mathsf{W}(a^\top) & -\mathsf{v}\mathsf{Y}(a^\top) \\ \hline & a(\mathsf{v}) \\ -\mathsf{v}^{-1}\mathsf{Y}(a) & \mathsf{W}(a) \end{array}\right].$$

PROOF. This is immediate from 5.2 by taking linear combinations. □

Considering $a \in \mathtt{U}(\mathfrak{A})$, and the natural extension to the unit group, $U(a,\pi/2,\mathsf{v}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{v}))\Lambda(\mathsf{v},\mathsf{Q})\widehat{\mathtt{U}}(a)\Lambda(\mathsf{v},\mathsf{Q})^{-1}$ can be written.

Proposition 5.4 (⇒ Statement 1.2)**.** *The continuous map*

$$K : \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]) \times \mathrm{S}^1 \to \mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{v}^{-1},\mathsf{v}]^{\mathrm{po}})$$

defined by

$$a,\theta \mapsto K(a,\theta,\mathsf{v}) = U(a,\theta,\mathsf{v})\Lambda(\mathsf{v},\mathsf{Q})U(a,\theta,1)^{-1}\Lambda(\mathsf{v},\mathsf{Q})^{-1}$$

is smooth in the variable θ. Here

$$K(a,0,\mathsf{v}) = \Lambda(\mathsf{v},\mathsf{B}(a))\Lambda(\mathsf{v},\mathsf{Q})^{-1}, \qquad K(a,\pi/2,\mathsf{v}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{v})a(1)^{-1}).$$

PROOF. The statement follows immediately from the previous lemma. □

Remark 5.5. When it comes to the linearization of not pointed loops but the "cocycle" $a(\mathsf{z})a(\mathsf{w})^{-1}$, then one can use the linearizing "cocycle" $K^{\mathrm{c}}(a,\theta,\mathsf{z},\mathsf{w}) = U(a,\theta,\mathsf{z})\Lambda(\mathsf{z}\mathsf{w}^{-1},\mathsf{Q})U(a,\theta,\mathsf{w})^{-1}$. It yields $K^{\mathrm{c}}(a,0,\mathsf{z},\mathsf{w}) = \Lambda(\mathsf{z}\mathsf{w}^{-1},\mathsf{B}(a))$ and $K^{\mathrm{c}}(a,\pi/2,\mathsf{z},\mathsf{w}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{z})a(\mathsf{w})^{-1})$. Then $K(a,\theta,\mathsf{v}) = K^{\mathrm{c}}(a,\theta,\mathsf{z},1)\Lambda(\mathsf{z},\mathsf{Q})^{-1}$.

It is notable that loops which are already linear will remain constant but stabilized: If $a(\mathsf{z}) = \Lambda(\mathsf{z},\tilde{Q})$ then $K^{\mathrm{c}}(a,\theta,\mathsf{z},\mathsf{w}) = \mathrm{Diag}(\ldots,\mathsf{z}\mathsf{w}^{-1}|\Lambda(\mathsf{z}\mathsf{w}^{-1},\tilde{Q}),1\ldots)$, independently from θ. Similarly, rapidly decreasing perturbations of a linear loop will linearize through rapidly decreasing perturbations of that linear loop.

Remark 5.6. For a locally convex algebra $\mathfrak{A}$ we can define

$$K_0(\mathfrak{A}) = \pi_0^{\mathrm{smooth}}(\mathfrak{Invol}(\mathsf{Q}+\mathcal{K}_{\mathbb{Z}}(\mathfrak{A}))),$$

the smooth path components of the involutions, which are perturbations of Q. Similarly, one can define

$$K_1(\mathfrak{A}) = \pi_0^{\mathrm{smooth}}(\mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A}))).$$

Now B, by this linearization argument, induces an isomorphism

$$\mathsf{B}_* : K_1(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{po}}) \to K_0(\mathfrak{A}).$$

This is the "hard part" of Bott periodicity in the complex case, when geometric loops can be represented by cyclic loops.

6. The contractibility of the pointed stable Toeplitz unit group

When we extend the stabilization procedure of Proposition 4.1 to Toeplitz algebras, the symbol suddenly appears in the result:

Proposition 6.1. *The continuous map*

$$T : \mathcal{T}_{\mathbb{N}}(\mathfrak{A}) \times \mathrm{S}^1 \to \mathcal{W}_{\mathbb{N}}(\mathfrak{A}) + \mathcal{K}_{\mathbb{N}}(\mathfrak{A})[\mathsf{v}^{-1}, \mathsf{v}] \subset \mathcal{T}_{\mathbb{N}}(\mathfrak{A})[\mathsf{v}^{-1}, \mathsf{v}]$$

defined by

$$A, \theta \mapsto T(A, \theta, \mathsf{v}) = \delta_{t,1} a(\mathsf{v}) \mathbf{e}_{0,0} + \delta_{t,-1} a(-\mathsf{v}) \mathbf{e}_{0,0} + \mathsf{V}^{-1} \mathsf{C}(\theta) \mathsf{V} A \mathsf{V}^{-1} \mathsf{C}(\theta)^{\dagger} \mathsf{V},$$

where $a = \sigma(A)$, is smooth in the variable θ. It yields a family of homomorphisms of $\mathcal{T}_{\mathbb{N}}(\mathfrak{A})$ to $\mathcal{T}_{\mathbb{N}}(\mathfrak{A})[\mathsf{v}^{-1}, \mathsf{v}]$. The map leaves the symbol invariant. For $\theta = 0$,

$$T(A, 0, \mathsf{v}) = A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix};$$

but for $\theta = \pm\pi/2$,

$$T(A, \pm\pi/2, \mathsf{v}) = \begin{bmatrix} a(\pm\mathsf{v}) & & & \\ & a_{11} & a_{12} & \cdots \\ & a_{21} & a_{22} & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. It follows from direct inspection of the matrices in question. □

As a corollary we obtain

Proposition 6.2. *The map $Z : \mathsf{U}(\mathcal{T}_{\mathbb{N}}(\mathfrak{A})) \times \mathrm{S}^1 \to \mathsf{U}(\mathcal{K}_{\mathbb{N}}(\mathfrak{A})[\mathsf{v}^{-1}, \mathsf{v}]^{\mathrm{po}})$ defined by $A, \theta \mapsto Z(A, \theta, \mathsf{v}) = T(A, \theta, \mathsf{v}) T(A, \theta, 1)^{-1}$ is smooth in θ. For $\theta = 0$ it yields $Z(A, 0, \mathsf{v}) = 1_{\mathbb{N}}$, but for $\theta = \pm\pi/2$ it yields $Z(A, \pm\pi/2, \mathsf{v}) = \mathsf{E}_{\mathbb{N}}(a(\pm\mathsf{v}) a(1)^{-1})$.*

Consequently, the symbols $a(\mathsf{z})$ of invertible Toeplitz algebra elements are stably homotopic to constant loops $a(1)$. If $\mathfrak{A} = \mathcal{K}_{\mathbb{N}}(\mathfrak{S})$, then (according to Corollary 4.2) stable homotopy implies the existence of ordinary homotopies. □

6.3. Suppose that Q is an involution, and $k \in \mathfrak{A}$. We will use the shorthand notation $k_Q^{+} = \frac{1}{2}(k + QkQ)$, $k_Q^{++} = \frac{1+Q}{2} k \frac{1+Q}{2}$, $k_Q^{+-} = \frac{1+Q}{2} k \frac{1-Q}{2}$, $k_Q^{-+} =$

$\frac{1-Q}{2}k\frac{1+Q}{2}$. Let us define

$$L(Q,k)=\mathsf{W}(\Lambda(\mathsf{z},Q))k\mathsf{W}(\Lambda(\mathsf{z}^{-1},Q))=\begin{bmatrix} k_Q^{++} & k_Q^{+-} & & \\ k_Q^{-+} & k_Q^{+} & k_Q^{+-} & \\ & k_Q^{-+} & k_Q^{+} & \ddots \\ & & \ddots & \ddots \end{bmatrix}.$$

This is a homomorphism in k, and we can extend it to $\tilde{k}=1+k$ by $L(Q,\tilde{k})=1_{\mathbb{N}}+L(Q,k)$. Notice that in this case, $\tilde{k}L(Q,\tilde{k}^{-1})$ has symbol $\tilde{k}\Lambda(\mathsf{z},Q)\tilde{k}^{-1}\Lambda(\mathsf{z},Q)^{-1}$.

6.4. Assume that $Q=\mathsf{Q}$ and $k\in\mathcal{T}_{\mathbb{Z}}(\mathfrak{S})$. Set

$$\widetilde{L}(k)=\left[\begin{array}{ccc|cccc} \ddots & \ddots & & & & & \\ \ddots & k_Q^{+} & k_Q^{-+} & & & & \\ & k_Q^{+-} & k_Q^{+} & k_Q^{-+}\mathbf{e}_{00} & k_Q^{-+}\mathbf{e}_{10} & k_Q^{-+}\mathbf{e}_{20} & \cdots \\ \hline & & \mathbf{e}_{00}k_Q^{+-} & \mathbf{e}_{00}k_Q^{++}\mathbf{e}_{00} & \mathbf{e}_{00}k_Q^{++}\mathbf{e}_{10} & \mathbf{e}_{00}k_Q^{++}\mathbf{e}_{20} & \cdots \\ & & \mathbf{e}_{01}k_Q^{+-} & \mathbf{e}_{01}k_Q^{++}\mathbf{e}_{00} & \mathbf{e}_{01}k_Q^{++}\mathbf{e}_{10} & \mathbf{e}_{01}k_Q^{++}\mathbf{e}_{20} & \cdots \\ & & \mathbf{e}_{02}k_Q^{+-} & \mathbf{e}_{02}k_Q^{++}\mathbf{e}_{00} & \mathbf{e}_{02}k_Q^{++}\mathbf{e}_{10} & \mathbf{e}_{02}k_Q^{++}\mathbf{e}_{20} & \cdots \\ & & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

What happens here, compared to $L(Q,k)$, is the following: We inflated the first row and column to infinitely many rows and columns, and reordered the matrix. Again, this is a homomorphism in k, and we can extend it to $\tilde{k}\in 1_{\mathbb{Z}}+\mathcal{T}_{\mathbb{Z}}(\mathfrak{S})$ by taking $\widetilde{L}(\tilde{k})=1_{\mathbb{Z}\times\mathbb{Z}}+\widetilde{L}(k)$. Assume now that $\tilde{k}\in\mathtt{U}(\mathcal{T}_{\mathbb{Z}}(\mathfrak{S}))$, and the symbol of its lower right quadrant is $a(\mathsf{z})$. Consider

$$\mathsf{U}(\Lambda(\mathsf{z},\mathsf{Q}))\widetilde{L}(\tilde{k})\Lambda(\tilde{k}^{-1},\mathsf{Q}_{\mathcal{T}_{\mathbb{Z}}(\mathfrak{S})})\mathsf{U}(\tilde{k}\Lambda(\mathsf{z},\mathsf{Q})^{-1}\tilde{k}^{-1})$$

$$=\left[\begin{array}{ccc|cc} \ddots & \ddots & & & \\ & \frac{1-\mathsf{Q}}{2} & \frac{1+\mathsf{Q}}{2} & & \\ \hline & & \frac{1-\mathsf{Q}}{2} & \frac{1+\mathsf{Q}}{2} & \\ & & & \ddots & \ddots \end{array}\right]\widetilde{L}(\tilde{k})\left[\begin{array}{ccc|cc} \ddots & \ddots & & & \\ & \frac{1+\mathsf{Q}}{2} & \frac{1-\mathsf{Q}}{2} & & \\ \hline & & \tilde{k}\frac{1+\mathsf{Q}}{2} & \tilde{k}\frac{1-\mathsf{Q}}{2} & \\ & & & \ddots & \ddots \end{array}\right]\tilde{k}^{-1}.$$

From the observation $\widetilde{L}(\tilde{k})\Lambda(\tilde{k}^{-1},\mathsf{Q}_{\mathcal{T}_{\mathbb{Z}}(\mathfrak{S})})\in\mathtt{U}(\mathcal{T}_{\mathbb{Z}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})))$, and a careful examination of the matrix product, we find that the resulting expression is of shape $\begin{bmatrix}1_{\mathbb{N}} & \\ & N(\tilde{k})\end{bmatrix}\in\mathtt{U}(\mathcal{T}_{\mathbb{Z}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})))$; where we introduced the notation $N(\tilde{k})$ for the lower right quadrant. Then the component $N(\tilde{k})\in\mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})))$ has symbol $\Lambda(\mathsf{z},\mathsf{Q})\mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\tilde{k}\Lambda(\mathsf{z},\mathsf{Q})^{-1}\tilde{k}^{-1}=\mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{z},\mathsf{Q})\tilde{k}\Lambda(\mathsf{z},\mathsf{Q})^{-1}\tilde{k}^{-1}$. Let us set $G(a)=N(\mathsf{U}(a))$. This yields

Proposition 6.5. *The continuous map* $G : \mathtt{U}(\mathfrak{S}[\mathsf{z}^{-1},\mathsf{z}]) \to \mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S})))$ *is such that the symbol of* $G(a)$ *is* $\mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})^{-1}\mathsf{U}(a)^{-1}$. □

Now, according to Proposition 6.2 and Corollary 4.4, the mere existence of the map above implies that the symbol $\mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})^{-1}\mathsf{U}(a)^{-1}$ is homotopic to $1_{\mathbb{Z}}$ for $a(\mathsf{z}) \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{po}})$. Hence, Proposition 6.5 can be considered as a reformulation of linearizability.

Proposition 6.6 (⇒ Statement 1.3)**.** *The unit group* $\mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S}))^{\mathrm{po}})$ *is smoothly contractible.*

Proof. We prove the statement up to stabilization. Then stabilization can be removed according to Corollary 4.4.

(a) First, consider any element $A \in \mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S}))^{\mathrm{po}})$. According to Proposition 6.2, its symbol a is (stably) homotopic to the constant loop 1. Applying Proposition 6.5 to this homotopy, we see that it is sufficient to prove that Toeplitz units with symbol $\mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)^{-1}\Lambda(\mathsf{z},\mathsf{Q})^{-1}$ can be contracted.

(b) Consider, again, A as above. Let

$$Q = \begin{bmatrix} -1_{\mathbb{Z}} & \\ & \mathsf{Q} \end{bmatrix} = \left[\begin{array}{cc||cc} -1_{\mathbb{N}} & & & \\ & -1_{\mathbb{N}} & & \\ \hline\hline & & -1_{\mathbb{N}} & \\ & & & 1_{\mathbb{N}} \end{array}\right];$$

here the double lines show how we decompose this block matrix of $\mathbb{Z}\times\mathbb{Z}$ matrices to a block matrix of $\mathbb{N}\times\mathbb{N}$ matrices. Furthermore, let

$$S(\theta) = \begin{bmatrix} s & & & -t \\ & 1 & & \\ & & 1 & \\ t & & & s \end{bmatrix} \begin{bmatrix} 1_{\mathbb{N}} & & & \\ & 1_{\mathbb{N}} & & \\ & & \mathsf{W}(a^{\top}) & \mathsf{Y}(a^{\top}) \\ & & \mathsf{Y}(a) & \mathsf{W}(a) \end{bmatrix}$$

$$\begin{bmatrix} 1_{\mathbb{N}} & & & \\ & \mathsf{W}(a^{-1}) & \mathsf{Y}(a^{-1}) & \\ & \mathsf{Y}((a^{-1})^{\top}) & \mathsf{W}((a^{-1})^{\top}) & \\ & & & 1_{\mathbb{N}} \end{bmatrix} \begin{bmatrix} 1_{\mathbb{N}} & & & \\ & A & & \\ & & 1_{\mathbb{N}} & \\ & & & A^{-1} \end{bmatrix} \begin{bmatrix} s & & & t \\ & 1 & & \\ & & 1 & \\ -t & & & s \end{bmatrix}$$

$$\in \mathtt{U}(\mathcal{K}_{\{1,2\}\times\mathbb{Z}}(\mathfrak{S})),$$

and take $S(\theta)L(Q,S(\theta)^{-1}) \in \mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathcal{K}_{\{1,2\}\times\mathbb{Z}}(\mathfrak{S})))$. This yields a homotopy between $S(0)L(Q,S(0)^{-1})$ and $S(\pi/2)L(Q,S(\pi/2)^{-1})$, which have symbols

$$S(0)\Lambda(\mathsf{z},Q)S(0)^{-1}\Lambda(\mathsf{z},Q)^{-1} = \begin{bmatrix} 1_{\mathbb{Z}} & \\ & \mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)^{-1}\Lambda(\mathsf{z},\mathsf{Q})^{-1} \end{bmatrix}$$

and

$$S(\pi/2)\Lambda(\mathsf{z},Q)S(\pi/2)^{-1}\Lambda(\mathsf{z},Q)^{-1} = \begin{bmatrix} 1_{\mathbb{Z}} & \\ & 1_{\mathbb{Z}} \end{bmatrix},$$

respectively. Thus, Toeplitz units with symbol $\Lambda(\mathsf{z}^{-1},\mathsf{Q})\mathsf{U}(a)\Lambda(\mathsf{z},\mathsf{Q})\mathsf{U}(a)^{-1}$ can be deformed to Toeplitz units with trivial symbols. According to part (a), it is sufficient to show that elements with trivial symbol can be contracted.

(c) Now suppose that the symbol of a Toeplitz unit A is 1. According to standard stabilization arguments, we can assume that $A = \mathsf{E}_{\mathbb{N}}(\tilde{k})$, where $\tilde{k} = \left[\begin{smallmatrix} k_0 & \\ & 1_{\mathbb{N}} \end{smallmatrix}\right] \in \mathtt{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{S}))$. Let $\tilde{k}(\theta) = \left[\begin{smallmatrix} s & t \\ -t & s \end{smallmatrix}\right] \begin{bmatrix} \tilde{k}_0 & \\ & 1_{\mathbb{N}} \end{bmatrix} \left[\begin{smallmatrix} s & -t \\ t & s \end{smallmatrix}\right]$. Then $\tilde{k}(\theta)L(\mathsf{Q},\tilde{k}(\theta))^{-1}$ yields a homotopy between $\tilde{k}(0)L(\mathsf{Q},\tilde{k}(0))^{-1} = \mathsf{E}_{\mathbb{N}}(\tilde{k}) = A$ and $\tilde{k}(\pi/2)L(\mathsf{Q},\tilde{k}(\pi/2))^{-1} = 1$. □

Remark 6.7. If the locally convex algebra $\mathfrak{A}$ is strong in the terminology of in [5], i.e. for all seminorm p there is a seminorm $\tilde{p}$ such that $p(X_1 \dots X_n) \leq \tilde{p}(X_1)\dots\tilde{p}(X_n)$ holds for all n, then the proof can be much simplified: In that case, the associated algebras are also strong, and the smooth homotopy lifting property holds for the symbol map. Then, using Proposition 6.2, the proof of the contractibility statement reduces to point (c) immediately, hence making points (a) and (b), and the construction of 6.4 unnecessary. One must note that Proposition 6.6 above is much easier to prove than Kuiper's Theorem about the contractibility of the unitary group. See, e.g. [9]. Stabilization was an important assumption in the previous statement. For example, $\mathtt{U}(\mathcal{T}_{\mathbb{N}}(\mathbb{C})^{\mathrm{po}})$ is not contractible, as it allows an extended, multiplicative determinant.

7. Possible modifications

Due to the nice properties of $U(a,\theta,\mathsf{v})$, Statement 1.4 can be seen in a rather straightforward manner. We remark that another such category is the category of Hilbert–Schmidt operators, used by Pressley and Segal, [8], Ch. 6. Furthermore, with some extra work, the transformation parameter θ (i.e. s and t jointly) can be replaced by t entirely, extending the constructions as formal homotopies.

8. Algebraically finite cyclic loops

A practical disadvantage of $\mathsf{B}(a)$ is that it is, in general, an infinite perturbation of Q. The exception is when $a \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathrm{f}})$, but this is a rather restrictive condition from geometrical viewpoint. We will show below that we can do well also in the case when a can be represented by finite loops but it is not in $\mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]^{\mathsf{f}})$.

8.1. For $m \leq 0 \leq n$, we say that the loop $a(\mathsf{z}) \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}])$ is an $L(m,n)$-finite loop if $a(\mathsf{z}) = \sum_{m\leq j\leq n} a_j \mathsf{z}^k$. A loop $a(\mathsf{z})$ is an $R(m,n)$-finite loop if its inverse $a(\mathsf{z})^{-1}$ is an $L(-n,-m)$-finite loop. For a finite sequence $F = \{(m_j,n_j)\}_{1\leq j\leq s}$, let

$$\mathfrak{A}_F = \{(a_s,\dots,a_1) : a_j \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]) \text{ is } L(m_j,n_j) \text{ or } R(m_j,n_j)\text{-finite}\}.$$

We say that $a \in \mathtt{U}(\mathfrak{A}[\mathsf{z}^{-1},\mathsf{z}]$ is algebraically finite of type F if $a = a_s \dots a_1$ for an element $(a_s,\dots,a_1) \in \mathfrak{A}_F$.

8.2. For $m \leq 0 \leq n$, we say that a matrix A is an $L(m,n)$-perturbation of A_0 if

$$A = A_0 + \sum_{m\leq i\leq n,\, j\in\mathbb{Z}} a_{i,j}\mathbf{e}_{i,j},$$

for $a_{i,j}$ chosen suitably. Similarly, we can define $R(m,n)$-perturbations by interchanging the role of i and j in the expression above. An (m,n)-perturbation is a matrix which is both an $L(m,n)$-perturbation and an $R(m,n)$-perturbation.

In what follows, we will always be concerned with perturbations of $\Lambda(\mathsf{s},\mathsf{Q})$, where s is equal to 1, -1, or another formal variable v. Both $L(m,n)$-perturbations and $R(m,n)$-perturbations of $\Lambda(\mathsf{s},\mathsf{Q})$ can be reduced to (m,n)-perturbations by taking direct cut-offs of unwanted matrix elements:

$$\begin{bmatrix} \mathsf{s} & & \\ L^- & M & L^+ \\ & & 1 \end{bmatrix} \xrightarrow{\mathrm{R}_{(m,n)}} \begin{bmatrix} \mathsf{s} & & \\ & M & \\ & & 1 \end{bmatrix} \xleftarrow{\mathrm{R}_{(m,n)}} \begin{bmatrix} \mathsf{s} & R^- & \\ & M & \\ & R^+ & 1 \end{bmatrix}.$$

The reduction $\mathrm{R}_{(m,n)}$ is essentially taking away the off-diagonal elements of a triangular block matrix (with respect to an appropriate ordering of the basis). Sometimes it is practical to use the partial reduction $\mathrm{R}^{[h]}_{(m,n)} = (1-h)\mathrm{Id} + h\,\mathrm{R}_{(m,n)}$, where h is assumed to be a scalar variable. Here the off-diagonal blocks are not taken away completely but multiplied by $1-h$. It is useful to notice that (partial) reduction is a homomorphism as long as we restrict our attention to matrices of appropriate block triangular shape. In particular, invertible elements / involutions are reduced to invertible elements / involutions.

8.3. The involutions Q and $\bar{Q}$ are unipotently related if $\frac{1}{2}(Q\bar{Q}+\bar{Q}Q)=1$ holds. In this case the expression $C(\bar{Q},Q)=\frac{1+\bar{Q}Q}{2}$ satisfies the identities

$$C(\bar{Q},Q)^{-1}=C(Q,\bar{Q}) \quad \text{and} \quad C(\bar{Q},Q)\,QC\,(\bar{Q},Q)^{-1}=\bar{Q}.$$

More generally, $C(\bar{Q},Q,h)=(1-h)1+h\frac{1+\bar{Q}Q}{2}$ satisfies the identities

$$C(\bar{Q},Q,h)^{-1}=C(Q,\bar{Q},h) \quad \text{and} \quad C(\bar{Q},Q,h)\,QC\,(\bar{Q},Q,h)^{-1}=(1-h)Q+h\bar{Q}.$$

This situation applies when, in the manner of the previous paragraph, an involution Q is reduced to an involution $\bar{Q}$.

Lemma 8.4. *If $a(\mathsf{z})=\sum_{m\le j\le n}a_j\mathsf{z}^k$, $m\le 0\le n$, then $U(a,\theta,\mathsf{v})$ is an (m,n)-perturbation of $\mathsf{U}(a)$.*

Proof. This is immediate from 5.2. □

Lemma 8.5. *Suppose that A is an (m',n')-perturbation of $\Lambda(\mathsf{s},\mathsf{Q})$, where $m'\le 0\le n'$. Then we claim:*

If a is an $L(m,n)$- or $R(m,n)$-finite loop, then $U(a,\theta_1,\mathsf{v})AU(a,\theta_2,\mathsf{w})^{-1}$ is an $L(m+m',n+n')$- or $R(m+m',n+n')$-perturbation of $\Lambda(\mathsf{s},\mathsf{Q})$, respectively.

Proof. The L case: Let $k>n+n'$ and $h=\mathsf{s}$ if $k<m+m'$. The special shape of the matrices implies

$$\mathbf{e}_k^\top U(a,\theta_1,\mathsf{v})A=\left(\sum_{m\le j\le n}a_j\mathbf{e}_{k-j}^\top\right)A=h\sum_{m\le j\le n}a_j\mathbf{e}_{k-j}^\top=h\mathbf{e}_k^\top U(a,\theta_2,\mathsf{w}),$$

from which $\mathbf{e}_k^\top U(a,\theta_1,\mathsf{v})AU(a,\theta_2,\mathsf{w})^{-1}=h\mathbf{e}_k^\top=\mathbf{e}_k^\top\Lambda(\mathsf{s},\mathsf{Q})$. This latter equality, which holds for appropriate k, is exactly the statement of having an $L(m+m',n+n')$-perturbation of $\Lambda(\mathsf{s},\mathsf{Q})$. The R case is similar. □

8.6. Next, we construct a linearization procedure which linearizes algebraically finite loops into finite perturbations: Let $F=\{(m_j,n_j)\}_{1\le j\le s}$ be a finiteness type, $\tilde{a}=(a_s,\dots,a_1)\in\mathfrak{A}_F$, and $a=a_s\dots a_1$. Set $M_k=m_1+\dots+m_k$, $N_k=n_1+\dots+n_k$. Let $|F|=(M_s,N_s)$. Also, let $\tilde{a}_k=(a_k,\dots,a_1)$, with appropriate finiteness type F_k. Then $|F_k|=(M_k,N_k)$. We define

$$\mathsf{B}_F(\tilde{a})=\mathrm{R}_{|F_s|}\left(\mathsf{U}(a_s)\dots\mathrm{R}_{|F_1|}\left(\mathsf{U}(a_1)\mathsf{Q}\mathsf{U}(a_1)^{-1}\right)\dots\mathsf{U}(a_s)^{-1}\right).$$

Then $\mathsf{B}_F(\tilde{a})$ is an involution, and an $|F|$-perturbation of Q. More generally, let

$$K^{\mathrm{c}}_F(\tilde a,\theta,\mathsf{v},\mathsf{w})=\mathrm{R}_{|F_s|}\Big(U(a_s,\theta,\mathsf{v})\ldots$$
$$\ldots\mathrm{R}_{|F_1|}\Big(U(a_1,\theta,\mathsf{v})\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})U(a_1,\theta,\mathsf{w})^{-1}\Big)\ldots U(a_s,\theta,\mathsf{w})^{-1}\Big).$$

Then, in particular, $K^{\mathrm{c}}_F(\tilde a,0,\mathsf{v},\mathsf{w})=\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{B}_F(\tilde a))$, and $K^{\mathrm{c}}_F(\tilde a,\pi/2,\mathsf{v},\mathsf{w})=\mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})\mathsf{E}_{\mathbb{Z}}(a(\mathsf{w})^{-1})$; which are immediate from the special shape of the matrices involved. This yields

Proposition 8.7. *The continuous map*

$$K_F:\mathfrak{A}_F\times\mathrm{S}^1\to\mathsf{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{v}^{-1},\mathsf{v}]^{\mathrm{po}})$$

defined by

$$\tilde a,\theta\mapsto K_F(\tilde a,\theta,\mathsf{v})=K^{\mathrm{c}}_F(\tilde a,\theta,\mathsf{v},1)\Lambda(\mathsf{v},\mathsf{Q})^{-1}$$

is smooth in the variable θ; and it is an $|F|$-perturbation of $1_{\mathbb{Z}}$. Here

$$K_F(\tilde a,0,\mathsf{v})=\Lambda(\mathsf{v},\mathsf{B}_F(\tilde a))\Lambda(\mathsf{v},\mathsf{Q})^{-1},\quad K_F(\tilde a,\pi/2,\mathsf{v})=\mathsf{E}_{\mathbb{Z}}(a(\mathsf{v})a(1)^{-1}).$$

In particular, as S^1 is restricted to $[0,\pi/2]$, it yields a linearizing homotopy of $a(\mathsf{z})a(1)^{-1}$ in the finite perturbation category. □

In the literature one finds comments about the possibly very large size of the matrices used in linearizing homotopies. The result above, however, shows the one can do reasonably well.

8.8. There is, however, a closer analogy between the non-finite and the finite cases: Let $Q_0=\mathsf{Q}$, and $Q_k=\mathrm{R}_{|F_k|}(\mathsf{U}(a_k)Q_{k-1}\mathsf{U}(a_k)^{-1})$ by recursion. Then $Q_k=\mathsf{B}_{F_k}(\tilde a_k)$. Using the notation $\prod_{i=1}^s x_i=x_n\ldots x_2x_1$, let

$$\mathsf{U}_F(\tilde a)=\prod_{i=1}^{s}\frac{\mathsf{U}(a_i)+Q_i\mathsf{U}(a_i)Q_{i-1}}{2}=\prod_{i=1}^{s}C\left(Q_i,\mathsf{U}(a_i)Q_{i-1}\mathsf{U}(a_i)^{-1}\right)\,\mathsf{U}(a_i).$$

According to our earlier observations,

$$\mathsf{B}_F(\tilde a)=\mathsf{U}_F(\tilde a)\mathsf{Q}\mathsf{U}_F(\tilde a)^{-1}.$$

We also define

$$U_F(\tilde a,\theta,\mathsf{v})=\mathrm{R}_{|F_s|}\left(U(a_s,\theta,\mathsf{v})\ldots\mathrm{R}_{|F_1|}\left(U(a_1,\theta,\mathsf{v})\mathsf{U}(a_1)^{-1}\right)\ldots\mathsf{U}(a_s)^{-1}\right)\mathsf{U}_F(\tilde a),$$

and

$$\widehat{\mathsf{U}}_F(\tilde a)=\mathrm{R}_{|F_s|}\left(\widehat{\mathsf{U}}(a_s)\ldots\mathrm{R}_{|F_1|}\left(\widehat{\mathsf{U}}(a_1)\mathsf{U}(a_1)^{-1}\right)\ldots\mathsf{U}(a_s)^{-1}\right)\mathsf{U}_F(\tilde a).$$

Then $U_F(\tilde{a},0,\mathsf{v}) = \mathsf{U}_F(\tilde{a})$, which is trivial; and, analogously to the original situation, $U_F(\tilde{a},\pi/2,\mathsf{v}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{v},\mathsf{Q})\widehat{\mathsf{U}}_F(\tilde{a})\Lambda(\mathsf{v},\mathsf{Q})^{-1}$, which follows from $\Lambda(\mathsf{v},\mathsf{B}_F(\tilde{a}))^{-1} = \mathsf{U}_F(\tilde{a})\Lambda(\mathsf{v},\mathsf{Q})^{-1}\mathsf{U}_F(\tilde{a})^{-1}$ and the homomorphism property of reduction. In fact,

$$K^{\mathrm{c}}_F(\tilde{a},\theta,\mathsf{v},\mathsf{w}) = U_F(\tilde{a},\theta,\mathsf{v})\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})U_F(\tilde{a},\theta,\mathsf{w})^{-1}$$

holds. Again, this follows from $\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{B}_F(\tilde{a})) = \mathsf{U}_F(\tilde{a})\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})\mathsf{U}_F(\tilde{a})^{-1}$ and the homomorphism property of reduction.

8.9. The constructions above can be expounded in order to show that the linearizations K and K_F can nicely be deformed into each other: Let

$$\mathsf{U}_F(\tilde{a},h) = \prod_{k=1}^{s}\left((1-h)\mathsf{U}(a_k) + h\mathsf{U}_{F_k}(\tilde{a}_k)\mathsf{U}_{F_{k-1}}(\tilde{a}_{k-1})^{-1}\right).$$

Here the product terms can also be written as $C(Q_k,\mathsf{U}(a_k)Q_{k-1}\mathsf{U}(a_k)^{-1},h)\mathsf{U}(a_k)$, which makes invertibility clear. Then $\mathsf{U}_F(\tilde{a},0) = \mathsf{U}(a)$, $\mathsf{U}_F(\tilde{a},1) = \mathsf{U}_F(\tilde{a})$. Let

$$\mathsf{B}_F(\tilde{a},h) = \mathsf{U}_F(\tilde{a},h)\mathsf{Q}\mathsf{U}_F(\tilde{a},h)^{-1}.$$

Notice that $\mathsf{B}_F(\tilde{a},0) = \mathsf{B}(a)$, $\mathsf{B}_F(\tilde{a},1) = \mathsf{B}_F(\tilde{a})$. Let

$$U_F(\tilde{a},h,\theta,\mathsf{v}) = \prod_{k=1}^{s}\left((1-h)U(a_k,\theta,\mathsf{v}) + hU_{F_k}(\tilde{a}_k,\theta,\mathsf{v})U_{F_{k-1}}(\tilde{a}_{k-1},\theta,\mathsf{v})^{-1}\right)$$
$$= \prod_{k=1}^{s}\mathrm{R}^{[h]}_{|F_k|}\left(U(a_k,\theta,\mathsf{v})\ldots\mathrm{R}_{|F_1|}\left(U(a_1,\theta,\mathsf{v})\mathsf{U}(a_1)^{-1}\right)\ldots\mathsf{U}(a_k)^{-1}\right)$$
$$C(Q_k,\mathsf{U}(a_k)Q_{k-1}\mathsf{U}(a_k)^{-1},h)\mathsf{U}(a_k)$$
$$\mathrm{R}_{|F_{k-1}|}\left(U(a_{k-1},\theta,\mathsf{v})\ldots\mathrm{R}_{|F_1|}\left(U(a_1,\theta,\mathsf{v})\mathsf{U}(a_1)^{-1}\right)\ldots\mathsf{U}(a_{k-1})^{-1}\right)^{-1}.$$

Again, the latter product form implies not only invertibility but that the inverses of the product terms are linear in h. In particular, it yields that the inverse is $U_F(\tilde{a},h,\theta,\mathsf{v})^{-1} = \prod_{k=1}^{s}(1-h)U(a_k,\theta,\mathsf{v})^{-1} + hU_{F_{k-1}}(\tilde{a}_{k-1},\theta,\mathsf{v})U_{F_k}(\tilde{a}_k,\theta,\mathsf{v})^{-1}$. This also shows that $U_F(\tilde{a},h,\theta,\mathsf{v})^{-1}$ is polynomial in h. We also define

$$\widehat{\mathsf{U}}_F(\tilde{a},h) = \prod_{k=1}^{s}\left((1-h)\widehat{\mathsf{U}}(a_k) + h\widehat{\mathsf{U}}_{F_k}(\tilde{a}_k)\widehat{\mathsf{U}}_{F_{k-1}}(\tilde{a}_{k-1})^{-1}\right).$$

One can see that the identities $U_F(\tilde{a},h,0,\mathsf{v}) = \mathsf{U}_F(\tilde{a},h)$ and $U_F(\tilde{a},h,\pi/2,\mathsf{v}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{z}))\Lambda(\mathsf{v},\mathsf{Q})\widehat{\mathsf{U}}_F(\tilde{a},h)\Lambda(\mathsf{v},\mathsf{Q})^{-1}$ hold. Furthermore, $U_F(\tilde{a},0,\theta,\mathsf{v}) = U(a,\theta,\mathsf{v})$, $U_F(\tilde{a},1,\theta,\mathsf{v}) = U_F(\tilde{a},\theta,\mathsf{v})$, and $\widehat{\mathsf{U}}_F(\tilde{a},0) = \widehat{\mathsf{U}}(a)$, $\widehat{\mathsf{U}}_F(\tilde{a},1) = \widehat{\mathsf{U}}_F(\tilde{a})$. We define

$$K_F^{\mathrm{ec}}(\tilde{a},h,\theta,\mathsf{v},\mathsf{w}) = U_F(\tilde{a},h,\theta,\mathsf{v})\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})U_F(\tilde{a},h,\theta,\mathsf{w})^{-1}.$$

From the earlier observations, the identities $K_F^{\mathrm{ec}}(\tilde{a},h,0,\mathsf{v},\mathsf{w}) = \Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{B}_F(\tilde{a},h))$ and $K_F^{\mathrm{ec}}(\tilde{a},h,\pi/2,\mathsf{v},\mathsf{w}) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{v}))\Lambda(\mathsf{v}\mathsf{w}^{-1},\mathsf{Q})\mathsf{E}_{\mathbb{Z}}(a(\mathsf{w}))^{-1}$ follow. Furthermore, $K_F^{\mathrm{ec}}(\tilde{a},0,\theta,\mathsf{v},\mathsf{w}) = K^{\mathrm{c}}(a,\theta,\mathsf{v},\mathsf{w})$ and $K_F^{\mathrm{ec}}(\tilde{a},1,\theta,\mathsf{v},\mathsf{w}) = K_F^{\mathrm{c}}(\tilde{a},\theta,\mathsf{v},\mathsf{w})$.

This yields

Proposition 8.10 (⇒ Statement 1.5)**.** *The continuous map*

$$K_F^{\mathrm{e}} : \mathfrak{A}_F \times \mathbb{R} \times \mathrm{S}^1 \to \mathsf{U}(\mathcal{K}_{\mathbb{Z}}(\mathfrak{A})[\mathsf{v}^{-1},\mathsf{v}]^{\mathrm{po}})$$

defined by

$$\tilde{a},h,\theta \mapsto K_F^{\mathrm{e}}(\tilde{a},h,\theta,\mathsf{v}) = K_F^{\mathrm{ec}}(\tilde{a},h,\theta,\mathsf{v},1)\Lambda(\mathsf{v},\mathsf{Q})^{-1}$$

is smooth in θ and polynomial in h. It has the properties

(i) $K_F^{\mathrm{e}}(\tilde{a},0,\theta) = K(a,\theta)$;

(ii) $K_F^{\mathrm{e}}(\tilde{a},1,\theta) = K_F(\tilde{a},\theta)$;

(iii) $K_F^{\mathrm{e}}(\tilde{a},h,0) = \Lambda(\mathsf{v},\mathsf{B}_F(\tilde{a},h)\Lambda(\mathsf{v},\mathsf{Q})^{-1}$;

(iv) $K_F^{\mathrm{e}}(\tilde{a},h,\pi/2) = \mathsf{E}_{\mathbb{Z}}(a(\mathsf{z})a(1)^{-1})$.

In particular, it connects the pullback homotopy $K|_{\mathfrak{A}_F}$ and homotopy K_F through other linearizing homotopies. □

References

[1] M. Atiyah and R. Bott, On the periodicity theorem for complex vector bundles, *Acta Math.* **112** (1964), 229–247.

[2] J. Cuntz, K-theory and C^*-algebras, in: Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math. 1046, (by A. Bak, ed.), *Springer Verlag, Berlin*, 1984, 55–79.

[3] J. Cuntz, Bivariant K-theory and the Weyl algebra, *K-Theory* **35** (2005), 93–137.

[4] J. Cuntz, R. Meyer and J. Rosenberg, Topological and bivariant K-theory, Oberwolfbach Seminars 36., *Birkhäuser Verlag, Basel*, 2007.

[5] Gy. Lakos, Smooth K-theory and locally convex algebras, PhD thesis, Massachusetts Institute of Technology, 2003.

[6] Gy Lakos, On the H-ring structure of infinite Grassmannians, *Method. Funct. Anal. Topology* **16** (2010), 242–258.

[7] R. B. Melrose and F. Rochon, Families index for pseudodifferential operators on manifolds with boundary, *Int. Math. Res. Not.* **22** (2004), 1115–1141.

[8] A. Pressley and G. Segal, Loop groups, Oxford Mathematical Monographs, *Oxford University Press, New York*, 1986.

[9] N. E. Wegge-Olsen, K-theory and C^*-algebras. A friendly approach, Oxford Science Publications, *The Clarendon Press, Oxford University Press, New York*, 1993.

GYULA LAKOS
DEPARTMENT OF GEOMETRY
EÖTVÖS UNIVERSITY
PÁZMÁNY PÉTER S. 1/C
BUDAPEST, H–1117
HUNGARY

E-mail: lakos@cs.elte.hu

(Received March 3, 2009; revised May 24, 2011)