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Spherically symmetric Finsler metrics in R^n

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Abstract. In this paper, we give the general form of spherically symmetric Finsler metrics in R^n and find that many well-known Finsler metrics belong to this class. Then we explicitly express projective metrics of this type. The necessary and sufficient conditions that projective Finsler metrics with spherical symmetry have constant flag curvature are also obtained.

1. Introduction

When studying Finsler geometry, one often encounters the intricacy of calculation which leads to many non-Riemann quantities uncomputable. One natural and important thought is to study those Finsler metrics with very nice symmetry which can make things much easier. In fact, S. Deng and Z. Hou have studied the isometric group of a general Finsler metric and proved that it must be a Lie group [6]. Furthermore, Z. I. Szabó studied n-dimensional Finsler spaces whose full group of isometries have dimension $\frac{1}{2}n(n-1)+1$ and gave a classification theorem [13].

In general relativity, when looking for a solution of the vacuum Einstein field equations describing the gravitational field which is spherically symmetric, one will obtain the Schwarzschild solution in the four dimensional time-space [1]. In the process, the condition of spherical symmetry plays a very important role: it can greatly simplify the computation. Motivated by this idea, we investigate spherically symmetric Finsler metrics in R^n in this paper.

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Similar to the definition in general relativity, a spherically symmetric Finsler metric means that it is invariant under any rotations in R^n. In other words, the vector fields generated by rotations are the Killing fields of the Finsler metric. Therefore we will firstly introduce the Killing field equation in Finsler geometry, which generalizes the Killing field equation in Riemannian case [10].

By solving the equations of Killing fields generated by rotations, we will determine the structure of spherically symmetric Finsler metrics $F(x,y)$ in R^n in Theorem 3.1: F must have the form $F = \phi(|x|, |y|, \langle x, y\rangle)$. Extensive well-known examples including the Bryant metrics [5] belong to this type. Furthermore, spherically symmetric Finsler metrics are not always (α, β) metrics. Thus it is of certain significance to study this type of Finsler metrics.

To characterize projective Finsler metrics in R^n is a very important problem. It relates to the Hilbert's Fourth problem [11]. We will discuss those spherically symmetric Finsler metrics $F(x,y)$ in R^n which are projective and express them explicitly by $F = \int f\big(\frac{v^2}{u^2} - r^2\big)du + g(r)v$ in Theorem 4.2 by using famous Rapcsák's lemma [8]. As we know, a projective Finsler metric is of scalar curvature. It is natural to ask which metrics have constant flag curvature among the projective spherically symmetric Finsler metrics. We will obtain a sufficient and necessary condition in Theorem 4.4, which is two partial differential equations. From these equations, perhaps one can find some new examples of Finsler metrics with constant flag curvature.

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2. The Killing field equation in Finsler geometry

Suppose F is a Finsler metric on an n-dimensional C^∞ manifold M. Let $\Phi : M \to M$ be a diffeomorphism and $\Phi_* : T_xM \to T_{\Phi(x)}M$ be the tangent map at point x. Φ is called smooth isometry if it satisfies

$$F(\Phi(x), \Phi_*(y)) = F(x,y)$$

where $y \in T_xM$.

A vector field X on M is called a Killing field if the 1-parameter group Φ_t generated by X are isometric. As we know in Riemann geometry, there is the following Killing field equation [10]

$$\mathcal{L}_X g = 0.$$

Here $\mathcal{L}_X g$ denotes the Lie derivative of Riemannian metric tensor g on M. In local coordinate $\{x^i, \frac{\partial}{\partial x^i}\}$, above equation can be written as

$$\frac{\partial g_{ij}}{\partial x^p}X^p + g_{pj}\frac{\partial X^p}{\partial x^i} + g_{ip}\frac{\partial X^p}{\partial x^j} = 0$$

where $g = g_{ij}dx^i \otimes dx^j$ and $X = X^i\frac{\partial}{\partial x^i}$.

Actually, in [9], S. F. Rutz has obtained a similar Killing field equation in Finsler geometry. For the sake of completeness, let us state it as the following theorem and prove it:

Theorem 2.1. *Let (M^n, F) be an n-dimensional smooth Finsler manifold. A vector field X is a Killing field on M. Then X satisfies the equation*

$$\frac{\partial g_{ij}}{\partial x^p}X^p + g_{pj}\frac{\partial X^p}{\partial x^i} + g_{ip}\frac{\partial X^p}{\partial x^j} + 2C_{ijp}\frac{\partial X^p}{\partial x^k}y^k = 0$$

where $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is coefficient of fundamental tensor, $C_{ijp} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^p}$ is coefficient of Cartan tensor and $X = X^i\frac{\partial}{\partial x^i}$ under the local coordinate.

Proof. Suppose Φ_t is the 1-parameter group of X. According to the definition of the Killing field in Finsler geometry, we have

$$F(\Phi_t(x), (\Phi_t)_*(y)) = F(x, y).$$

Under the local coordinate that means

$$g_{ij}(\Phi_t(x), (\Phi_t)_*(y))\frac{\partial \Phi_t^i}{\partial x^k}y^k\frac{\partial \Phi_t^j}{\partial x^l}y^l = g_{ij}(x, y)y^iy^j.$$

Notice that $X_x = \frac{d\Phi_x(t)}{dt}|_{t=0}$ and Φ_0 is identity. So taking derivative with respect to t in above equation and set $t = 0$, we get

$$\frac{\partial g_{ij}}{\partial x^p}X^py^iy^j + \frac{\partial g_{ij}}{\partial y^p}\frac{\partial X^p}{\partial x^k}y^ky^iy^j + g_{ij}y^p\frac{\partial X^i}{\partial x^p}y^j + g_{ij}y^i\frac{\partial X^j}{\partial x^k}y^k = 0.$$

It is equivalent to

$$\frac{\partial g_{ij}}{\partial x^p}X^p + \frac{\partial g_{ij}}{\partial y^p}\frac{\partial X^p}{\partial x^k}y^k + g_{pj}\frac{\partial X^p}{\partial x^i} + g_{ip}\frac{\partial X^p}{\partial x^j} = 0.$$

By the definition of Cartan torsion, we obtain the result immediately. □

Remark. In the case of Riemannian metric, the Cartan torsion vanishes. Hence the equation in Theorem 2.1 coincides with $\mathcal{L}_X g = 0$.

Corollary 2.2. *Let (M^n, F) be an n-dimensional smooth Finsler manifold. If a vector field X is a Killing field on M, then it satisfies*

$$\frac{\partial F}{\partial x^i}X^i + \frac{\partial F}{\partial y^i}\frac{\partial X^i}{\partial x^j}y^j = 0.$$

Proof. The conclusion can be proved straight forward by contracting the equation in Theorem 2.1 with y^i and y^j. □

3. The general form of spherically symmetric Finsler metrics in R^n

We denote Ω a convex domain in R^n and F a Finsler metric on Ω. (Ω, F) is called spherically symmetric if orthogonal group $O(n)$ is isometry of (Ω, F). This definition is equivalent to say that (Ω, F) is invariant under all rotations in R^n. Hence there is a natural question: what is the restriction on the metric F if it has spherical symmetry? We have the following theorem:

Theorem 3.1. *Let $F(x,y)$ be a Finsler metric on a convex domain $\Omega \subseteq R^n$. $F(x,y)$ is spherically symmetric if and only if there exists a positive function $\phi(r,u,v)$ s.t.*

$$F(x,y) = \phi(|x|, |y|, \langle x, y\rangle)$$

where $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$, $|y| = \sqrt{(y^1)^2 + \cdots + (y^n)^2}$ *and* $\langle x, y\rangle = x^1y^1 + \cdots + x^ny^n$.

Proof. Suppose $F(x,y)$ is spherically symmetric on $\Omega \subseteq R^n$. Choose $\{e_1, \dots, e_n\}$ is the standard orthonormal base in R^n and denote X_iOX_j the coordinate plane spanned by $\{e_i, e_j\}$. Consider a family rotations θ_t on coordinate plane X_iOX_j:

$$\begin{aligned} &\theta_t(x^1, \dots, x^i, \dots, x^j, \dots, x^n) \\ &= (x^1, \dots, x^i \cos t + x^j \sin t, \dots, -x^i \sin t + x^j \cos t, \dots, x^n). \end{aligned}$$

Obviously θ_t is a 1-parameter group and isometric. So a Killing vector field X generated by θ_t is

$$X = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}.$$

By corollary 2.2, we have the following equation

$$\frac{\partial F}{\partial x^i} x^j - \frac{\partial F}{\partial x^j} x^i + \frac{\partial F}{\partial y^i} y^j - \frac{\partial F}{\partial y^j} y^i = 0. \tag{1}$$

This equation is a first order linear partial differential equation. The characteristic equation is given by

$$\frac{dx^i}{x^j} = -\frac{dx^j}{x^i} = \frac{dy^i}{y^j} = -\frac{dy^j}{y^i}.$$

Thus

$$(x^i)^2 + (x^j)^2 = c_1, \quad (y^i)^2 + (y^j)^2 = c_2, \quad x^iy^i + x^jy^j = c_3$$

are three independent first integrals. Hence the solution of equation (1) is

$$F = \overline{\phi}(x^1, \dots, \widehat{x^i}, \dots, \widehat{x^j}, \dots, x^n, (x^i)^2 + (x^j)^2, (y^i)^2 + (y^j)^2, x^iy^i + x^jy^j).$$

Here $\widehat{x^i}$ means omitting the variable x^i. For i, j are arbitrary numbers from 1 to n, so there are $\frac{n(n-1)}{2}$ Killing field equations like (1). Therefore, F must have the following form

$$\begin{aligned} F(x,y) &= \widetilde{\phi}((x^1)^2 + \cdots + (x^n)^2, (y^1)^2 + \cdots + (x^n)^2, x^1y^1 + \cdots + x^ny^n) \\ &= \phi(|x|, |y|, \langle x, y\rangle). \end{aligned}$$

The converse is obvious by a direct computation. □

Remark. In [9], S. F. Rutz has yielded the form of spherically symmetric Finsler metric in 3 dimension by using the spherical coordinate.

Let $F = \phi(|x|, |y|, \langle x, y\rangle)$ where $\phi(r, u, v)$ is a positive C^∞ function with homogeneous of degree one with respect to variable u and v, let us find the condition for the positivity of $(g_{ij}) := \big(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} qbig)$. It is easy to compute g_{ij}:

$$\begin{aligned} g_{ij} = &\frac{\phi\phi_u}{u}\delta_{ij} + (\phi_v^2 + \phi\phi_{vv})x^ix^j + \left(\frac{\phi_u^2 + \phi\phi_{uu}}{u^2} - \frac{\phi\phi_u}{u^3}\right)y^iy^j \\ &+ \left(\frac{\phi_u\phi_v + \phi\phi_{uv}}{u}\right)(x^iy^j + x^jy^i) \end{aligned}$$

where $u := |y|$. Thus we can obtain [3]

$$\det(g_{ij}) = \left(\frac{\phi}{u}\right)^{n+1}\phi_u^{n-2}\left[\phi_u + \big(|x|^2|y|^2 - \langle x, y\rangle^2\big)\frac{\phi_{vv}}{u}\right].$$

Lemma 3.2. *Suppose a positive C^∞ function $\phi(r, u, v)$ is homogeneous of degree one with respect to u and v. If ϕ satisfies that*

$$\phi_u > 0, \phi_{uu} \geq 0$$

when $u > 0$ and $r \geq 0$, then $F = \phi(|x|, |y|, \langle x, y\rangle)$ is a Finsler metric.

Proof. Since $\phi(r, u, v)$ is homogeneous of degree one, we have

$$\phi_{uu}u + \phi_{uv}v = 0.$$

So

$$\phi_{uu} = -\frac{v}{u}\phi_{uv}.$$

Similarly we have

$$\phi_{uv} = -\frac{v}{u}\phi_{vv}.$$

Thus we obtain that

$$\phi_{uu} = \left(\frac{v}{u}\right)^2\phi_{vv}.$$

Hence the condition $\phi_u > 0, \phi_{uu} \geq 0$ is equivalent to

$$\phi_u > 0, \phi_{vv} \geq 0.$$

By the formula of $\det(g_{ij})$ computed above, one can easily see that the matrix (g_{ij}) is positive. □

In fact, many classical Finsler metrics are spherically symmetric [2], [3].

Example 3.3 (Klein model). *Let $B^n \subset R^n$ be the standard unit ball and let*

$$\alpha(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}, \quad y \in T_xB^n.$$

$\alpha(x,y)$ is a Riemannnian metric on B^n. It is projective and has constant flag curvature $K = -1$.

Example 3.4 (Funk metric). *A Randers metric F is defined on the standard unite ball B^n:*

$$F(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2}.$$

It is also projective and has constant flag curvature $K = -\frac{1}{4}$.

Example 3.5 (Berwald metric). *An (α,β) metric F is also defined on B^n:*

$$F(x,y) := \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}.$$

F is projective and has constant flag curvature $K = 0$.

Example 3.6 (projective spherical model). *Let $S^n \subset R^{n+1}$ be the standard unit sphere. The standard inner product $\langle\,,\rangle$ in R^{n+1} induced a Riemannian metric on S^n: for $x \in S^n$, let*

$$\alpha := |y|, \quad y \in T_xS^n \subset R^{n+1}.$$

Let S^n_+ denote the upper hemisphere and let $\psi_+ : R^n \to S^n_+$ be the projection map defined by

$$\psi_+(x) := \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}\right).$$

The pull-back metric on R^n from S^n_+ by ψ_+ is given by

$$F(x,y) := \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+|x|^2}, \quad y \in T_xR^n$$

$(R^n, F(x,y))$ is projectively flat and has constant flag curvature $K = 1$.

Example 3.7 (Bryant metric)**.** *Denote*

$$A := (\cos(2\alpha)|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2))^2 + (\sin(2\alpha)|y|^2)^2,$$
$$B := \cos(2\alpha)|y|^2 + (|x|^2|y|^2 - \langle x,y\rangle^2),$$
$$C := \sin(2\alpha)\langle x,y\rangle,$$
$$D := |x|^4 + 2\cos(2\alpha)|x|^2 + 1.$$

For an angle α with $0 \leq \alpha < \frac{\pi}{2}$, Bryant metric F is defined by

$$F := \sqrt{\frac{\sqrt{A}+B}{2D} + \left(\frac{C}{D}\right)^2} + \frac{C}{D}$$

on the whole region R^n. As we know it is projective and has constant flag curvature $K = 1$.

From above examples, we can see that spherically symmetric Finsler metrics do not always belong to (α,β) metrics. So it is meaningful to study projective metrics of this type with constant flag curvature [7].

4. Projective spherically symmetric Finsler metrics in R^n

A Finlser metric F in R^n is called projective metric, if its geodesics are straight lines. Since spherically symmetric Finsler metrics have very nice symmetry, when discussing projective metrics of this type in R^n, we can obtain a quite simple result compared with projectively flat (α,β) metrics [12]. Before stating our result, we need an important lemma about projective Finsler metrics.

Lemma 4.1 (Rapcsák [8])**.** *Let $F(x,y)$ be a Finsler metric on an open subset $\mathcal{U} \in R^n$. $F(x,y)$ is projective on $\mathcal{U}$ if and only if it satisfies*

$$F_{x^k y^l} y^k = F_{x^l}.$$

In this case, the projective factor $P(x,y)$ is given by

$$P = \frac{F_{x^k} y^k}{2F}.$$

Theorem 4.2. *Suppose F is a spherically symmetric Finsler metric on a convex domain $\Omega \in R^n$, F is projective if and only if there exist smooth functions $f(t) > 0$ and $g(r)$ s.t.*

$$\phi(r,u,v) = \int f\left(\frac{v^2}{u^2} - r^2\right) du + g(r)v$$

where $F(x,y) = \phi(|x|, |y|, \langle x, y\rangle)$.

Proof. By lemma 4.1, F is projective if and only if F satisfies

$$F_{x^l} = F_{y^l x^k} y^k. \tag{2}$$

If F is spherically symmetric, then there exists $\phi(r,u,v)$ s.t.

$$F(x,y) = \phi(|x|, |y|, \langle x, y\rangle).$$

So

$$F_{x^l} = \phi_r \frac{x^l}{|x|} + \phi_v y^l = \phi_r \frac{x^l}{r} + \phi_v y^l$$

and

$$\begin{aligned}
F_{y^l x^k} y^k &= (\phi_{y^l})_{x^k} y^k = \left(\phi_u \frac{y^l}{|y|} + \phi_v x^l\right)_{x^k} y^k \\
&= \sum_{k=1}^n \left(\phi_{ru} \frac{x^k}{|x|}\frac{y^l}{|y|} + \phi_{uv} y^k \frac{y^l}{|y|} + \phi_{rv}\frac{x^k}{|x|} x^l + \phi_{vv} y^k x^l + \phi_v \delta_k^l\right) y^k \\
&= \phi_{ru} \frac{\langle x, y\rangle}{|x||y|} y^l + \phi_{uv}|y| y^l + \phi_{rv}\frac{\langle x, y\rangle}{|x|} x^l + \phi_{vv}|y|^2 x^l + \phi_v y^l \\
&= \left(\phi_{rv}\frac{v}{r} + \phi_{vv} u^2\right) x^l + \left(\phi_{ru}\frac{v}{ru} + \phi_{uv} u + \phi_v\right) y^l
\end{aligned}$$

where $r := |x|$, $u := |y|$, $v := \langle x, y\rangle$. Thus (2) holds if and only if ϕ satisfies

$$\begin{cases} \phi_{rv}\dfrac{v}{r} + \phi_{vv} u^2 = \dfrac{\phi_r}{r} \\ \phi_{uv} u + \phi_{ru}\dfrac{v}{ru} = 0. \end{cases} \tag{3}$$

In fact, the two equations in (3) are equivalent by only noticing that

$$\phi_{rv} = \frac{\phi_r - u\phi_{ru}}{v}, \quad \phi_{vv} = -\frac{u\phi_{uv}}{v}$$

for ϕ being homogeneous of degree one. Hence F is projective if and only if ϕ satisfies

$$\phi_{uv} u + \phi_{ru}\frac{v}{ru} = 0. \tag{4}$$

This is a first order linear partial differential equation with respect to ϕ_u. The characteristic equation is

$$\frac{dv}{u} = \frac{dr}{\frac{v}{ru}}.$$

So

$$\phi_u = \tilde{f}(v^2 - u^2r^2, u^2) = \tilde{f}\left(\frac{v^2}{u^2} - r^2, 1\right) = f\left(\frac{v^2}{u^2} - r^2\right)$$

is the solution of equation (4). Here it should be pointed out that in above equation, we use the homogeneous property of ϕ_u. Thus there exists a function $c(r, v)$ s.t.

$$\phi = \int f\left(\frac{v^2}{u^2} - r^2\right) du + c(r, v).$$

Again noticing the homogeneity of ϕ, we conclude that

$$c(r, v) = g(r)v.$$

Thus complete the proof. □

As we know, projective Finsler metric is of scalar curvature. So from above theorem, we can find many spherically symmetric Finsler metrics having scalar curvature. Now let us study those projective Finsler metrics which have constant flag curvature among this type in R^n. We need a lemma first.

Lemma 4.3 ([11])**.** *Suppose $F = F(x, y)$ is a projective Finlser metric on a convex domain $\Omega \subseteq R^n$. Then F has constant flag curvature $K = \lambda$ if and only if projective factor P satisfies*

$$P_{x^k} = PP_{y^k} - \lambda FF_{y^k}$$

where $P := \frac{F_{x^m}y^m}{2F}$.

With this lemma, we have the following conclusion.

Theorem 4.4. *Let a spherically symmetric Finsler metric $F = \phi(|x|, |y|, \langle x, y\rangle)$ be projective on a convex domain $\Omega \subseteq R^n$. Then F has constant flag curvature $K = \lambda$ if and only if $\phi(r, u, v)$ satisfies*

$$\begin{cases} 4\lambda r\phi^4\phi_u + r\phi_u Q^2 - 4ru\phi\phi_v Q + 4u\phi^2\phi_r = 0 \\ 4\lambda r\phi^4\phi_v + r\phi_v Q^2 + 2\phi^2 Q_r - 4\phi\phi_r Q = 0 \end{cases} \tag{5}$$

where $Q := \frac{v}{r}\phi_r + u^2\phi_v$.

Proof. From Lemma 4.3, F has constant flag curvature $K=\lambda$ if and only if

$$P_{x^k} = PP_{y^k} - \lambda F F_{y^k} \tag{6}$$

where projective factor $P=\frac{F_{x^m}y^m}{2F}$. Now $F=\phi(r,u,v)$, $r=|x|$, $u=|y|$ and $v=\langle x,y\rangle$. Hence

$$\begin{aligned}
P &= \frac{1}{2\phi}\left(\frac{v}{r}\phi_r + u^2\phi_v\right),\\
P_{x^k} &= -\frac{1}{2\phi^2}\left(\frac{\phi_r}{r}x^k + \phi_v y^k\right)\left(\frac{v}{r}\phi_r + u^2\phi_v\right)\\
&\quad + \frac{1}{2\phi}\left(\phi_{rr}\frac{v}{r^2}x^k + \phi_{rv}\frac{v}{r}y^k + \frac{\phi_r}{r}y^k - \phi_r\frac{v}{r^3}x^k + \phi_{rv}\frac{u^2}{r}x^k + \phi_{vv}u^2y^k\right),\\
PP_{y^k} &= P\left(-\frac{1}{2\phi^2}\right)\left(\frac{\phi_u}{u}y^k + \phi_v x^k\right)\left(\frac{v}{r}\phi_r + u^2\phi_v\right)\\
&\quad + \frac{P}{2\phi}\left(\phi_{ru}\frac{v}{ru}y^k + \phi_{rv}\frac{v}{r}x^k + \frac{\phi_r}{r}x^k + u\phi_{uv}y^k + \phi_{vv}u^2x^k + 2\phi_v y^k\right),\\
\lambda F F_{y^k} &= \lambda\phi\left(\frac{\phi_u}{u}y^k + \phi_v x^k\right).
\end{aligned}$$

Substituting above equations into (6), we can know that (6) holds if and only if

$$\begin{aligned}
&-\frac{\phi_v}{2\phi^2}\left(\frac{v}{r}\phi_r + u^2\phi_v\right) + \frac{1}{2\phi}\left(\phi_{rv}\frac{v}{r} + \frac{\phi_r}{r} + \phi_{vv}u^2\right)\\
&\qquad = -\frac{P}{2\phi^2}\frac{\phi_u}{u}\left(\phi_r\frac{v}{r} + \phi_v u^2\right) + \frac{P}{2\phi}(\phi_{ru}\frac{v}{ru} + u\phi_{uv} + 2\phi_v) - \lambda\phi\frac{\phi_u}{u}
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&-\frac{\phi_r}{2\phi^2 r}\left(\frac{v}{r}\phi_r + u^2\phi_v\right) + \frac{1}{2\phi}\left(\phi_{rr}\frac{v}{r^2} - \phi_r\frac{v}{r^3} + \phi_{rv}\frac{u^2}{r}\right)\\
&\qquad = -\frac{P}{2\phi^2}\phi_v\left(\frac{v}{r}\phi_r + u^2\phi_v\right) + \frac{P}{2\phi}\left(\phi_{rv}\frac{v}{r} + \frac{\phi_r}{r} + \phi_{vv}u^2\right) - \lambda\phi\phi_v.
\end{aligned} \tag{8}$$

Noticing that F is projective, so ϕ satisfies equation (3) in Theorem 4.2

$$\begin{cases}\phi_{rv}\dfrac{v}{r} + \phi_{vv}u^2 = \dfrac{\phi_r}{r}\\[2mm] \phi_{uv}u + \phi_{ru}\dfrac{v}{ru} = 0.\end{cases}$$

Using above two equations and substituting the formula of P, we can simplify equation (7), (8) and obtain the result. □

Remark.

1) It can be verified that the examples in section 3 satisfy our Theorem 4.2 and Theorem 4.4 by Maple.
2) Very recently, the author solved the equations (5) in Theorem 4.4 by using a new parameter and gave the classification Theorems [14].

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