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Semi-invariant submanifolds of $\mathcal{K}$-manifolds

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Abstract. We are concerned with $\mathcal{K}$-manifolds which are a natural generalization of metric quasi-Sasakian manifolds. They are Riemannian manifolds with a compatible f-structure which admits a parallelizable kernel, have closed Sasaki 2-form and verify a certain normality condition. We study semi-invariant submanifolds of a $\mathcal{K}$-manifold and investigate the integrability of the various distributions involved. We also study the normality of semi-invariant submanifolds and present a significant example.

1. Introduction

We consider a Riemannian manifold $\widetilde{M}$ of dimension $2n+s$ equipped with an f-structure φ of rank $2n$ with parallelizable kernel which is compatible with the Riemannian metric. These manifolds are known as *f.pk-manifolds* or *globally framed f-manifolds* (cf. [16], [17]) and naturally generalize almost contact metric manifolds. When certain further conditions are satisfied we obtain more specific structures that D. E. Blair in [5] calls $\mathcal{K}$- and $\mathcal{S}$-structures that naturally generalize quasi-Sasakian and Sasakian structures (e.g. cf. [5], [13], [12]).

There are many examples of such structures, (cf. [5], [15]), even of even dimensional manifolds which are never Kähler but which admit $\mathcal{S}$-structures; in [15] an $\mathcal{S}$-structure on the 4-dimensional manifold $U(2)$ is constructed.
The study of semi-invariant submanifolds was started by A. Bejancu in [1] for the Kählerian case and then intensively continued by several geometers (cf. e.g. [2], [3], [4], [21]) in both the Hermitian and the Sasakian case. Generalizations to the case of $\mathcal{S}$-manifolds can be found in literature (cf. eg. [8], [19]). C. Calin

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(cf. [9], [10]) investigated the case of semi-invariant submanifolds of quasi-Sasakian manifolds. The present paper generalizes this case: in fact, it deals with semi-invariant submanifolds of the $\mathcal{K}$-manifolds. It is organized in the following way. Section 2 recalls the definitions and results that will be used in the paper. In Section 3 we generalize in a natural way the notion of semi-invariant submanifold of a $\mathcal{K}$-manifolds, exhibit a pertinent example and study the integrability of the distributions involved in this structure: the invariant the anti-invariant and their direct sums with $\ker\varphi$. Finally, in Section 4 we present the concept of normality for a semi-invariant submanifold and give two characterizations.
All manifolds and distributions considered are smooth i.e. of the class C^∞; we denote by $\Gamma(-)$ the set of all sections of the corresponding bundle.

2. $\mathcal{K}$- and $\mathcal{S}$-manifolds

Let $\widetilde{M}$ be a $(2n+s)$-dimensional manifold equipped with an f-structure φ, vector fields $\xi_1,\ldots,\xi_s$ and 1-forms $\eta^1,\ldots,\eta^s$ such that for all $i,j\in\{1,\ldots,s\}$, $\varphi(\xi_i)=0$, $\eta^i\circ\varphi=0$, $\eta^i(\xi_j)=\delta^i_j$ and $\varphi^2=-\mathrm{Id}+\sum_{j=1}^s\eta^j\otimes\xi_j$. The set $(\widetilde{M},\varphi,\xi_i,\eta^j)$, $i,j\in\{1,\ldots,s\}$, is called an *f-manifold with parallelizable kernel* (shortly: *$f.pk$-manifold*). If g is a Riemannian metric compatible with the structure, that is satisfies $g(\varphi X,\varphi Y)=g(X,Y)-\sum_{i=1}^s\eta^i(X)\eta^i(Y)$, for any $X,Y\in\Gamma(TM)$, the set $(\widetilde{M},\varphi,\xi_i,\eta^j,g)$, $i,j\in\{1,\ldots,s\}$, is called a metric *$f.pk$-manifold*. The distribution $\mathcal{D}=\Im\varphi$ is clearly orthogonal to $\ker\varphi=\langle\xi_1,\ldots,\xi_s\rangle$. With a metric $f.pk$-manifold there is naturally associated the Sasaki 2-form $F:=g(-,\varphi-)$ and the tensor N of type $(1,2)$ such that $N:=[\varphi,\varphi]+2\sum_{i=1}^s d\eta^i\otimes\xi_i$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. When $N=0$ we say that $\widetilde{M}$ is *normal*. Moreover, if the $f.pk$-manifold $\widetilde{M}$ is normal and has closed Sasaki 2-form we say that it is a $\mathcal{K}$-manifold (cf. [5]). Clearly in the case $s=1$ we get a quasi-Sasakian manifold. If moreover $d\eta^1=\cdots=d\eta^s=F$ then the $\mathcal{K}$-manifold is called an $\mathcal{S}$-manifold and for $s=1$ we have a Sasakian manifold.
S. Kanemaki obtained in [18] an important characterization of the quasi-Sasakian manifolds. In [13], the authors proved the following generalization of Kanemaki's result.

Theorem 2.1 ([13]). *Let $(\widetilde{M},\varphi,\xi_i,\eta^j,g)$, $i,j\in\{1,\ldots,s\}$, be an f.pk-manifold. Then it is a $\mathcal{K}$-manifold if and only if*

a) $\mathcal{L}_{\xi_i}\eta^j=0$ *for all* $i,j\in\{1,\ldots,s\}$
b) *there exists a family* $A_1,\ldots,A_s$ *of tensor fields of type* $(1,1)$ *such that*

(1) $(\nabla_X\varphi)Y = \sum_{i=1}^{s}\{g(A_iX,Y)\xi_i - \eta^i(Y)A_iX\}$

(2) $A_i \circ \varphi = \varphi \circ A_i$

(3) $g(A_iX,Y) = g(X,A_iY)$.

Remark 2.1. In the proof of this theorem one meets the family of tensor fields $\underline{A}_i = \varphi \circ \nabla\xi_i$, $i \in \{1,\dots,s\}$, verifying b) of Theorem 2.1. Moreover, the family $\bar{A}_i = \underline{A}_i + \eta^i \otimes \xi_i$, $i \in \{1,\dots,s\}$ is called *the family of indicators* and satisfy b) of Theorem 2.1 and $\bar{A}_i\xi_j = \delta_{ij}\xi_j$ (cf. [13]).

Remark 2.2. It is well known that on an $\mathcal{S}$-manifold $(\widetilde{M},\varphi,\xi_i,\eta^i,g)$, $i,j \in \{1,\dots,s\}$, the following identity holds (cf. [7])

$$(\nabla_X\varphi)Y = g(\varphi X,\varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X). \tag{2.1}$$

On the other hand in [14] it is proven that the validity on an $f.pk$-manifold $\widetilde{M}$ of (2.1) together with $\mathcal{L}_{\xi_i}\eta^j = 0$, $i,j \in \{i,\dots,s\}$ and $\xi_1,\dots,\xi_s$ Killing, implies that $(\widetilde{M},\varphi,\xi_i,\eta^i,g)$, $i,j \in \{1,\dots,s\}$, is an $\mathcal{S}$-manifold. Then we can conclude that on an $\mathcal{S}$-manifold a family of $(1,1)$-tensor fields verifying b) of Theorem 2.1 is given by $A_1 = \cdots = A_s = -\varphi^2$.

In the sequel we will denote by $A_1,\dots,A_s$ a family of $(1,1)$-tensor fields verifying b) of Theorem 2.1.

Taking ξ_k in place of Y in b) 1. of Theorem 2.1 and applying φ to both the sides for each $k \in \{1,\dots,s\}$, $X \in \Gamma(T\widetilde{M})$ we get

$$\widetilde{\nabla}_X\xi_k = -\varphi(A_kX) + \sum_{i=1}^{s}\eta^i(\widetilde{\nabla}_X\xi_k)\xi_i. \tag{2.2}$$

Then we again apply φ to both sides of the last identity and get

$$A_kX = \varphi(\widetilde{\nabla}_X\xi_k) + \sum_{i=1}^{s}\eta^i(A_kX)\xi_i. \tag{2.3}$$

On the other hand, taking in (2.2) ξ_j, $j \in \{1,\dots,s\}$, in place of X and using $\varphi \circ A_k = A_k \circ \varphi$ we get

$$\widetilde{\nabla}_{\xi_j}\xi_k = \sum_{i=1}^{s}\eta^i(\widetilde{\nabla}_{\xi_j}\xi_k)\xi_i, \tag{2.4}$$

that is

$$\widetilde{\nabla}_{\xi_j}\xi_k \in \langle\xi_1,\dots,\xi_k\rangle. \tag{2.5}$$

Then by (2.3) we have

$$A_k\xi_j = \sum_{i=1}^{s} \eta^i(A_k\xi_j)\xi_i, \tag{2.6}$$

that is also $A_k\xi_j \in \ker\varphi$.

Lemma 2.1. *Let* $(\widetilde{M}, \varphi, \xi_i, \eta^j, g)$, $i, j \in \{1, \ldots, s\}$, *be a* $\mathcal{K}$*-manifold. Then for each* $i \in \{1, \ldots, s\}$ *we have*

$$\widetilde{\nabla}_{\xi_i}\varphi = 0 \tag{2.7}$$

Proof. Using identity b)1. of Theorem 2.1 and (2.6) we get

$$(\widetilde{\nabla}_{\xi_i}\varphi)X = \sum_{j,k=1}^{s} \eta^i(X)\{\eta^j(A_k\xi_i) - \eta^k(A_j\xi_i)\}\xi_k. \tag{2.8}$$

If in particular we write (2.8) using the indicators $\bar{A}_i$, $i \in \{1, \ldots, s\}$, since $\bar{A}_i\xi_j = \delta_{ij}\xi_j$ (cf. Remark 2.1), we obtain that $\widetilde{\nabla}_{\xi_i}\varphi = 0$. □

3. Semi-invariant submanifolds of $\mathcal{K}$-manifolds

Definition 3.1. Let $(\widetilde{M}, \varphi, \xi_i, \eta^j, g)$, $i, j \in \{1, \ldots, s\}$ be a $\mathcal{K}$-manifold and M be a submanifold of $\widetilde{M}$. We say that M is a *semi-invariant submanifold* of $\widetilde{M}$ if there exist two distributions D and $D^\perp$ on M such that the following conditions are verified

a) $TM = D \oplus D^\perp \oplus \langle \xi_1, \ldots, \xi_s \rangle$

b) $\varphi(D) \subset D$

c) $\varphi(D^\perp) \subset TM^\perp$

where $TM^\perp$ is the bundle normal to M. D is called the *invariant distribution*, $D^\perp$ the *anti-invariant distribution.* The semi-invariant submanifold is said to be *proper* if both D and $D^\perp$ are non-zero distributions.

From the definition it follows that the distributions D and $D^\perp$ are orthogonal. Certainly D has even dimension as φ is an almost complex structure on it. If $D = \{0\}$ then M is an anti-invariant submanifold of $\widetilde{M}$, i.e. for each $x \in M$ $\varphi(T_xM) \subset T_xM^\perp$; if $D^\perp = \{0\}$ then M is an invariant submanifold of $\widetilde{M}$, i.e. for each $x \in M$ $\varphi(T_xM) \subset T_xM$.

Any vector field X tangent to the semi-invariant submanifold M we can write as

$$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i, \quad \text{where} \quad PX \in \Gamma(D),\ QX \in \Gamma(D^\perp)$$

We give now an example based on the Lie theory. For more details about Lie groups and subgroups see, for example, [20].

Example 3.1. Let us consider a nilpotent Lie algebra $\mathfrak{n}$, and let N be the simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}$. Then if $\mathfrak{n}$ has rational coefficients, the Lie group N admits a cocompact subgroup Γ - the quotient space $\Gamma/N = M(N,\Gamma)$ is a compact manifold.

Consider the following nilpotent Lie algebra n_8 with the basis

$$\{Z_0, Z_1, X_1, X_2, X_3, Y_1, Y_2, Y_3\}$$

and the bracket

$$[X_i, Y_i] = a_i Z_0 + b_i Z_1$$

where the numbers $a_1, a_2, a_3, b_1, b_2, b_3$ are rational and not zero, and the other brackets are zero.

Define the linear transformation $\varphi : n_8 \to n_8$ by the formula

$$\varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i \quad \varphi(Z_j) = 0.$$

The total space of the simply connected Lie group N_8 admits a left invariant Riemannian metric g for which the left-invariant vector fields

$$X_1^*, X_2^*, X_3^*, Y_1^*, Y_2^*, Y_3^*, Z_0^*, Z_1^*$$

are orthonormal, i.e.

$$g(A^*, B^*) = g_0(A, B)$$

for any $A, B \in n_8$ where g_0 is a scalar product on n_8. As the vectors Z_0 and Z_1 commute with all vectors of n_8, so $[Z_0^*, A^*] = [Z_1^*, A^*] = 0$ for any $A \in n_8$. It also means that the vector fields Z_0^* and Z_1^* are Killing vector fields of the Riemannian manifold (N_8, g).

Let us define an f.pk-structure on N_8 for s=2,

$$(N_8, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$$

where $\xi_1 = Z_0^*, \xi_2 = Z_1^*, \eta^1 = g(Z_0^*, .), \eta^2 = g(Z_1^*, .), \varphi(A^*) = \varphi(A)^*$.

It is easy to verify that this structure is normal. Using the structure equations of the Lie algebra n_8 we get that it is a $\mathcal{K}$-manifold.

First, notice that for an invariant k-form η

$$d\eta(A_1^*, \ldots, A_{k+1}^*) = \Sigma_{i<j}(-1)^{i+j}\eta_e([A_i, A_j], A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j \ldots A_{k+1}).$$

Therefore simple calculations show that our manifold is a $\mathcal{K}$-manifold. Moreover,

$$d\eta^1 = d\eta^2$$

iff $a_i = b_i$ for i=1,2,3.

This f.pk structure descends to the compact manifold $M(N_8, \Gamma)$ - we denote the corresponding tensors on this manifold by the same letters.

The vectors $\{Z_0, Z_1, X_1, X_2, Y_1\}$ define a 5-dimensional subalgebra of n_8, and the corresponding simply connected Lie group N_5 is a closed Lie subgroup of N_8, in particular a closed submanifold. Take the distribution D spanned by X_1^*, Y_1^* over N_5. Its orthogonal complement in TN_5 is spanned by X_2. Thus we have constructed a proper semi-invariant submanifold of N_8. The whole setup descends to the compact manifold $M(N_8, \Gamma)$, and the manifold $\Gamma \cap N_5/N_5$ is its proper semi-invariant submanifold. However, it needn't be a closed submanifold.

We recall that a $\mathcal{D}$-homothetic deformation on $\widetilde{M}$ of constant $a > 0$ is a change of the structure in the following way (cf. [11]):

$$\widetilde{\varphi} = \varphi, \quad \widetilde{\xi}_i = \frac{1}{a}\xi_i, \quad \widetilde{\eta}^i = a\eta, \quad \widetilde{g} = ag + a(a-1)\sum_{i=1}^{s} \eta^i \otimes \eta^i.$$

It is easy to see that $(\widetilde{\varphi}, \widetilde{\xi}_i, \widetilde{\eta}^j, \widetilde{g})$, $i, j \in \{1, \dots, s\}$ is a $\mathcal{K}$-structure on $\widetilde{M}$. Moreover, if $\widetilde{M}$ carries an $\mathcal{S}$-structure, then $(\widetilde{\varphi}, \widetilde{\xi}_i, \widetilde{\eta}^j, \widetilde{g})$, $i, j \in \{1, \dots, s\}$ is an $\mathcal{S}$-structure on $\widetilde{M}$.

Proposition 3.1. *Semi-invariant submanifolds are invariant under $\mathcal{D}$-homothetic deformations.*

Proof. Let M be a semi-invariant submanifold of a $\mathcal{K}$-manifold $(\widetilde{M}, \varphi, \xi_i, \eta^j, g)$, $i, j \in \{1, \dots, s\}$ and let $(\widetilde{\varphi}, \widetilde{\xi}_i, \widetilde{\eta}^j, \widetilde{g})$, $i, j \in \{1, \dots, s\}$ be a $\mathcal{K}$-structure obtained on $\widetilde{M}$ by a $\mathcal{D}$-homothetic deformation of constant a. Then for each $x \in M$, $T_xM^{\perp_g} = T_xM^{\perp_{\widetilde{g}}}$. In fact, for each $X \in T_xM$, $Y \in T_xM^{\perp_g}$ we have $\widetilde{g}(X, Y) = ag(X, Y) + a(a-1)\sum_{i=1}^{s} \eta^i(X)\eta^i(Y) = 0$ and then $Y \in T_xM^{\perp_{\widetilde{g}}}$; on the other hand if we take $Z \in T_xM^{\perp_{\widetilde{g}}}$, then we get $ag(X, Z) = \widetilde{g}(X, Z) - a(a-1)\sum_{i=1}^{s} \eta^i(X)\eta^i(Z) = (a-1)\sum_{i=1}^{s} \eta^i(X)\widetilde{\eta}^i(Z) = 0$ so that $Z \in T_xM^{\perp_g}$. Now, it is obvious that D, $D^\perp$ verify Definition 3.1 with respect to the $\mathcal{D}$-homothetic deformed structure. □

We recall the Gauss and Weingarten equations

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \text{for each } X, Y \in \Gamma(TM)$$

$$\widetilde{\nabla}_X N = -\mathcal{A}_N X + \nabla^\perp_X N, \quad \text{for each } X \in \Gamma(TM),\ N \in \Gamma(TM^\perp).$$

Moreover, the second fundamental form h and the Weingarten operator $\mathcal{A}_N$ are related by the well known identity

$$g(\mathcal{A}_N X, Y) = g(h(X,Y), N). \tag{3.1}$$

By (2.5) it follows that

$$\nabla_{\xi_j}\xi_k \in \langle \xi_1, \ldots, \xi_k \rangle, \quad h(\xi_k, \xi_j) = 0. \tag{3.2}$$

Let us fix some notation: we put for each $X \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$, $Z \in \Gamma(T\widetilde{M})$

$$\varphi X = \tau X + \omega X, \quad \text{where } \tau X \in \Gamma(TM),\ \omega X \in \Gamma(TM^\perp) \tag{3.3}$$

$$\varphi N = BN + CN, \quad \text{where } BN \in \Gamma(TM),\ CN \in \Gamma(TM^\perp) \tag{3.4}$$

$$A_i Z = \alpha_i Z + \beta_i Z, \quad \text{where } \alpha_i Z \in \Gamma(TM),\ \beta_i Z \in \Gamma(TM^\perp). \tag{3.5}$$

Remark 3.1. It follows immediately by (3.3), (3.4) that

$$\omega = \varphi \circ Q. \tag{3.6}$$

Moreover, from the antisymmetry of φ with respect to g we obtain that τ and C are antisymmetric as well. Furthermore, by $\varphi^2 = -Id + \sum_{i=1}^s \eta^i \otimes \xi_i$ we get

$$\tau^2 = -Id + B \circ \omega + \sum_{i=1}^s \eta^i \otimes \xi_i, \tag{3.7}$$

$$C^2 = -Id - \omega \circ B, \tag{3.8}$$

$$\omega \circ \tau = C \circ \omega = B \circ C = \tau \circ B = 0. \tag{3.9}$$

Applying (3.7) to τX, for any $X \in \Gamma(TM)$, we get that $\tau^3 X = -\tau X$, and then τ is an f-structure on the tangent bundle TM; analogously, applying (3.8) to CN, for any $N \in \Gamma(TM^\perp)$, we get $C^3 = -C$, that is C is an f-structure on $TM^\perp$.

In the remaining results of the present section we always suppose that a semi-invariant submanifold M of a $\mathcal{K}$-manifold $(\widetilde{M}, \varphi, \xi_i, \eta^j, g)$, $i, j \in \{1, \ldots, s\}$, is fixed.

Lemma 3.1. *For any vector field tangent to M and $k \in \{1, \ldots, s\}$ we have:*

$$\alpha_k(X) = \tau(\nabla_X \xi_k) + Bh(X, \xi_k) + \sum_{i,j=1}^s \eta^j(X)\eta^j(A_k\xi_i)\xi_i \tag{3.10}$$

$$\beta_k(X) = \omega(\nabla_X \xi_k) + Ch(X, \xi_k). \tag{3.11}$$

PROOF. By (2.3), (3.5), the Gauss equation and (3.3) we get $\alpha_k(X) + \beta_k(X) = \tau(\nabla_X \xi_k) + \omega(\nabla_X \xi_k) + Bh(X, \xi_k) + Ch(X, \xi_k) + \sum_{i=1}^{s} g(X, A_k \xi_i)\xi_i$. Then we use (2.6) and compare the tangent and the normal part to obtain (3.10), (3.11). □

Proposition 3.2. *Let M be a semi-invariant submanifold of a $\mathcal{K}$-manifold $\widetilde{M}$. Then $\Gamma(TM)$ is invariant under A_k, $k \in \{1, \ldots, s\}$, that is $A_k(\Gamma(TM)) \subset \Gamma(TM)$, if and only if*

$$\omega(\nabla_X \xi_k) = 0 \quad \text{and} \quad Ch(X, \xi_k) = 0. \tag{3.12}$$

Furthermore, if $\Gamma(TM)$ is invariant under A_k then both $\Gamma(\mathcal{D})$ and $\Gamma(D^\perp)$ are invariant under A_k.

PROOF. If $X \in \Gamma(TM)$ then $A_k X$ is tangent to M if and only if $\beta_k(X) = 0$. Hence by Lemma 3.1

$$\omega(\nabla_X \xi_k) + Ch(X, \xi_k) = 0. \tag{3.13}$$

Since C is antisymmetric, by (3.9) for each $Y \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$, we have $g(\omega Y, CN) = -g(C\omega Y, N) = 0$ and then the two summands in (3.13) are orthogonal. Hence $\beta_k X = 0$ if and only if each summand in (3.13) is zero, that is (3.12).
To prove the second part, first notice that by (2.6) $g(A_k X, \xi_i) = g(X, A_k \xi_i) = 0$, for any $X \in \Gamma(D)$ or $X \in \Gamma(D^\perp)$. Then to show the invariance of $\Gamma(D)$ under A_k, it is enough to observe that $X' = -\varphi X \in \Gamma(D)$ and for each $Z \in \Gamma(D^\perp)$ $g(A_k X, Z) = g(A_k(\varphi X'), Z) = -g(A_k X', \varphi Z) = 0$. Finally, we observe that $g(A_k Z, X) = g(Z, A_k X) = 0$, due to the just proved invariance of $\Gamma(D)$ under A_k. Hence we have invariance of $\Gamma(D^\perp)$ under A_k. □

We recall that the covariant derivatives of τ, ω, B and C are defined respectively by $(\nabla_X \tau)Y = \nabla_X(\tau Y) - \tau(\nabla_X Y)$, $(\overset{*}{\nabla}_X \omega)Y = \nabla^\perp_X \omega Y - \omega(\nabla_X Y)$, $(\overset{*}{\nabla}_X B)N = \nabla_X BN - B(\nabla^\perp_X N)$ and $(\nabla^\perp_X C)N = \nabla^\perp_X CN - C(\nabla^\perp_X N)$, for each $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$.

Lemma 3.2. *We have the following explicit expressions of the covariant derivatives*

$$(\nabla_X \tau)Y = \sum_{i=1}^{s} \big\{ g(A_i X, Y)\xi_i - \eta^i(Y)\alpha_i(X) \big\} + \mathcal{A}_{\omega Y} X + Bh(X, Y)$$

$$(\overset{*}{\nabla}_X \omega)Y = -\sum_{i=1}^{s} \eta^i(Y)\beta_i X - h(X, \tau Y) + Ch(X, Y)$$

$$(\overset{*}{\nabla}_X B)N = \sum_{i=1}^{s} g(A_i X, N)\xi_i + \mathcal{A}_{CN}X - \tau(\mathcal{A}_N X)$$

$$(\nabla^{\perp}_X C)N = -h(X, BN) - \omega(\mathcal{A}_N X)$$

Proof. By (3.3) and by the Gauss and Weingarten equations we get

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y &= \nabla_X(\tau Y) + h(X, \tau Y) - \mathcal{A}_{\omega Y}X + \nabla^{\perp}_X(\omega Y)\\ &\quad - \tau(\nabla_X Y) - \omega(\nabla_X Y) - Bh(X,Y) - Ch(X,Y).\end{aligned} \tag{3.14}$$

On the other hand by b)1. of Theorem 2.1 and (3.5) we have

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s}\big\{g(A_i X, Y)\xi_i - \eta^i(Y)\alpha_i X - \eta^i(Y)\beta_i X\big\}. \tag{3.15}$$

Then we get the first two claimed identities comparing (3.14) and (3.15) and taking separately the tangent and the normal summands.
Analogously, using the Gauss and Weingarten equations we have

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)N &= \nabla_X(BN) + h(X, BN) - \mathcal{A}_{CN}X + \nabla^{\perp}_X CN\\ &\quad + \tau(\mathcal{A}_N X) + \omega(\mathcal{A}_N X) - B(\nabla^{\perp}_X N) - C(\nabla^{\perp}_X N)\end{aligned}$$

while by b)1. of Theorem 2.1 we get $(\widetilde{\nabla}_X \varphi)N = \sum_{i=1}^{s} g(A_i X, N)\xi_i$. Then the last two claimed identities follow by comparing the two expressions of $(\widetilde{\nabla}_X \varphi)N$ and taking first the tangent and then the normal summands. □

Lemma 3.3. *For each $X, Y \in \Gamma(D^{\perp})$, $U \in \Gamma(TM)$, $V \in \Gamma(D)$ we have*

$$\mathcal{A}_{\varphi X}Y = \mathcal{A}_{\varphi Y}X \tag{3.16}$$

$$g(h(U,V), \varphi X) = g(\nabla_U X, \varphi V). \tag{3.17}$$

Proof. Using (3.1), the Gauss equation, compatibility of φ with respect to g and Weingarten equation we get

$$\begin{aligned}g(\mathcal{A}_{\varphi X}Y, U) &= g(h(Y,U), \varphi X) = g(\widetilde{\nabla}_U Y, \varphi X) - g(\widetilde{\nabla}_U(\varphi Y), X)\\ &= g(\mathcal{A}_{\varphi Y}U, X) = g(\mathcal{A}_{\varphi Y}X, U),\end{aligned}$$

that is (3.16).
By the Gauss equation, the parallelism of g with respect to $\widetilde{\nabla}$ and b)1. of Theorem 2.1

$$\begin{aligned}g(h(U,V), \varphi X) &= -g(V, \widetilde{\nabla}_U \varphi X) = -g(V, \varphi(\widetilde{\nabla}_U X))\\ &= g(\varphi V, \nabla_U X + h(U,X)) = g(\varphi V, \nabla_U X)\end{aligned}$$

that is (3.17). □

We would like to establish some necessary and sufficient conditions for the integrability of various distributions involved in the semi-invariant submanifold. Before going further we need the following

Lemma 3.4. *We have*

$$g([X,Y],Z)=0 \quad \forall\ X,Y\in\Gamma(D^{\perp}),\ \ Z\in\Gamma(D). \tag{3.18}$$

Proof. By c) of Definition 3.1 we have $\tau X=\tau Y=0$ and then $\varphi X=\omega X$, $\varphi Y=\omega Y$. Hence

$$\begin{aligned} g([X,Y],Z) &= g(\varphi[X,Y],\varphi Z)=g(\tau[X,Y],\varphi Z)=-g((\nabla_X\tau)Y-(\nabla_Y\tau)X,\varphi Z)\\ &= g(\mathcal{A}_{\omega X}Y,\varphi Z)-g(\mathcal{A}_{\omega Y}X,\varphi Z)=0. \end{aligned}$$

Here in the last but one equality we use the first identity of Lemma 3.2 and in the last we use (3.16). □

Theorem 3.1. *The distribution $D^{\perp}$ is integrable if and only if for all $i\in\{1,\dots,s\}$ $A_i(\Gamma(D^{\perp}))$ is orthogonal to $\varphi(\Gamma(D^{\perp}))$.*

Proof. Let $X,Y\in\Gamma(D^{\perp})$, $Z\in\Gamma(D)$. By (2.2) $g(\nabla_XY,\xi_i)=g(\widetilde{\nabla}_XY,\xi_i)-g(Y,\widetilde{\nabla}_X\xi_i)=g(Y,\varphi(A_iX))=-g(\varphi Y,A_iX)$ and hence

$$g([X,Y],\xi_i)=-2g(A_iX,\varphi Y).$$

We conclude that $[X,Y]$ is orthogonal to $\ker\varphi$ if and only if for all $i\in\{1,\dots,s\}$ $A_i(\Gamma(D^{\perp}))$ and $\varphi(\Gamma(D^{\perp}))$ are orthogonal to each other. By (3.18) we get our claim. □

Remark 3.2. Obviously by Proposition 3.2 if for each $i\in\{1,\dots,s\}$ $\Gamma(TM)$ is invariant under A_i then $D^{\perp}$ is integrable.

By Remark 2.2 and Theorem 3.1 it follows

Corollary 3.1. *Let M be a semi-invariant submanifold of an $\mathcal{S}$-manifold $\widetilde{M}$. Then the distribution $D^{\perp}$ is integrable.*

Proof. In fact, $\varphi(\Gamma(D^{\perp}))$ is orthogonal to $-\varphi^2(\Gamma(D^{\perp}))=A_i(\Gamma(D^{\perp}))$. □

Theorem 3.2. *The distribution $D^{\perp}\oplus\ker\varphi$ is always integrable.*

Proof. Let $X,Y\in\Gamma(D^{\perp})$, $Z\in\Gamma(D)$. Since we know by (3.18) that $[X,Y]$ is normal to Z it is sufficient to prove that $[X,\xi_i]$ is orthogonal to Z, for each $i\in\{1,\dots,s\}$. In fact we have

$$g([X,\xi_i],Z)=g(\varphi[X,\xi_i],\varphi Z)=-g(\varphi(\widetilde{\nabla}_X\xi_i),\varphi Z)+g(\varphi(\widetilde{\nabla}_{\xi_i}X),\varphi Z)$$

$$
\begin{aligned}
&= g((\widetilde{\nabla}_X\varphi)\xi_i), \varphi Z) + g(\widetilde{\nabla}_{\xi_i}(\varphi Z), \varphi X)\\
&= g(A_iX, \varphi Z) + g(h(\xi_i, \varphi Z), \varphi X)\\
&= g(A_iX, \varphi Z) + g(\varphi X, \widetilde{\nabla}_{\varphi Z}\xi_i)\\
&= -g(A_iX, \varphi Z) - g(\varphi X, \varphi A_i\varphi X) = 0.
\end{aligned}
$$

Here we use (2.7), b)1. of Theorem 2.1, the Gauss equation and (2.2). The last case is obvious as $[\xi_i, \xi_j] = 0$ (cf. [5]). □

Theorem 3.3. *The distribution $D \oplus \ker\varphi$ is integrable if and only if*

$$h(X, \varphi Y) = h(\varphi X, Y), \textit{for each } X, Y \in \Gamma(D). \tag{3.19}$$

Proof. For each $Z \in \Gamma(D^\perp)$, $i \in \{1, \dots, s\}$, by the compatibility of φ with the metric, (2.7), b)1. of Theorem 2.1 and (2.2) we have

$$
\begin{aligned}
g([X, \xi_i], Z) &= -g((\widetilde{\nabla}_X\varphi)\xi_i, \varphi Z) - g(\widetilde{\nabla}_{\xi_i}(\varphi X), \varphi Z)\\
&= g(A_iX, \varphi Z) - g(\widetilde{\nabla}_{\varphi X}\xi_i, \varphi Z) = g(A_iX, \varphi Z) + g(\varphi A_i\varphi X, \varphi Z) = 0.
\end{aligned}
$$

On the other hand, from the expression of $\overset{*}{\nabla}\omega$ in Lemma 3.2, we have

$$(\overset{*}{\nabla}_X\omega)Y - (\overset{*}{\nabla}_Y\omega)X = -h(X, \varphi Y) + Ch(X, Y) + h(Y, \varphi X) - Ch(Y, X),$$

as for each $i \in \{1, \dots, s\}$ $\eta^i(X) = \eta^i(Y) = 0$ and $\omega X = \omega Y = 0$, that is $\varphi X = \tau X$, $\varphi Y = \tau Y$. Hence $\omega([X, Y]) = h(X, \varphi Y) - h(Y, \varphi X)$. If $D \oplus \ker\varphi$ is integrable then $[X, Y] \in \Gamma(D \oplus \ker\varphi)$ and hence $\omega[X, Y] = 0$. Vice versa, if $h(X, \varphi Y) = h(\varphi X, Y)$ then by (3.6) $\varphi(Q[X, Y]) = \omega[X, Y] = 0$ so that $Q[X, Y] = 0$. Hence $[X, Y] \in \Gamma(D \oplus \ker\varphi)$. □

Theorem 3.4. *The distribution D is integrable if and only if* (3.19) *is verified and, moreover, for each $i \in \{1, \dots, s\}$ $A_i(\Gamma(D))$ and $\Gamma(D)$ are orthogonal.*

Proof. From the proof of Theorem 3.3 we get that for each $X, Y \in \Gamma(D)$, $[X, Y]$ is orthogonal to $\Gamma(D^\perp)$ if and only if (3.19) is verified. Furthermore, by (2.2) we obtain $g([X, Y], \xi_i) = -g(Y, \widetilde{\nabla}_X\xi_i) + g(X, \widetilde{\nabla}_Y\xi_i) = g(Y, \varphi A_iX) - g(X, \varphi A_iY) = 2g(A_iX, \varphi Y)$ and this completes the proof. □

Corollary 3.2. *If there exists $i \in \{1, \dots, s\}$ such that A_i is an automorphism of $\Gamma(TM)$ and D is integrable then M is an anti-invariant submanifold.*

Proof. The hypotheses and Proposition 3.2 imply $A_i(\Gamma(D)) = \Gamma(D)$. Then by Theorem 3.4 it follows that $D = \{0\}$. □

The following Corollary is a simple consequence of Remark 2.2 and Theorem 3.4.

Corollary 3.3. *Let M be a semi-invariant submanifold of an $\mathcal{S}$-manifold $\widetilde{M}$. Then the distribution D is never integrable.*

Proof. In fact, if D is integrable then $\Gamma(D)$ is orthogonal to $-\varphi^2(\Gamma(D))$, a contradiction. □

4. Normal semi-invariant submanifolds of $\mathcal{K}$-manifolds

The concept of normality for semi-invariant submanifolds of Kählerian manifolds is well-known (e.g. cf. [21]). Furthermore Bejancu and Papaghiuc (cf. [4]) gave the definition of normal semi-invariant submanifold of a Sasakian manifold and Calin extended the definition to a semi-invariant submanifold of a quasi-Sasakian manifold. Now we give a natural generalization of this definition for a semi-invariant submanifold of a $\mathcal{K}$-manifold.

Definition 4.1. Let M be a semi-invariant submanifold of a $\mathcal{K}$-manifold $\widetilde{M}$. We say that M is *normal* if the (1,2)-tensor field S on M defined for each $X, Y \in \Gamma(TM)$ by

$$S(X,Y) = [\tau,\tau](X,Y) - 2Bd\omega(X,Y) + \sum_{i=1}^{s}\{F(\alpha_i X, Y) - F(\alpha_i Y, X)\}\xi_i, \quad (4.1)$$

and called the *torsion of the semi-invariant structure*, vanishes identically.

Lemma 4.1. *For each $X, Y \in \Gamma(TM)$, $k \in \{1,\dots,s\}$ the following identities hold*

$$d\eta^k(X,Y) = g(\beta_k X, \omega Y) + F(\alpha_k X, Y) + \sum_{i=1}^{s}\eta^i(\nabla_X \xi_k)\eta^i(Y) \quad (4.2)$$

$$d\eta^k(\varphi X, \tau Y) = g(A_k X, \tau Y) \quad (4.3)$$

$$F(\tau X, \tau Y) = F(X,Y) \quad (4.4)$$

Proof. Since for each $k \in \{1,\dots,s\}$ ξ_k is Killing (cf. [5]) we have for any $X, Y \in \Gamma(TM)$

$$d\eta^k(X,Y) = g(Y, \widetilde{\nabla}_X \xi_k). \quad (4.5)$$

Then we easily get (4.2) from (2.2), (3.5) and the Gauss equation.
By (4.5) and (2.2) we obtain

$$d\eta^k(\varphi X, \tau Y) = -g(\varphi A_k \varphi X, \tau Y) = g(A_k X, \tau Y) \quad (4.6)$$

that is (4.3).
We observe that for each $X \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$ we have

$$g(\omega X, N) = -g(X, BN). \tag{4.7}$$

Hence by (3.9), (3.7), (4.7) and the antisymmetry of τ we infer that

$$\begin{aligned} F(\tau X, \tau Y) &= g(\tau X, \varphi\tau Y) = g(\tau X, \tau^2 Y) = -g(\tau X, Y) + g(\tau X, B\omega Y) \\ &= g(X, \tau Y) - g(\omega\tau X, \omega Y) = g(X, \varphi Y) = F(X, Y). \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Thus (4.4) has been proved. □

Proposition 4.1. *We have the following expression for the torsion*

$$\begin{aligned} S(X,Y) &= \mathcal{A}_{\omega Y}\tau X - \mathcal{A}_{\omega X}\tau Y - \tau(\mathcal{A}_{\omega Y}X - \mathcal{A}_{\omega X}Y) \\ &\quad + \sum_{i=1}^{s}\{\eta^i(X)\alpha_i\omega(Y) - \eta^i(Y)\alpha_i\omega(X)\}. \end{aligned} \tag{4.8}$$

for any $X, Y \in \Gamma(TM)$.

Proof. By a direct computation, for each $X, Y \in \Gamma(TM)$ we get

$$[\tau,\tau](X,Y) = (\nabla_{\tau X}\tau)Y - (\nabla_{\tau Y}\tau)X + \tau\left((\nabla_Y\tau)X - (\nabla_X\tau)Y\right). \tag{4.9}$$

On the other hand, by Lemma 3.2 we have that

$$\begin{aligned} 2d\omega(X,Y) &= (\overset{*}{\nabla}_X\omega)Y - (\overset{*}{\nabla}_Y\omega)X = \sum_{i=1}^{s}\{\eta^i(X)\beta_i(Y) - \eta^i(Y)\beta_i(X)\} \\ &\quad - h(X,\tau Y) + h(Y,\tau X). \end{aligned} \tag{4.10}$$

Hence from (4.9), (4.10) it follows that the tensor field S can be written as

$$\begin{aligned} S(X,Y) &= (\nabla_{\tau X}\tau)Y - (\nabla_{\tau Y}\tau)X + \tau\left((\nabla_Y\tau)X - (\nabla_X\tau)Y\right) \\ &\quad + \sum_{i=1}^{s}\big\{\eta^i(Y)B\beta_i(X) - \eta^i(X)B\beta_i(Y) + (F(\alpha_i X, Y) - F(\alpha_i Y, X))\xi_i\big\} \\ &\quad - Bh(Y,\tau X) + Bh(X,\tau Y). \end{aligned} \tag{4.11}$$

(4.9) Lemma 3.2, the symmetry of h and of $A_1, \dots, A_s$ and (4.3) assure that

$$\begin{aligned} [\tau,\tau] &= \mathcal{A}_{\omega Y}\tau X - \mathcal{A}_{\omega X}\tau Y - \tau(\mathcal{A}_{\omega Y}X - \mathcal{A}_{\omega X}Y) \\ &\quad + \sum_{i=1}^{s}\big\{\eta^i(Y)(\tau\alpha_i X - \alpha_i\tau X) - \eta^i(X)(\tau\alpha_i Y - \alpha_i\tau Y) \end{aligned}$$

$$+ (d\eta^i(\varphi Y, \tau X) - d\eta^i(\varphi X, \tau Y))\xi_i\big\} + Bh(\tau X, Y) - Bh(\tau Y, X). \quad (4.12)$$

Furthermore, for each $i \in \{i, \dots, s\}$ $\alpha_i\tau X - \tau\alpha_i X - B\beta_i X$ is the tangent part of $A_i\tau X - \varphi\alpha_i X - \varphi\beta_i X = A_i\tau X - \varphi A_i X = A_i\tau X - A_i\varphi X = A_i(\tau - \varphi)X = -A_i\omega X$. Then

$$\alpha_i\tau X - \tau\alpha_i X - B\beta_i X = -\alpha_i\omega X. \quad (4.13)$$

From (4.3) we easily get that

$$d\eta^i(\varphi X, \tau Y) = F(\alpha_i X, Y). \quad (4.14)$$

for each $i \in \{i, \dots, s\}$. Using (4.11), (4.12), (4.14) and (4.10) we obtain

$$\begin{aligned} S(X,Y) &= \mathcal{A}_{\omega Y}\tau X - \mathcal{A}_{\omega X}\tau Y - \tau(\mathcal{A}_{\omega Y}X - \mathcal{A}_{\omega X}Y) \\ &\quad + \sum_{i=1}^{s}\big\{\eta^i(Y)(\tau\alpha_i X - \alpha_i\tau X + B\beta_i X) - \eta^i(X)(\tau\alpha_i Y - \alpha_i\tau Y + B\beta_i Y) \\ &\quad + d\eta^i(\varphi Y, \tau X) - d\eta^i(\varphi X, \tau Y) - F(\alpha_i Y, X) + F(\alpha_i X, Y)\big\}. \quad (4.15) \end{aligned}$$

Hence (4.8) is a consequence of (4.15) and (4.13). □

Lemma 4.2. *For all $i, j \in \{1, \dots, s\}$*

$$g(\alpha_i X, Y) = g(X, \alpha_i Y) \qquad \forall\ X, Y \in \Gamma(TM) \quad (4.16)$$

$$g(\beta_i V, W) = g(V, \beta_i W) \qquad \forall\ V, W \in \Gamma(TM^\perp) \quad (4.17)$$

$$g(X, \alpha_i V) = g(\beta_i X, V) \qquad \forall\ X \in \Gamma(TM), V \in \Gamma(TM^\perp) \quad (4.18)$$

$$g(\beta_i X, \omega Y) = -g(\tau X, \alpha_i Y) \quad \forall\ X \in \Gamma(D),\ Y \in \Gamma(D^\perp) \quad (4.19)$$

$$g(\omega X, \beta_i\xi_j) = 0 \qquad \forall\ X \in \Gamma(D^\perp). \quad (4.20)$$

Proof. (4.16), (4.17) and (4.18) are obvious; (4.19), (4.20) can be easily derived from the identity $g(\alpha_i X, \tau Y) + g(\beta_i X, \omega Y) - g(\tau X, \alpha_i Y) - g(\omega X, \beta_i Y)$. □

Theorem 4.1. *A semi-invariant submanifold M of a $\mathcal{K}$-manifold $\widetilde{M}$ is normal if and only if the distribution $D^\perp$ is integrable and*

$$\mathcal{A}_{\omega Y}\tau X = \tau\mathcal{A}_{\omega Y}X \quad \forall\ X \in \Gamma(D),\ Y \in \Gamma(D^\perp). \quad (4.21)$$

Proof. The identity (4.8) assures that for any $j \in \{1, \dots, s\}$, $Y \in \Gamma(D^\perp)$

$$S(\xi_j, Y) = \alpha_j\omega Y - \tau\mathcal{A}_{\omega Y}\xi_j \quad (4.22)$$

and then by the antisymmetry of τ, for each $Z \in \Gamma(D^\perp)$

$$g(S(\xi_j, Y), Z) = g(\alpha_j \omega Y, Z). \tag{4.23}$$

Moreover, from (4.8) we obtain

$$S(X,Y) = \mathcal{A}_{\omega Y}\tau X - \tau\mathcal{A}_{\omega Y}X, \qquad X \in \Gamma(D),\ Y \in \Gamma(D^\perp). \tag{4.24}$$

Now, if $S = 0$, from (4.23) and Theorem 3.1 it follows that the distribution $D^\perp$ is integrable. Furthermore, (4.21) is clearly verified by virtue of (4.24).
Vice versa, first we observe that by (4.8) $S(X,Y) = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D^\perp)$ or $X = \xi_i$, $i \in \{1, \dots, s\}$ and $Y \in \Gamma(D)$. Then from the integrability of $D^\perp$ and (4.23) we get that for all $Y \in \Gamma(D^\perp)$ $S(\xi_i, Y)$ is normal to $D^\perp$. On the other hand for each $Z \in \Gamma(D)$, by (4.22), we have the antisymmetry of τ and (4.18)

$$\begin{aligned} g(S(\xi_i, Y), Z) &= g(\alpha_i \omega Y - \tau\mathcal{A}_{\omega Y}\xi_i, Z) = g(\alpha_i \omega Y, Z) + g(\mathcal{A}_{\omega Y}\xi_i, \tau Z) \\ &= g(\omega Y, \beta_i Z) + g(\xi_i, \mathcal{A}_{\omega Y}\tau Z) = g(\omega Y, \beta_i Z) = 0 \end{aligned}$$

since by (4.19), the symmetry of each A_i, (2.2) and (4.21)

$$\begin{aligned} g(\omega Y, \beta_i Z) &= -g(\alpha_i Y, \tau Z) = -g(Y, A_i \tau Z) - g(\varphi Y, \varphi A_i \tau Z) \\ &= g(\omega Y, \widetilde{\nabla}_{\tau Z}\xi_i) - g(\widetilde{\nabla}_{\tau Z}\omega Y, \xi_i) = g(\mathcal{A}_{\omega Y}\tau Z, \xi_i) - g(\nabla^\perp_{\tau Z}\omega Y, \xi_i) = 0. \end{aligned}$$

Finally, (4.22), (4.18), (4.20) ensure that for any $j \in \{1, \dots, s\}$, $g(S(\xi_i, Y), \xi_j) = g(\alpha_i \omega Y, \xi_j) - g(\tau\mathcal{A}_{\omega Y}\xi_i, \xi_j) = 0$. Hence for all $Y \in \Gamma(D)$, $i \in \{1, \dots, s\}$, $S(\xi_i, Y) = 0$, as it is obviously normal to $TM^\perp$ by virtue of (4.22). □

Remark 4.1. As $\varphi(D^\perp)$ is a vector subbundle of $TM^\perp$ we can consider its orthogonal complement μ. Then $\varphi(\mu) = \mu$. In fact, by (4.7) $g(\varphi N, X) = 0$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(\mu)$, that is $\varphi(\mu) \subset TM^\perp$. Moreover, $g(\varphi N, \varphi X) = 0$ for any $X \in \Gamma(D^\perp)$ and $N \in \Gamma(\mu)$, and then $\varphi(\mu) \subset \mu$. The opposite inclusion is obvious.

Another characterization of the normality of semi-invariant submanifolds of $\mathcal{K}$-manifolds is given by the following result.

Theorem 4.2. *A semi-invariant submanifold M of a $\mathcal{K}$-manifold $\widetilde{M}$ is normal if and only if*

$$h(\tau X, W) \in \Gamma(\mu) \qquad \forall X \in \Gamma(D),\ W \in \Gamma(D^\perp) \tag{4.25}$$

$$h(X, \tau Y) + h(\tau X, Y) \in \Gamma(\mu) \quad \forall X, Y \in \Gamma(D) \tag{4.26}$$

$$A_i(D^\perp) \subseteq \mu \oplus D^\perp \qquad \forall i \in \{1, \dots, s\}. \tag{4.27}$$

Proof. We observe that $\forall X, Y \in \Gamma(D),\ Z, W \in \Gamma(D^{\perp})$, the antisymmetry of τ and (3.1) assure that

$$g(\mathcal{A}_{\omega Z}\tau X - \tau\mathcal{A}_{\omega Z}X, W) = g(\mathcal{A}_{\omega Z}\tau X, W) = g(h(\tau X, W), \omega Z) \tag{4.28}$$

$$g(\mathcal{A}_{\omega Z}\tau X - \tau\mathcal{A}_{\omega Z}X, Y) = g(h(\tau X, Y) + h(X, \tau Y), \omega Z). \tag{4.29}$$

Furthermore, for each $X \in \Gamma(D^{\perp})$, $Y \in \Gamma(D)$ we have

$$\begin{aligned} g(\varphi Y, A_i X) &= g(A_i\tau Y, X) = g(\varphi A_i \tau Y, \varphi X) = -g(\omega X, \widetilde{\nabla}_{\tau Y}\xi_i) = g(\widetilde{\nabla}_{\tau Y}\omega X, \xi_i) \\ &= g(\mathcal{A}_{\omega X}\tau Y, \xi_i) = g(\mathcal{A}_{\omega X}\tau Y - \tau\mathcal{A}_{\omega X}Y, \xi_i). \end{aligned} \tag{4.30}$$

If the semi-invariant submanifold is normal then using Theorem 4.1 from (4.29), (4.28) we easily derive (4.26) and (4.25). To prove (4.27), first we take $X \in \Gamma(D^{\perp})$, $Y \in \Gamma(D)$ and observe that from (4.30) and Theorem 3.4 it follows that $g(\varphi Y, A_i X) = 0$ and then $0 = g(\varphi Y, A_i X) = -g(\alpha_i X + \beta_i X, \varphi Y) = -g(\alpha_i X, \varphi Y)$ so that $\alpha_i X \in \Gamma(D^{\perp} \oplus \langle \xi_1, \ldots, \xi_s\rangle)$. Furthermore, by (3.10), (4.16) and (3.2), $g(\alpha_i X, \xi_j) = g(X, \alpha_i \xi_j) = g(X, \tau\nabla_{\xi_j}\xi_i) + g(X, Bh(\xi_i, \xi_j)) = 0$. Hence

$$\alpha_i X \in \Gamma(D^{\perp}). \tag{4.31}$$

On the other hand, $g(\beta_i X, \varphi Z) = g(A_i X, \varphi Z) = 0$, for any $Z \in \Gamma(D^{\perp})$, as by Theorem 4.1 $D^{\perp}$ is integrable, so

$$\beta_i(X) \in \Gamma(\mu). \tag{4.32}$$

The properties (4.31), (4.32) ensure (4.27).
Conversely, for any $X, Y \in \Gamma(D^{\perp})$, from (4.27) one obtains that $g(A_i X, \varphi Y) = g(\alpha_i X, \varphi Y) + g(\beta_i X, \varphi Y) = 0$, that is $D^{\perp}$ is integrable. Moreover, by (4.29), (4.28), (4.26) and (4.25) it follows that $\mathcal{A}_{\omega Z}\tau X - \tau\mathcal{A}_{\omega Z}X$ is normal to $D \oplus D^{\perp}$ for all $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$; on the other hand, $g(A_i Z, \varphi X) = 0$ as $A_i Z \in \Gamma(\mu \oplus D^{\perp})$ and $\varphi X \in \Gamma(D)$. Then by (4.30) $\mathcal{A}_{\omega Z}\tau X - \tau\mathcal{A}_{\omega Z}X$ is orthogonal to $\langle \xi_1, \ldots, \xi_s\rangle$. Hence we have (4.21). □

Definition 4.2. We say that a submanifold M of a $\mathcal{K}$-manifold is *anti-holomorphic* if M is a semi-invariant submanifold such that $\dim(TM^{\perp}) = \dim(D^{\perp})$.

Remark 4.2. If M is a normal anti-holomorphic semi-invariant submanifold of a $\mathcal{K}$-manifold, then $\mu = \{0\}$. Hence Theorem 4.2 assures that

Corollary 4.1. *Let M be an anti-holomorphic submanifold of a $\mathcal{K}$-manifold $\widetilde{M}$. Then M is a normal semi-invariant submanifold of $\widetilde{M}$ if and only if*

$$\begin{aligned} &h(\tau X, W) = 0 && \forall X \in \Gamma(D),\ W \in \Gamma(D^\perp) \\ &h(X, \tau Y) + h(\tau X, Y) = 0 && \forall X, Y \in \Gamma(D) \\ &A_i(D^\perp) \subseteq D^\perp && \forall i \in \{1, \ldots, s\}. \end{aligned}$$

Finally, let us return to the example.

i) The distribution $D \oplus \langle \xi_1, \xi_2 \rangle = \langle Z_0^*, Z_1^*, X_1^*, Y_1^* \rangle$ is integrable, its integral submanifolds are invariant submanifolds.

ii) The distribution $D^\perp \oplus \langle \xi_1, \xi_2 \rangle = \langle Z_0^*, Z_1^*, X_1^* \rangle$, is integrable, its integral submanifolds are anti-invariant submanifolds.

iii) The submanifold $M(N_5, \Gamma)$ is normal. The normality of this submanifold can be checked on the level of the universal covering space, i.e. N_8, using the characterization given in Theorem 4.1. In this case the distribution $D = \langle X_1^*, Y_1^* \rangle$ and $D^\perp = \langle X_2^* \rangle$. The subbundle $D^\perp$ being 1-dimensional is integrable. Therefore it remains to check that

$$\mathcal{A}_{\omega Y} \tau X = \tau \mathcal{A}_{\omega Y} X$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Therefore it is sufficient to check that equality holds for $X = X_1^*, Y_1^*$ and $Y = X_2^*$. It is an easy calculation that in these cases both sides of the equation are zero.

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