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Weighted composition operators on Dirichlet-type spaces and related Q_p spaces

By YUAN CHENG (Tianjin), SANJAY KUMAR (Jammu) and ZE-HUA ZHOU (Tianjin)

Abstract. We generalize Gallardo-Gutiérrez and Partington's results on BMOA to the Dirichlet-type spaces with similar strategy. That is, we use the generalized Nevanlinna counting function associated to the weight function to characterize the boundedness and compactness of weighted composition operators on the Dirichlet-type spaces.

1. Introduction

Let D denote the open unit disk of the complex plane $\mathbb{C}$. For $p \in (0, \infty)$, the Dirichlet-type space $\mathcal{D}_p$ is the Hilbert space of holomorphic functions on D for which the norm

$$\|f\|_{\mathcal{D}_p} = |f(0)| + \left(\int_D |f'(z)|^2 (1 - |z|^2)^p dA(z) \right)^{1/2} < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ is the normalized Lebesgue measure on the unit disk. The inner product of f and g in $\mathcal{D}_p$ is given by

$$\langle f, g \rangle = |f(0)g(0)| + \left(\int_D f'(z)\overline{g'(z)} (1 - |z|^2)^p dA(z) \right)^{1/2} < \infty.$$

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$\mathcal{D}_p$ is a reproducing kernel Hilbert space with the kernel function $K_w(z) = K(z,w) = (1-\overline{w}z)^{-p}$. The space $\mathcal{D}_p$ has been extensively studied in number of papers, e.g., [13], [14], [17], [18]. It is well known that when $p=1$, $\mathcal{D}_p$ is the classic Hardy space H^2 and when $p>1$, $\mathcal{D}_p$ is the weighted Bergman space with weight $(1-|z|^2)^{p-2}$. In particular, $\mathcal{D}_2$ is the classic Bergman space L^2_a. See [17], for example.

The space of Möbius bounded functions in $\mathcal{D}_p$ is written as Q_p. That is, Q_p is the Banach space of function $f \in \mathcal{D}_p$ with the norm

$$\|f\|_{Q_p} = |f(0)| + \sup_{w\in D} \|f\circ\varphi_w - f\|_{\mathcal{D}_p} < \infty,$$

where $\varphi_w(z) = (w-z)/(1-\overline{w}z)$. For different $p \in (0,\infty)$, $Q_{p_1} \subset Q_{p_2}$ when $0 < p_1 < p_2 \le 1$. In particular, $Q_1 = BMOA$, the bounded mean oscillation space of analytic functions; and when $p>1$, $Q_p = \mathcal{B}$, the Bloch space on D. See [21], [22], and the reference therein.

For φ, a non-constant holomorphic map of the unit disk into itself, the composition operator C_φ with the symbol φ is defined by

$$C_\varphi(f) = f\circ\varphi,$$

where f is a holomorphic function in D. It is well known that C_φ on $\mathcal{D}_p$ is always bounded when $p \ge 1$ and may be unbounded when $0<p<1$. For u holomorphic on D, the weighted composition operator uC_φ is defined by $uC_\varphi(f) = u\cdot f\circ\varphi$. Apparently, if $u=1$, the weighted composition operator uC_φ becomes C_φ; and if $\varphi(z)=z$ for $z\in D$, uC_φ becomes the multiplier M_u.

It is a long story to characterize the properties of (weighted) composition operator on several function spaces. The boundedness, compactness, Schatten class property are usually related to the pullback Carleson measure, Nevanlinna counting function and generalized Berezin transform, see [3], [4], [12], [15], [16].

Recently, Gallardo-Gutiérrez and Partington characterized the boundedness and compactness of weighted composition operators on Hardy and weighted Bergman spaces in [6] and [7]. Their characterization involved a condition related to a Nevanlinna counting function associated to the symbols φ and the weight u. Their method is based on the analyzing the weighted composition operator acting the the normalized kernel function. It is shown that the spaces BMOA and VMOA (resp. the Bloch space and little Bloch space) play key roles in the boundedness and compactness of weighted composition operators on the Hardy spaces (reps. the weighted Bergman spaces).

This manuscript is a generalization of Gallardo-Gutiérrez and Partington's work [6]. Consider the Hardy spaces and weighted Bergman spaces as the Dirichlet-type spaces $\mathcal{D}_p$, [6] is about the case $p \geq 1$. We characterize the case for all $p \in (0, \infty)$. The idea is the same as that of [6]. We show that the Q_p plays the role on the boundedness of weighted composition operators on $\mathcal{D}_p$ as $BMOA$ related to the Hardy space.

Notation: Throughout the paper, we will denote $a \approx b$ whenever there exist two positive universal constants c and C, such that $cb \leq a \leq Cb$. Further, for the sake of simplicity, C will always denote an independent constant, which can be different from one display to another.

2. The main result

Let T denote the boundary of D. Recall that a Carleson disk in D centered at $\zeta \in T$ of radius $r \in (0,1)$ is given by $D(\zeta, r) = \{z \in D : |z - \zeta| < r\}$. It is easy to check that $A(D(\zeta, r)) \approx r^2$. For $p \in (0, \infty)$, a positive Borel measure μ on D is called a *p-Carleson measure*, if

$$\|\mu\|_p = \sup_{\zeta \in T} \frac{\mu(D(\zeta, r))}{r^p} < \infty.$$

When $p = 1$, we get the standard definition of the original Carleson measure.

The main result of this manuscript is the following theorem.

Theorem 1. *Let φ be an analytic self-map of D. For $p \in (0, \infty)$, suppose $u \in Q_p$. Then uC_φ is bounded on $\mathcal{D}_p$ if and only if*

$$\sup_{\zeta \in T} \int_{D(\zeta, r)} \left(\sum_{\varphi(z) = v} |u(z)|^2 (1 - |z|)^p \right) dA(v) = O(r^{2+p}). \tag{1}$$

The case for $p \geq 1$, has been proved in [6], so we focus on the case for $p \in (0, 1)$. To give the proof we need some preliminary results. The following lemmas are quoted from [21] or [2].

Lemma 1. *Let $p \in (0, \infty)$ and let μ be a positive Borel measure on D. Then μ is a p-Carleson measure if and only if*

$$\sup_{w \in D} \int_D \left(\frac{1 - |w|}{|1 - z\overline{w}|^2} \right)^p d\mu(z) < \infty.$$

Lemma 2. *Let $p \in (0, \infty)$ and f holomorphic on D with $d\mu_{f,p}(z) = |f'(z)|^2 \times (1 - |z|)^p dA(z)$. Then $f \in Q_p$ if and only if $\mu_{f,p}$ is a p-Carleson measure.*

PROOF OF THEOREM 1. We only consider $p \in (0,1)$. Firstly fix a Carleson disk $D(\zeta_0, r_0)$ in D, centered at $\zeta_0 \in T$ of radius r_0. Consider $w_0 = (1-r_0)\zeta_0$. Then, if k_{w_0} denotes the normalized reproducing kernel in $\mathcal{D}_p$ at $w_0 \in D$, that is,

$$k_{w_0}(z) = \frac{K_{w_0}(z)}{\|K_{w_0}\|_{\mathcal{D}_p}} = \frac{(1-|w_0|^2)^{p/2}}{(1-\overline{w_0}z)^p}.$$

For all $w \in D$, it is well known that $\{k_w\}$ spans a dense subset of $\mathcal{D}_p$. Consider uC_φ acting on $k_{w_0}(z)$, we have

$$\|uC_\varphi(k_{w_0})\|^2_{\mathcal{D}_p} = |u(0)|^2|k_{w_0}(\varphi(0))|^2 + \int_D |(u(k_{w_0} \circ \varphi))'(z)|^2(1-|z|^2)^p dA(z).$$

It follows that

$$\begin{aligned} |(u(k_{w_0} \circ \varphi))'(z)|^2 &= |u'(z)k_{w_0}(\varphi(z)) + u(z)k'_{w_0}(\varphi(z))\varphi'(z)|^2 \\ &= |u'(z)k_{w_0}(\varphi(z))|^2 + |u(z)k'_{w_0}(\varphi(z))\varphi'(z)|^2 \\ &\quad + 2\Re u'(z)k_{w_0}(\varphi(z))\overline{u(z)k'_{w_0}(\varphi(z))\varphi'(z)} = T_1 + T_2 + T_3. \end{aligned}$$

If we write $I_j = \int_D T_j(1-|z|^2)^p dA(z)$ for $j = 1,2,3$, the Cauchy–Schwarz inequality implies that $I_3^2 \le 4I_1I_2$. We have

$$\begin{aligned} I_1 &= \int_D |u'(z)k_{w_0}(\varphi(z))|^2(1-|z|^2)^p dA(z) \\ &= \int_D \left|\frac{(1-|w_0|^2)^{p/2}}{(1-\overline{w_0}\varphi(z))^p}\right|^2 |u'(z)|^2(1-|z|^2)^p dA(z) \end{aligned}$$

and the Area Formula of change variables gives that

$$\begin{aligned} I_2 &= \int_D |u(z)k'_{w_0}(\varphi(z))\varphi'(z)|^2(1-|z|^2)^p dA(z) \\ &= \int_D |k'_{w_0}(v)|^2\left(\sum_{\varphi(z)=v}|u(z)|^2(1-|z|^2)^p\right)dA(v). \end{aligned}$$

Now, since $u \in Q_p$, it follows from Lemma 2 that $|u'(z)|^2(1-|z|^2)^p dA(z)$ is a p-Carleson measure. So

$$I_1 \le C\int_D \left|\frac{(1-|w_0|^2)^{p/2}}{(1-\overline{w_0}\varphi(z))^p}\right|^2 dA(z) = C\|C_\varphi(k_{w_0})\|^2_{L^2_a}.$$

Because $k_{w_0} \in \mathcal{D}_p \subset L^2_a$ is bounded and C_φ is bounded on L^2_a, we conclude that I_1 is bounded.

If further (1) holds, denote

$$\Delta_n(w_0) = \{v \in D : 2^{n-1} r_0 \leq |\zeta_0 - v| < 2^n r_0\}$$

for $n \geq 1$ and denote $\Delta_0(w_0) = D(\zeta_0, r_0)$. Notice that for any $v \in \Delta_n(w_0)$ one has $|1 - \overline{w}v| \approx 2^n r_0$. We can get

$$\begin{aligned} I_2 &= \int_D |k'_{w_0}(v)|^2 \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \\ &= \int_{\cup \Delta_n(w_0)} |k'_{w_0}(v)|^2 \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \\ &= \sum_{n=0}^{\infty} \int_{\Delta_n(w_0)} |k'_{w_0}(v)|^2 \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \\ &\approx \sum_{n=0}^{\infty} \frac{(1-|w|^2)^p}{(2^n r_0)^{2+2p}} \int_{\Delta_n(w_0)} \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v). \end{aligned}$$

Since $1 - |w|^2 \approx r_0$, $\Delta_n(w_0) \subset D(\zeta_0, 2^n r_0)$ and $A(D(\zeta_0, 2^n r_0)) \approx (2^n r_0)^2$. Combine these to (1) we have that

$$\int_{\Delta_n(w_0)} \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \leq C (2^n r_0)^{2+p},$$

and thus

$$I_2 \leq C \sum_{n=0}^{\infty} \frac{1}{2^{np}}.$$

The above series converges when $p > 0$. Since k_w spans a dense set of $\mathcal{D}_p$, we conclude that if (1) holds, then uC_φ is bounded.

Now suppose uC_φ is bounded, then I_1 and $I_1 + I_2$ are bounded. Therefore,

$$I_2 = \int_D |k'_{w_0}(v)|^2 \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v).$$

is bounded. So

$$\begin{aligned} &\int_{D(\zeta_0, r_0)} |k'_{w_0}(v)|^2 \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \\ = &\int_{D(\zeta_0, r_0)} \frac{(1-|w_0|^2)^p}{(1-\overline{w_0}v)^{2+2p}} \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|^2)^p \right) dA(v) \leq C. \end{aligned}$$

For $v \in D(\zeta_0, r_0)$, using the fact $|1 - \overline{w_0}v| \approx 1 - |w|^2 \approx r_0$ again, we have

$$\sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|)^p \right) dA(v) = O(r^{2+p}).$$

This means that we can get (1) from the boundedness of uC_φ. □

Let $u = 1$, we have the following corollary on the boundedness of composition operators on $\mathcal{D}_p$

Corollary 1. *Let φ be an analytic self-map of D and $p \in (0, \infty)$. Then C_φ is bounded on $\mathcal{D}_p$ if and only if*

$$\sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} (1-|z|)^p \right) dA(v) = O(r^{2+p}).$$

If we let $\varphi(z) = z$ for all $z \in D$, then the following corollary follows immediately.

Corollary 2. *Let $u \in \mathcal{D}_p$. For $p \in (0, \infty)$, consider the following three properties that u may possess:*

(a) *$u \in Q_p$;*

(b) *M_u is bounded on $\mathcal{D}_p$;*

(c) $$\sup_{\zeta \in T} \int_{D(\zeta,r)} |u(z)|^2 (1-|z|)^p dA(z) = O(r^{2+p}).$$

Then any two of these properties implies the third.

Proof. Theorem 1 has shown that (b) and (c) are equivalent under the assumption (a). We need to prove that (b) and (c) imply (a). Indeed, according to [17], u is a multiplier on $\mathcal{D}_p$ if and only if u is bounded and $|u'(z)|^2(1-|z|)^p dA(z)$ is a p-Carleson measure. It follows from Lemma 2 that $u \in Q_p$. This completes the proof. □

If $Q_{p,0}$ stands for the space of all functions $f \in Q_p$ with

$$\lim_{|w| \to 1} \int_D |f'(z)|^2 (1 - |\varphi_w(z)|^2)^p dA(z) = 0,$$

then $Q_{p,0}$ is a closed subspace of Q_p. Since k_w converges to 0 weakly in $\mathcal{D}_p$, we have the little-o version of Theorem 1.

Theorem 2. *Let φ be an analytic self-map of D. For $p \in (0, \infty)$, suppose $u \in Q_{p,0}$. Then uC_φ is compact on $\mathcal{D}_p$ if and only if*

$$r^{-2} \sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} |u(z)|^2 (1-|z|)^p \right) dA(v) = o(r^p) \quad \text{as } r \to 0.$$

Corollary 3. *Let φ be an analytic self-map of D and $p \in (0, \infty)$. Then C_φ is compact on $\mathcal{D}_p$ if and only if*

$$r^{-2} \sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} (1-|z|)^p \right) dA(v) = o(r^p) \quad \text{as } r \to 0.$$

Corollary 4. *Let $u \in \mathcal{D}_p$. For $p \in (0, \infty)$, consider the following three properties that u may possess:*

(a) *$u \in Q_{p,0}$;*

(b) *M_u is compact on $\mathcal{D}_p$;*

(c) $$r^{-2} \sup_{\zeta \in T} \int_{D(\zeta,r)} |u(z)|^2 (1-|z|)^p dA(z) = o(r^p) \quad \text{as } r \to 0.$$

Then any two of these properties implies the third.

3. Weighted composition operators on $\mathcal{D}_p^q$ spaces

For $0 < q < \infty$ and $-1 < p < \infty$, the spaces of Dirichlet type $\mathcal{D}_p^q$ consist of those functions f holomorphic on D such that

$$\|f\|_{\mathcal{D}_p^q} = \left(|f(0)|^q + \int_{\mathbf{D}} |f'(z)|^q (1-|z|^2)^p dA(z) \right)^{1/q} < \infty.$$

Obviously, $\mathcal{D}_p^q$ is a generalization of $\mathcal{D}_p$ and in particular, when $q = 2$, $\mathcal{D}_p^q$ is $\mathcal{D}_p$. For more details about $\mathcal{D}_p^q$, see, [1], [19], [20]. Further, $\mathcal{D}_p^{q_1} \subset \mathcal{D}_p^{q_2}$, if $1 \le q_2 < q_1$. For detail about Dirichlet type spaces one can refer to [8], [9], [10] and the references therein.

In this section we characterize boundedness and compactness of $\mathcal{D}_p^q$ by taking different values of p and q.

Theorem 3. *Let φ be an analytic self-map of D. For $p \in (0, \infty)$, suppose $u \in \mathcal{D}_{p-1}^p$ with $|u'(z)|^p (1-|z|)^{p-1} dA(z)$ is a Carleson measure. Then uC_φ is bounded on $\mathcal{D}_p$ if and only if*

$$\sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} |u(z)|^p (1-|z|)^{p-1} \right) dA(v) = O(r^{2+p}). \tag{2}$$

Theorem 4. *Let φ be an analytic self-map of D. For $0 < q < \infty$, $q < p+2$ and $p > -1$, let $u \in \mathcal{D}_p^q$ with $|u'(z)|^q(1-|z|)^p dA(z)$ is a Carleson measure. Then uC_φ is bounded operator on $\mathcal{D}_p^q$ if and only if*

$$\sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} |u(z)|^q (1-|z|)^p \right) dA(v) = O(r^{2+p}). \tag{3}$$

Theorem 5. *Let φ be an analytic self-map of D. For $0 < q < \infty, q = p+2$ and $p > -1$, let $u \in \mathcal{D}_p^q$ with $|u'(z)|^q(1-|z|^2)^p dA(z)$ is a Carleson measure. Then uC_φ is bounded operator on $\mathcal{D}_p^q$ if and only if*

$$\sup_{\zeta \in T} \int_{D(\zeta,r)} \left(\sum_{\varphi(z)=v} |u(z)|^q (1-|z|)^p \right) dA(v) = O(r^{2+p} \log \frac{1}{r^2}). \tag{4}$$

The proof of Theorem 3, Theorem 4 and Theorem 5 follows on similar lines as proof of Theorem 1, with the following minor modifications:

For different values of p and q , we have the different test functions. If $0 < q < \infty, p > -1$:

- For $q < p+2$ with $q \neq p+1$, the test function is

$$f_z(w) = \frac{(1-|z|^2)^{\frac{p+2-q}{q}}}{(1-\bar{z}w)^{\frac{2p+4-2q}{q}}}, \quad |z| < 1.$$

- For $q = p+1$, the test function is

$$f_z(w) = \frac{(1-|z|^2)^{\frac{1}{q}}}{(1-\bar{z}w)^{\frac{2}{q}}}, \quad |z| < 1.$$

- If $q = p+2$, then

$$f_z(w) = \frac{\log \frac{1}{1-\bar{z}w}}{(\log \frac{1}{1-|z|^2})^{\frac{1}{q}}}, \quad |z| < 1.$$

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YUAN CHENG
INSTITUTE OF MATHEMATICS
SCHOOL OF SCIENCE
TIANJIN UNIVERSITY OF
TECHNOLOGY AND EDUCATION
TIANJIN, 300222
P.R. CHINA

E-mail: yuancheng1984@163.com

SANJAY KUMAR
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JAMMU
JAMMU 180 006
INDIA

E-mail: sanjaykmath@gmail.com

ZE-HUA ZHOU
DEPARTMENT OF MATHEMATICS
TIANJIN UNIVERSITY
TIANJIN 300072
P.R. CHINA

E-mail: zehuazhou2003@yahoo.com.cn

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