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On φ-convexity

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Abstract. In this paper, approximate convexity and approximate midconvexity properties, called φ-convexity and φ-midconvexity, of real valued function are investigated. Various characterizations of φ-convex and φ-midconvex functions are obtained. Furthermore, the relationship between φ-midconvexity and φ-convexity is established.

1. Introduction

The stability theory of functional inequalities started with the paper [13] of HYERS and ULAM who introduced the notion of ε-convex function: If D is a convex subset of a real linear space X and ε is a nonnegative number, then a function $f : D \to \mathbb{R}$ is called *ε-convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \tag{1}$$

for all $x, y \in D$, $t \in [0,1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is of finite dimension then f can be written as $f = g + h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n\varepsilon$, where the positive constant k_n depends only on the dimension n of the underlying space X. Hyers and Ulam proved that $k_n \leq (n(n+3))/(4(n+1))$. GREEN [11], CHOLEWA [5] obtained much better

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estimations of k_n showing that asymptotically k_n is not bigger than $(\log_2(n))/2$. Laczkovich [22] compared this constant to several other dimension-depending stability constants and proved that it is not less than $(\log_2(n/2))/4$. This result shows that there is no analogous stability results for infinite dimensional spaces X. A counterexample in this direction was earlier constructed by Casini and Papini [4]. The stability aspects of ε-convexity are discussed by Ger [10]. An overview of results on δ-convexity can be found in the book of Hyers, Isac, and Rassias [12].

If $t = 1/2$ and (1) holds for all $x, y \in D$, then f is called an *ε-Jensen-convex function*. There is no analogous decomposition for ε-Jensen-convex functions by the counterexample given by Cholewa [5]. However, one can get Bernstein–Doetsch type regularity theorems which show that ε-Jensen-convexity and local upper boundedness imply 2ε-convexity. This result is due to Bernstein and Doetsch [2] for $\varepsilon = 0$, and to Ng and Nikodem [28] in the case $\varepsilon \geq 0$. For some recent extensions of these results to more general convexity concepts, see [29]. For locally upper bounded ε-Jensen-convex functions one can obtain the existence of an analogous stability constant j_n (defined similarly as k_n above). The sharp value of this stability constant has recently been found by Dilworth, Howard, and Roberts [6] who have shown that

$$j_n = \frac{1}{2}\Big([\log_2(n)] + 1 + \frac{n}{2^{[\log_2(n)]}}\Big) \leq 1 + \frac{1}{2}\log_2(n)$$

is the best possible value for j_n. (Here $[\cdot]$ denotes the integer-part function). The connection between ε-Jensen-convexity and ε-$\mathbb{Q}$-convexity has been investigated by Mrowiec [26].

If $D \subset \mathbb{R}$ and (1) is supposed to be valid for all $x, y \in D$ except a set of 2-dimensional Lebesgue measure zero then one can speak about *almost ε-convexity*. Results in this direction are due to Kuczma [20] (the case $\varepsilon = 0$) and Ger [9] (the case $\varepsilon \geq 0$).

In a recent paper [30], the second author introduced a more general notion than ε-convexity. Let ε and δ be nonnegative constants. A function $f : D \to \mathbb{R}$ is called (ε, δ)-convex, if

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) + \delta + \varepsilon t(1-t)\|x - y\|$$

for every $x, y \in D$ and $t \in [0, 1]$. The main results of the paper [30] obtain a complete characterization of (ε, δ)-convexity if $D \subseteq \mathbb{R}$ is an open real interval by showing that these functions are of the form $f = g + h + \ell$, where g is convex, h is bounded with $\|h\| \leq \delta/2$ and ℓ is Lipschitzian with Lipschitz modulus $\mathrm{Lip}(\ell) \leq \varepsilon$.

In the papers [17], [18], the notion of (ε, p)-convexity and (ε, p)-midconvexity were introduced: If $\varepsilon, p \geq 0$ and $t \in [0,1]$, then a function $f : D \to \mathbb{R}$ is called (ε, p, t)-*convex*, if

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) + \varepsilon(t(1-t)\|x-y\|)^p$$

for every $x, y \in D$. If the above property holds for $t = 1/2$ and for all $t \in [0,1]$, then we speak about (ε, p)-*midconvexity* and (ε, p)-*convexity*, respectively. The main result in [18] shows that, for locally upper bounded functions, (ε, p)-midconvexity implies $(c\varepsilon, p)$-convexity for some constant c.

Another, but related, notion of approximate convexity, the concept of so-called paraconvexity was introduced by Rolewicz [33], [34], [35] in the late 70s. It also turned out that Takagi-like functions appear naturally in the investigation of approximate convexity, see, for example, Boros [3], Házy [15], [16], Házy and Páles [17], [18], [19], Makó and Páles [24], [25], Mrowiec, Tabor and Tabor [27], Tabor and Tabor [36], [37], Tabor, Tabor, and Żołdak [39], [38].

The aim of this paper is to offer a unified framework for most of the mentioned approximate convexity notions by introducing the notions of φ-convexity and φ-midconvexity and to extend the previously known results to this more general setting. We also introduce the relevant Takagi type functions which appear naturally in the description of the connection of φ-convexity and φ-midconvexity.

2. φ-convexity and φ-midconvexity

Throughout the paper $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the sets of real, nonnegative real, and natural numbers, respectively. Assume that D is a nonempty convex subset of a real normed space X and denote $D^+ := \{\|x-y\| : x, y \in D\}$. Let $\varphi : D^+ \to \mathbb{R}_+$ be a given function.

Definition 1. A function $f : D \to \mathbb{R}$ is called φ-*convex* on D, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\varphi\big((1-t)\|x-y\|\big) + (1-t)\varphi\big(t\|x-y\|\big) \quad (2)$$

holds for all $t \in [0,1]$ and for all $x, y \in D$. If (2) holds for $t = 1/2$, i.e., if, for all $x, y \in D$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \varphi\left(\left\|\frac{x-y}{2}\right\|\right), \quad (3)$$

then we say that f is φ-*midconvex*.

In the case $\varphi \equiv 0$, the meaning of inequalities (2) and (3) is the convexity and midconvexity (Jensen-convexity) of f, respectively.

An important particular case occurs when $\varphi : D^+ \to \mathbb{R}_+$ is of the form $\varphi(x) := \varepsilon x^p$, where $p, \varepsilon \geq 0$ are arbitrary constants. Then the function f is called (ε, p)-*convex* and (ε, p)-*midconvex* on D, respectively (cf. [30]).

The next results describe the structure of φ-convex functions and φ-midconvex functions.

Proposition 2.

(i) *If, for $j = 1, \dots, n$, $\varphi_j : D^+ \to \mathbb{R}_+$, the function $f_j : D \to \mathbb{R}$ is φ_j-convex and c_j is a nonnegative number, then $c_1 f_1 + \dots + c_n f_n$ is $(c_1\varphi_1 + \dots + c_n\varphi_n)$-convex. In particular, the set of φ-convex functions on D is convex.*

(ii) *Let $\{f_\gamma : D \to \mathbb{R} \mid \gamma \in \Gamma\}$ be a family of φ-convex functions. Assume, for all $x \in D$, that $f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x) < +\infty$. Then f is φ-convex.*

(iii) *Let $\{f_\gamma : D \to \mathbb{R} \mid \gamma \in \Gamma\}$ be a downward directed family of φ-convex functions in the following sense: for all $\gamma_1, \gamma_2 \in \Gamma$ and $x_1, x_2 \in D$, there exists $\gamma \in \Gamma$ such that $f_\gamma(x_i) \leq f_{\gamma_i}(x_i)$ for $i = 1, 2$. Assume, for all $x \in D$, that $f(x) := \inf_{\gamma\in\Gamma} f_\gamma(x) > -\infty$. Then f is φ-convex.*

Proof. (i) is easy to prove.

(ii) Let $x, y \in D$ and $t \in [0, 1]$. For all $\gamma \in \Gamma$, we have

$$\begin{aligned} f_\gamma(tx + (1-t)y) &\leq tf_\gamma(x) + (1-t)f_\gamma(y) + t\varphi((1-t)\|x-y\|) + (1-t)\varphi(t\|x-y\|) \\ &\leq tf(x) + (1-t)f(y) + t\varphi((1-t)\|x-y\|) + (1-t)\varphi(t\|x-y\|). \end{aligned}$$

Thus,

$$\begin{aligned} f(tx + (1-t)y) &= \sup_{\gamma\in\Gamma} f_\gamma(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) + t\varphi\big((1-t)\|x-y\|\big) + (1-t)\varphi\big(t\|x-y\|\big). \end{aligned}$$

Hence f is φ-convex.

(iii) Let $x, y \in D$ and $t \in [0, 1]$. Let $\delta > 0$ be arbitrary. Then $f(x) < f(x) + \delta$ and $f(y) < f(y) + \delta$. Thus there exist γ_1, γ_2, such that $f_{\gamma_1}(x) < f(x) + \delta$ and $f_{\gamma_2}(y) < f(y) + \delta$. By the conditions of the proposition, there exists $\gamma \in \Gamma$, such that

$$f_\gamma(x) \leq f_{\gamma_1}(x) < f(x) + \delta, \qquad f_\gamma(y) \leq f_{\gamma_2}(y) < f(y) + \delta.$$

Then we get

$$\begin{aligned} f(tx + (1-t)y) &\leq f_\gamma(tx + (1-t)y) \\ &\leq tf_\gamma(x) + (1-t)f_\gamma(y) + t\varphi\big((1-t)\|x-y\|\big) + (1-t)\varphi\big(t\|x-y\|\big) \\ &\leq tf(x) + (1-t)f(y) + \delta + t\varphi\big((1-t)\|x-y\|\big) + (1-t)\varphi\big(t\|x-y\|\big). \end{aligned}$$

This proves that f is φ-convex. □

The following statements concern midconvex functions, they are analogous to those of Proposition 2.

Proposition 3.

(i) *If, for $j = 1, \dots, n$, $\varphi_j : D^+ \to \mathbb{R}_+$, the function $f_j : D \to \mathbb{R}$ is φ_j-midconvex and c_j is a nonnegative number, then $c_1 f_1 + \dots + c_n f_n$ is $(c_1\varphi_1 + \dots + c_n\varphi_n)$-midconvex. In particular, the set of φ-midconvex functions on D is convex.*

(ii) *Let $\{f_\gamma : D \to \mathbb{R} \mid \gamma \in \Gamma\}$ be a family of φ-midconvex functions. Assume, for all $x \in D$, that $f(x) := \sup_{\gamma\in\Gamma} f_\gamma(x) < +\infty$. Then f is φ-midconvex.*

(iii) *Let $\{f_\gamma : D \to \mathbb{R} \mid \gamma \in \Gamma\}$ be a downward directed family of φ-midconvex functions in the following sense: for all $\gamma_1, \gamma_2 \in \Gamma$ and $x_1, x_2 \in D$, there exists $\gamma \in \Gamma$ such that $f_\gamma(x_i) \le f_{\gamma_i}(x_i)$ for $i = 1, 2$. Assume, for all $x \in D$, that $f(x) := \inf_{\gamma\in\Gamma} f_\gamma(x) > -\infty$. Then f is φ-midconvex.*

Definition 4. A function $f : D \to \mathbb{R}$ is said to be of *φ-Hölder class* on D or briefly f is called *φ-Hölder* on D if there exists a nonnegative constant H such that, for all $x, y \in D$,

$$|f(x) - f(y)| \le H\varphi(\|x - y\|). \tag{4}$$

The smallest constant H such that (4) holds is said to be the *φ-Hölder modulus* of f and is denoted by $H_\varphi(f)$.

A relationship between the φ-Hölder property and φ-convexity is obtained in the following result.

Proposition 5. *Let $f : D \to \mathbb{R}$ be of φ-Hölder class on D. Then f is $\big(H_\varphi(f) \cdot \varphi\big)$-convex on D.*

Proof. Let $x, y \in D$ and let $t \in [0, 1]$. Then

$$\begin{aligned} &f(tx + (1-t)y) - tf(x) - (1-t)f(y) \\ &\quad = t\big(f(tx + (1-t)y) - f(x)\big) + (1-t)\big(f(tx + (1-t)y) - f(y)\big) \\ &\quad \le tH_\varphi(f)\varphi(\|tx + (1-t)y - x\|) + (1-t)H_\varphi(f)\varphi(\|tx + (1-t)y - y\|), \end{aligned}$$

which is equivalent to the $\big(H_\varphi(f) \cdot \varphi\big)$-convexity of f. □

For functions $\varphi : D^+ \to \mathbb{R}$, we introduce the following subadditivity-type property:

Definition 6. We say that φ is *increasingly subadditive on* D^+ if, for all $u, v, w \in D^+$ with $u \leq v + w$,

$$\varphi(u) \leq \varphi(v) + \varphi(w) \tag{5}$$

holds.

Clearly, if $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is nondecreasing and subadditive then it is also increasingly subadditive on $\mathbb{R}^+$.

Proposition 7. *Assume that $\varphi : D^+ \to \mathbb{R}$ is increasingly subadditive. Then, for all $z \in D$, the map $x \mapsto -\varphi(\|x - z\|)$ is of φ-Hölder class on D with φ-Hölder modulus* 1*, and therefore, it is also φ-convex on D.*

Proof. Let $z \in D$ be fixed. To prove the φ-Hölder property of the map $x \mapsto -\varphi(\|x - z\|)$, let $x, y \in D$. Then $u = \|x - z\|$, $v = \|x - y\|$, and $w = \|y - z\|$ are elements of D^+ such that (5) holds. Therefore, by the increasing subadditivity, we get

$$\varphi(\|x - z\|) - \varphi(\|y - z\|) \leq \varphi(\|x - y\|) + \varphi(\|y - z\|) - \varphi(\|y - z\|) = \varphi(\|x - y\|).$$

Interchanging x and y, we also have $\varphi(\|y - z\|) - \varphi(\|x - z\|) \leq \varphi(\|y - x\|)$. These two inequalities imply

$$\big|\varphi(\|x - z\|) - \varphi(\|y - z\|)\big| \leq \varphi(\|x - y\|),$$

which means that the map $x \mapsto -\varphi(\|x - z\|)$ is φ-Hölder on D with φ-Hölder modulus 1. □

The next lemma is well known, for completeness we provide its short proof.

Lemma 8. *Let $0 \leq p \leq 1$ be an arbitrary constant. Then the map $x \mapsto x^p$ is subadditive and nondecreasing on $\mathbb{R}_+$ and hence it is also increasingly subadditive on $\mathbb{R}^+$.*

Proof. For $s \in]0, 1[$, we have $s \leq s^p$. Hence

$$1 = s + (1 - s) \leq s^p + (1 - s)^p.$$

If $x, y \in \mathbb{R}_+$ then, with $s := \frac{x}{x+y} \in]0, 1[$, we get

$$1 \leq \Big(\frac{x}{x + y}\Big)^p + \Big(\frac{y}{x + y}\Big)^p,$$

which shows the subadditivity of the function $x \mapsto x^p$. □

Definition 9. Let $0 < p \leq 1$ be an arbitrary constant. For all $t \in D^+$ let $\varphi(t) := t^p$, then if $f : D \to \mathbb{R}$ is a φ-Hölder function, then it is called (classical) *p-Hölder functions*. In this case the φ-Hölder modulus is called *p-Hölder modulus*

of f and it is denoted by $H_p(f)$.

The next corollary gives a relationship between the p-Hölder functions and the p-convex functions.

Corollary 10. *Let $0 < p \leq 1$ be an arbitrary constant and $z \in X$. Then $x \mapsto -\|x - z\|^p$ is of p-Hölder class on X with the p-Hölder modulus 1, and therefore, it is $(1, p)$-convex on X.*

The subsequent theorem, which is one of the main results of this paper, offers equivalent conditions for φ-convexity. It generalizes the result of [30, Theorem 1].

Theorem 11. *Let D be an open real interval and $f : D \to \mathbb{R}$. Then the following conditions are equivalent.*

(i) *f is φ-convex on D.*

(ii) *For $x, u, y \in D$ with $x < u < y$,*

$$\frac{f(u) - f(x) - \varphi(u - x)}{u - x} \leq \frac{f(y) - f(u) + \varphi(y - u)}{y - u}. \tag{6}$$

(iii) *There exists a function $a : D \to \mathbb{R}$ such that, for $x, u \in D$,*

$$f(x) - f(u) \geq a(u)(x - u) - \varphi(|x - u|). \tag{7}$$

Proof. (i) $\Rightarrow$ (ii) Assume that $f : D \to \mathbb{R}$ is φ-convex and let $x < u < y$ be arbitrary elements of D. Choose $t \in [0, 1]$ such that $u = tx + (1 - t)y$, that is let $t := \frac{y-u}{y-x}$. Then, applying the φ-convexity of f, we get

$$f(u) \leq \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y) + \frac{y-u}{y-x} \varphi\left(\frac{u-x}{y-x}(y-x)\right) + \frac{u-x}{y-x} \varphi\left(\frac{y-u}{y-x}(y-x)\right),$$

which is equivalent to

$$(y-u)\big(f(u) - f(x) - \varphi(u-x)\big) \leq (u-x)\big(f(y) - f(u) + \varphi(y-u)\big).$$

Dividing by $(y - u)(u - x) > 0$, we arrive at (6).

(ii) $\Rightarrow$ (iii) Assume that (ii) holds and, for $u \in D$, define

$$a(u) := \inf_{y \in D, u < y} \frac{f(y) - f(u) + \varphi(y - u)}{y - u}.$$

Then in view of (ii), we get

$$\frac{f(x) - f(u) - \varphi(u - x)}{x - u} \leq a(u) \leq \frac{f(y) - f(u) + \varphi(y - u)}{y - u}, \tag{8}$$

for all $x < u < y$ in D. The left-hand side inequality in (8) yields (7) in the case $x < u$, and analogously, the right-hand side inequality (with the substitution $y := x$) reduces to (7) in the case $x > u$. The case $x = u$ is obvious.

(iii) $\Rightarrow$ (i) Let $x, y \in D$, $t \in [0,1]$, and set $u := tx + (1-t)y$. Then, by (iii), we have

$$f(x) - f(u) \geq a(u)(x-u) - \varphi(|x-u|), \qquad f(y) - f(u) \geq a(u)(y-u) - \varphi(|y-u|).$$

Multiplying the first inequality by t and the second inequality by $1-t$ and adding up the inequalities so obtained, we get (2). □

Remark 12. In the vector variable setting (i.e., when D is an open convex subset of a normed space X), instead of condition (iii), the following analogous property can be formulated:

(iii)* There exists a function $a : D \to X^*$ such that, for $x, u \in D$,

$$f(x) - f(u) \geq a(u)(x-u) - \varphi(\|x-u\|).$$

One can easily see (by the same argument as above) that (iii)* implies (i), that is, the φ-convexity of f. The validity of the reversed implication is an open problem.

The next theorem gives another characterization of φ-convex functions, if φ is increasingly subadditive.

Theorem 13. *Let D be an open real interval and let $\varphi : D^+ \to \mathbb{R}_+$ be increasingly subadditive. Then a function $f : D \to \mathbb{R}$ is φ-convex if and only if there exist two functions $a : D \to \mathbb{R}$ and $b : D \to \mathbb{R}$ such that*

$$f(x) = \sup_{u \in D} \big(a(u)x + b(u) - \varphi(|x-u|)\big), \tag{9}$$

for all $x \in D$.

Proof. Assume that f is φ-convex. By Theorem 11, there exists a function $a : D \to \mathbb{R}$ such that

$$f(x) \geq f(u) + a(u)(x-u) - \varphi(|x-u|),$$

for all $u, x \in D$. Define $b(u) := f(u) - a(u)u$, for $u \in D$. Thus, for $u, x \in D$,

$$f(x) \geq a(u)x + b(u) - \varphi(|x-u|)$$

and we have equality for $u = x$. Therefore, (9) holds.

Conversely, assume that (9) is valid for $x \in D$. By Proposition 7, for fixed $u \in D$, the mapping $x \mapsto -\varphi(|x-u|)$ is φ-convex. The map $x \mapsto a(u)x + b(u)$ is affine, and hence the function $f_u : D \to \mathbb{R}$ defined by $f_u(x) := a(u)x + b(u) - \varphi(|x-u|)$ is φ-convex for all fixed $u \in D$. Now applying (ii) of Proposition 2, we obtain that f is φ-convex. □

Remark 14. In the vector variable setting (i.e., when D is an open convex subset of a normed space X), the following implication can be formulated: If $\varphi : D^+ \to \mathbb{R}_+$ is increasingly subadditive and there exist two function $a : D \to X^*$ and $b : D \to \mathbb{R}$ such that, for $x \in D$,

$$f(x) = \sup_{u \in D} \big(a(u)(x) + b(u) - \varphi(\|x - u\|)\big),$$

then f is φ-convex. The validity of the reversed implication is an open problem.

Corollary 15. *Let D be an open real interval and let $0 < p \leq 1$ and $\varepsilon \geq 0$ be arbitrary constants. Then a function $f : D \to \mathbb{R}$ is (ε, p)-convex if and only if there exist two functions $a : D \to \mathbb{R}$ and $b : D \to \mathbb{R}$ such that*

$$f(x) = \sup_{u \in D} \big(a(u)x + b(u) - \varepsilon|x - u|^p\big),$$

for all $x \in D$.

The subsequent theorem offers a sufficient condition for the φ-midconvexity. The result is analogous to the implication (iii)⇒(i) of Theorem 11. Unfortunately, we were not able to obtain the necessity of this condition, i.e., the reversed implication.

Theorem 16. *Let $f : D \to \mathbb{R}$ and assume that, for all $u \in D$, there exists an additive function $A_u : X \to X$ such that*

$$f(x) - f(u) \geq A_u(x - u) - \varphi(\|x - u\|) \quad (x \in D). \tag{10}$$

Then, f is φ-midconvex.

Proof. Let $x, y \in D$ and set $u := \frac{x+y}{2}$. Then, by (10), we have

$$f(x) - f(u) \geq A_u(x - u) - \varphi(\|x - u\|) = A_u\left(\frac{x - y}{2}\right) - \varphi\left(\left\|\frac{x - y}{2}\right\|\right),$$

$$f(y) - f(u) \geq A_u(y - u) - \varphi(\|y - u\|) = A_u\left(\frac{y - x}{2}\right) - \varphi\left(\left\|\frac{y - x}{2}\right\|\right).$$

Adding up the inequalities and multiplying the inequality so obtained by $\frac{1}{2}$, we get (3). □

The following result is analogous to Theorem 13, however it offers only a sufficient condition for φ-midconvexity.

Theorem 17. *Let $\varphi : D^+ \to \mathbb{R}_+$ be increasingly subadditive and let $f : D \to \mathbb{R}$. Assume that, for all $u \in D$, there exists an additive function $A_u : X \to X$ and there exists a function $b : D \to \mathbb{R}$ such that*

$$f(x) = \sup_{u \in D} \big(A_u(x) + b(u) - \varphi(\|x - u\|)\big), \tag{11}$$

for all $x \in D$. Then f is φ-midconvex.

Proof. Assume that (11) is valid for $x \in D$. By Proposition 7, for fixed $u \in D$, the mapping $x \mapsto -\varphi(\|x - u\|)$ is φ-convex, so it is φ-midconvex. The map $x \mapsto A_u(x) + b(u)$ is affine, and hence the function $f_u : D \to \mathbb{R}$ defined by $f_u(x) := A_u(x) + b(u) - \varphi(\|x - u\|)$ is φ-midconvex for all fixed $u \in D$. Now applying (ii) of Proposition 3, we obtain that f is φ-midconvex. □

Henceforth we search for relations between the local upper-bounded φ-midconvex functions and φ-convex functions with the help of the results from the papers [14] and [18] by Házy and Páles.

Define the function $d_{\mathbb{Z}} : \mathbb{R} \to \mathbb{R}_+$ by

$$d_{\mathbb{Z}}(t) = \operatorname{dist}(t, \mathbb{Z}) := \min\{|t - k| : k \in \mathbb{Z}\}.$$

It is immediate to see that $d_{\mathbb{Z}}$ is 1-periodic and symmetric with respect to $t = 1/2$, i.e., $d_{\mathbb{Z}}(t) = d_{\mathbb{Z}}(1-t)$ holds for all $t \in \mathbb{R}$. For a fixed $\varphi : \frac{1}{2}D^+ \to \mathbb{R}_+$, we introduce the Takagi type function $\mathcal{T}_\varphi : \mathbb{R} \times D^+ \to \mathbb{R}_+$ by

$$\mathcal{T}_\varphi(t, u) := \sum_{n=0}^{\infty} \frac{\varphi\big(d_{\mathbb{Z}}(2^n t)u\big)}{2^n} \quad ((t, u) \in \mathbb{R} \times D^+). \tag{12}$$

Applying the estimate $0 \le d_{\mathbb{Z}} \le \frac{1}{2}$, one can easily see that $\mathcal{T}_\varphi(t, u) \le 2\varphi\big(\frac{u}{2}\big)$ for $u \in D^+$ whenever φ is nondecreasing.

For $p \ge 0$, we also define the Takagi type function $T_p : \mathbb{R} \to \mathbb{R}_+$ by

$$T_p(t) := \sum_{n=0}^{\infty} \frac{\big(d_{\mathbb{Z}}(2^n t)\big)^p}{2^n} \quad (t \in \mathbb{R}). \tag{13}$$

In the case when φ is of the form $\varphi(t) = \varepsilon|t|^p$ for some constants $\varepsilon \ge 0$ and $p \ge 0$, the following identity holds:

$$\mathcal{T}_\varphi(t, u) = \varepsilon T_p(t) u^p \quad ((t, u) \in \mathbb{R} \times D^+).$$

Observe that $\mathcal{T}_\varphi$ and T_p are also 1-periodic and symmetric with respect to $t = 1/2$ in their first variables.

In order to obtain lower and upper estimates for the functions $\mathcal{T}_\varphi$ and T_p defined above, we need to recall de Rham's classical theorem [31]. By $\mathcal{B}(\mathbb{R},\mathbb{R})$ we denote the space of bounded functions $f:\mathbb{R}\to\mathbb{R}$ equipped with the supremum norm.

Theorem 18. *Let $\psi\in\mathcal{B}(\mathbb{R},\mathbb{R})$, $a,b\in\mathbb{R}$, $|a|<1$. Let $F_\psi:\mathcal{B}(\mathbb{R},\mathbb{R})\to\mathcal{B}(\mathbb{R},\mathbb{R})$ be an operator defined as follows*

$$\big(F_\psi f\big)(t):=af(bt)+\psi(t)\quad \text{for } f\in\mathcal{B}(\mathbb{R},\mathbb{R}),\ t\in\mathbb{R}.$$

Then

(i) *F_ψ is a contraction on $\mathcal{B}(\mathbb{R},\mathbb{R})$ with a unique fixed point f_ψ which is given by the formula*

$$f_\psi(t)=\sum_{n=0}^{\infty}a^n\psi(b^nt)\qquad (t\in\mathbb{R});$$

(ii) *if $a\geq 0$ and the functions $g,h\in\mathcal{B}(\mathbb{R},\mathbb{R})$ satisfy the inequalities $g\leq F_\psi g$ and $F_\psi h\leq h$, then $g\leq f_\psi\leq h$.*

Remark 19. In view of the first assertion of this theorem, observe that the functions $\mathcal{T}_\varphi(\cdot,u)$ and T_p defined in (12) and (13) are the fixed points of the operator:

$$\big(F_\psi f\big)(t):=\frac{1}{2}f(2t)+\psi(t)\quad \text{for } f\in\mathcal{B}(\mathbb{R},\mathbb{R}),\ t\in\mathbb{R} \tag{14}$$

where $\psi\in\mathcal{B}(\mathbb{R},\mathbb{R})$ is given by $\psi(t):=\varphi\big(d_{\mathbb{Z}}(t)u\big)$ and $\psi(t):=\big(d_{\mathbb{Z}}(t)\big)^p$, respectively.

In the results below, we establish upper and lower bounds for $\mathcal{T}_\varphi$ in terms of the function $\tau_\varphi:\mathbb{R}\times D^+\to\mathbb{R}$ defined by

$$\tau_\varphi(t,u):=d_{\mathbb{Z}}(t)\varphi\big((1-d_{\mathbb{Z}}(t))u\big)+(1-d_{\mathbb{Z}}(t))\varphi\big(d_{\mathbb{Z}}(t)u\big)\quad ((t,u)\in\mathbb{R}\times D^+).$$

Observe that, for $t\in[0,1]$, we have

$$\tau_\varphi(t,u):=t\varphi\big((1-t)u\big)+(1-t)\varphi\big(tu\big)\quad (u\in D^+),$$

which is exactly the error term related to φ-convexity.

Proposition 20. *Let $\varphi:D^+\to\mathbb{R}_+$ be subadditive. Then, for all $(t,u)\in\mathbb{R}\times D^+$,*

$$\tau_\varphi(t,u)\leq\mathcal{T}_\varphi(t,u). \tag{15}$$

PROOF. Let $u \in D^+$ be arbitrarily fixed. By the 1-periodicity and symmetry with respect to the point $t = 1/2$, it suffices to show that (15) holds for all $t \in \left[0, \frac{1}{2}\right]$. If $t = 0$ then (15) is obvious. Now assume that $0 < t \leq \frac{1}{2}$. Then there exists a unique $k \in \mathbb{N}$ such that $\frac{1}{2^{k+1}} < t \leq \frac{1}{2^k}$. Then, one can easily see that

$$d_{\mathbb{Z}}(t) = t, \quad d_{\mathbb{Z}}(2t) = 2t, \quad \ldots, \quad d_{\mathbb{Z}}(2^{k-1}t) = 2^{k-1}t, \quad d_{\mathbb{Z}}(2^k t) = 1 - 2^k t. \quad (16)$$

On the other hand, by the well-known identity $\sum_{j=0}^{k-1} 2^j = 2^k - 1$, we have

$$(1-t)u = tu + 2tu + \cdots + 2^{k-1}tu + (1 - 2^k t)u.$$

Then, by the subadditivity of φ, and by $t \leq \frac{1}{2^k} < \frac{1}{2^{k-1}} < \cdots < \frac{1}{2}$, it follows that

$$\begin{aligned} t\varphi((1-t)u) &\leq t\varphi(tu) + t\varphi(2tu) + \cdots + t\varphi(2^{k-1}tu) + t\varphi((1-2^k t)u) \\ &\leq t\varphi(tu) + \frac{\varphi(2tu)}{2} + \cdots + \frac{\varphi(2^{k-1}tu)}{2^{k-1}} + \frac{\varphi((1-2^k t)u)}{2^k}. \end{aligned}$$

Adding $(1-t)\varphi(tu)$ to the previous inequality and using (16), we get

$$\begin{aligned} \tau_\varphi(t,u) := t\varphi((1-t)u) + (1-t)\varphi(tu) &\leq \varphi(tu) + \frac{\varphi(2tu)}{2} + \ldots \\ + \frac{\varphi(2^{k-1}tu)}{2^{k-1}} + \frac{\varphi((1-2^k t)u)}{2^k} &= \sum_{j=0}^{k} \frac{\varphi(d_{\mathbb{Z}}(2^j t)u)}{2^j} \leq \mathcal{T}_\varphi(t,u). \end{aligned}$$

Which completes the proof of (15). □

Proposition 21. *Let $\varphi : D^+ \to \mathbb{R}_+$ be nondecreasing with $\varphi(s) > 0$ for $s > 0$ and assume that*

$$\gamma_\varphi := \sup_{0 < s \in \frac{1}{2}D^+} \frac{\varphi(2s)}{\varphi(s)} < 2.$$

Then, for all $(t,u) \in \mathbb{R} \times D^+$,

$$\mathcal{T}_\varphi(t,u) \leq \frac{2}{2-\gamma_\varphi}\tau_\varphi(t,u) \quad (17)$$

holds.

PROOF. To prove (17), we fix an arbitrary element $u \in D^+$. By Remark 19, the function $\mathcal{T}_\varphi(\cdot, u)$ is the fixed point of the operator

$$(F_\varphi f)(t) = \frac{1}{2}f(2t) + \varphi(d_{\mathbb{Z}}(t)u).$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) := \frac{2}{2-\gamma_\varphi}\tau_\varphi(t,u)$. In view of Theorem 18, in order to prove inequality (17), it is enough to show that

$$(F_\varphi g)(t) \le g(t) \qquad (t \in \mathbb{R}). \tag{18}$$

Since g is periodic by 1 and symmetric with respect to $t = 1/2$, it suffices to prove that (18) is satisfied on $\left[0, \frac{1}{2}\right]$. Trivially, $\gamma_\varphi \ge 1$, hence the inequality (18) is obvious for $t = 0$ or for $u = 0$. Thus, we may assume that $u > 0$ and $0 < t \le \frac{1}{2}$. By the definition of the constant γ_φ, we have that

$$\varphi(tu)\Big(1 - \frac{\gamma_\varphi}{2}\Big) \le \varphi(tu) - \frac{\varphi(2tu)}{2}. \tag{19}$$

Since $t \le 2t$ and $1 - 2t \le 1 - t$ and φ is nondecreasing we also have that

$$0 \le t(\varphi((1-t)u) - \varphi((1-2t)u)) + t\varphi(2tu) - t\varphi(tu).$$

Adding $\varphi(tu) - \frac{\varphi(2tu)}{2}$ to the previous inequality and also using (19), we obtain

$$\begin{aligned}\varphi(tu)\Big(1 - \frac{\gamma_\varphi}{2}\Big) &\le \varphi(tu) - \frac{\varphi(2tu)}{2} \\ &\le t\big(\varphi((1-t)u) - \varphi((1-2t)u)\big) + (1-t)\varphi(tu) - \Big(\frac{1}{2} - t\Big)\varphi(2tu).\end{aligned}$$

Rearranging this inequality, we finally obtain that

$$\begin{aligned}\frac{1}{2-\gamma_\varphi}\big(2t\varphi\big((1-2t)u\big) + (1-2t)\varphi\big(2tu\big)\big) + \varphi(tu) \\ \le \frac{2}{2-\gamma_\varphi}\big(t\varphi\big((1-t)u\big) + (1-t)\varphi\big(tu\big)\big),\end{aligned}$$

which means that (18) is satisfied for all $0 < t \le \frac{1}{2}$. □

Let μ be a nonnegative finite Borel measure on $[0,1]$ and let $\operatorname{supp}\mu$ denote the support of μ.

Lemma 22. *Let μ be a nonnegative and nonzero finite Borel measure on $[0,1]$ and let $\chi :]0,\infty[\to \mathbb{R}_+$ be defined by*

$$\chi(s) = \frac{\int_{[0,1]}(2s)^p d\mu(p)}{\int_{[0,1]} s^p d\mu(p)}.$$

Then χ is nondecreasing on $]0,\infty[$ and

$$\lim_{s\to\infty} \chi(s) = 2^{p_0}, \tag{20}$$

where $p_0 := \sup(\operatorname{supp}\mu)$.

PROOF. The function $x \mapsto 2^x$ is strictly increasing, hence, for $p, q \in \mathbb{R}$, we have $(2^p - 2^q)(p - q) \geq 0$. It suffices to show that $\chi' \geq 0$. For $s > 0$, we obtain

$$\begin{aligned}
\chi'(s) &= \frac{\int_{[0,1]} 2^p p s^{p-1} d\mu(p) \cdot \int_{[0,1]} s^p d\mu(p) - \int_{[0,1]} 2^p s^p d\mu(p) \cdot \int_{[0,1]} p s^{p-1} d\mu(p)}{\big(\int_{[0,1]} s^p d\mu(p)\big)^2} \\
&= \frac{\int_{[0,1]} 2^p p s^{p-1} d\mu(p) \cdot \int_{[0,1]} s^q d\mu(q) + \int_{[0,1]} 2^q q s^{q-1} d\mu(q) \cdot \int_{[0,1]} s^p d\mu(p)}{2\big(\int_{[0,1]} s^p d\mu(p)\big)^2} \\
&\quad - \frac{\int_{[0,1]} 2^p s^p d\mu(p) \cdot \int_{[0,1]} q s^{q-1} d\mu(q) + \int_{[0,1]} 2^q s^q d\mu(q) \cdot \int_{[0,1]} p s^{p-1} d\mu(p)}{2\big(\int_{[0,1]} s^p d\mu(p)\big)^2} \\
&= \frac{\int_{[0,1]} \int_{[0,1]} (2^p - 2^q)(p - q) s^{p+q-1} d\mu(p) d\mu(q)}{2\big(\int_{[0,1]} s^p d\mu(p)\big)^2} \geq 0,
\end{aligned}$$

which proves that χ is nondecreasing.

Using $\operatorname{supp} \mu \subseteq [0, p_0]$, for $s > 0$, we obtain

$$\int_{[0,1]} s^p d\mu(p) = \int_{[0,1]} 2^p \Big(\frac{s}{2}\Big)^p d\mu(p) \leq 2^{p_0} \int_{[0,1]} \Big(\frac{s}{2}\Big)^p d\mu(p),$$

which proves that $\chi(s) \leq 2^{p_0}$, and hence, $\lim_{s\to\infty} \chi(s) \leq 2^{p_0}$.

To show that in (20) the equality is valid, assume that $\lim_{s\to\infty} \chi(s) < 2^{p_0}$. Choose $q < q_0 < p_0$ so that $\lim_{s\to\infty} \chi(s) \leq 2^q$. Then, for all $s > 0$,

$$\int_{[0,1]} (2s)^p d\mu(p) \leq 2^q \int_{[0,1]} s^p d\mu(p),$$

i.e., for all $s \geq 1$,

$$\begin{aligned}
0 &\leq \int_{[0,1]} (2^q - 2^p) s^p d\mu(p) \\
&= \int_{[0,q[} (2^q - 2^p) s^p d\mu(p) + \int_{[q,q_0[} (2^q - 2^p) s^p d\mu(p) + \int_{[q_0,1]} (2^q - 2^p) s^p d\mu(p) \\
&\leq \int_{[0,q[} (2^q - 2^p) s^p d\mu(p) + \int_{[q_0,1]} (2^q - 2^p) s^p d\mu(p) \\
&\leq \int_{[0,q[} (2^q - 2^p) s^p d\mu(p) + \int_{[q_0,1]} (2^q - 2^p) s^{q_0} d\mu(p).
\end{aligned}$$

Therefore, for $s \geq 1$,

$$0 \leq \int_{[0,q[} (2^q - 2^p) s^{p-q_0} d\mu(p) + \int_{[q_0,1]} (2^q - 2^p) d\mu(p).$$

The first integrand converges uniformly to 0 on $[0,q[$ as $s\to\infty$. Thus, by taking the limit $s\to\infty$, we get

$$0\le\int_{[q_0,1]}(2^q-2^p)d\mu(p). \tag{21}$$

On the other hand, the inequality $q_0<p_0=\sup(\operatorname{supp}\mu)$ implies $\mu([q_0,1])>0$ and, obviously, $2^q-2^p<0$ for $p\in[q_0,1]$. Hence the right hand side of (21) is negative. The contradiction so obtained proves (20). □

Proposition 23. *Let μ be a nonnegative and nonzero finite Borel measure on $[0,1]$. Denote $\alpha:=\sup D^+$ and $p_0:=\sup(\operatorname{supp}\mu)$ and define $\varphi:D^+\to\mathbb{R}_+$ by*

$$\varphi(s):=\int_{[0,1]}s^p d\mu(p)\quad \textit{for all } s\in D^+.$$

Then φ is subadditive and nondecreasing, furthermore,

$$\gamma_\varphi=\begin{cases}\dfrac{\int_{[0,1]}\alpha^p d\mu(p)}{\int_{[0,1]}(\alpha/2)^p d\mu(p)}, & \text{if } \alpha<\infty,\\[2ex] 2^{p_0}, & \text{if } \alpha=\infty\end{cases} \tag{22}$$

and $\gamma_\varphi<2$ if either $\alpha<\infty$ and μ is not concentrated at the singleton $\{1\}$ or $p_0<1$. In addition, for all $t\in[0,1]$ and $u\in D^+$,

$$\int_{[0,1]}\big[t(1-t)^p+(1-t)t^p\big]u^p d\mu(p)\le\int_{[0,1]}T_p(t)u^p d\mu(p) \tag{23}$$

and, provided that $\gamma_\varphi<2$,

$$\int_{[0,1]}T_p(t)u^p d\mu(p)\le\frac{2}{2-\gamma_\varphi}\int_{[0,1]}\big[t(1-t)^p+(1-t)t^p\big]u^p d\mu(p). \tag{24}$$

Proof. It can be easily seen that φ is nondecreasing. The subadditivity is a consequence of Lemma 8.

Let $\alpha<\infty$. Then, by Lemma 22, the map $s\mapsto\frac{\varphi(2s)}{\varphi(s)}=\frac{\int_{[0,1]}(2s^p)d\mu(p)}{\int_{[0,1]}s^p d\mu(p)}=\chi(s)$ is nondecreasing on $\frac12 D^+$, so it attains its supremum at $\alpha/2$. Thus, in this case, $\gamma_\varphi=\frac{\int_{[0,1]}\alpha^p d\mu(p)}{\int_{[0,1]}(\alpha/2)^p d\mu(p)}$. To prove that $\gamma_\varphi<2$, we use the inequality $2^p<2$ for $p\in[0,1]$ to obtain:

$$\int_{[0,1]}\alpha^p d\mu(p)=\int_{[0,1]}2^p\Big(\frac{\alpha}{2}\Big)^p d\mu(p)<2\int_{[0,1]}\Big(\frac{\alpha}{2}\Big)^p d\mu(p).$$

In the case $\alpha = \infty$, by Lemma 22, we have that $\gamma_\varphi = \lim_{s\to\infty} \chi(s) = 2^{p_0}$. Obviously, $\gamma_\varphi < 2$ if $p_0 < 1$.

The inequalities (23) and (24) are immediate consequences of Proposition 20 and Proposition 21, respectively. □

In the case when the measure μ is concentrated at a singleton $\{p\}$, Proposition 23 simplifies to the following result.

Corollary 24. *Let $0 \le p \le 1$ be an arbitrary constant. Then, for all $t \in [0,1]$,*

$$t(1-t)^p + (1-t)t^p \le T_p(t)$$

and, provided that $p < 1$,

$$T_p(t) \le \frac{2}{2-2^p}\big(t(1-t)^p + (1-t)t^p\big).$$

The proof of the next theorem is analogous to that in [18].

Theorem 25. *Let $\varphi : D^+ \to \mathbb{R}_+$ be nondecreasing. If $f : D \to \mathbb{R}$ is φ-midconvex and locally bounded from above at a point of D, then f is locally bounded from above on D.*

The following theorem generalizes the analogous result of the paper [18] obtained for (ε, p)-convexity. A similar result was also established by Tabor and Tabor [36], [37].

Theorem 26. *Let $f : D \to \mathbb{R}$ be locally bounded from above at a point of D and let $\varphi : \frac{1}{2}D^+ \to \mathbb{R}_+$ be nondecreasing. Then f is φ-midconvex on D, i.e.,* (3) *holds for all $x, y \in D$ if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\varphi(t, \|x-y\|) \tag{25}$$

for all $x, y \in D$ and $t \in [0,1]$.

Proof. Assume that f is φ-midconvex on D and locally bounded from above at a point of D. From Theorem 25, it follows that f is locally bounded from above at each point of D. Thus f is bounded from above on each compact subset of D, in particular, for each fixed $x, y \in D$, f is bounded from above on $[x,y] = \{tx + (1-t)y \mid t \in [0,1]\}$. Denote by $K_{x,y}$ a finite upper bound of the function

$$t \mapsto f(tx + (1-t)y) - tf(x) - (1-t)f(y) \qquad (t \in [0,1]). \tag{26}$$

We are going to show, by induction on n, that

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\frac{K_{x,y}}{2^n}+\sum_{j=0}^{n-1}\frac{\varphi\big(d_{\mathbb{Z}}(2^j t)\|x-y\|\big)}{2^j} \tag{27}$$

for all $x, y \in D$ and $t \in [0,1]$. For $n=0$, the statement follows from the definition of $K_{x,y}$ (with the convention that the summation for $j=0$ to (-1) is equal to zero).

Now assume that (27) is true for some $n \in \mathbb{N}$. Assume that $t \in [0,1/2]$. Then, due to the φ-midconvexity of f, we get

$$\begin{aligned} f(tx+(1-t)y) &= f\left(\frac{y+(2tx+(1-2t)y)}{2}\right) \\ &\le \frac{f(y)+f(2tx+(1-2t)y)}{2}+\varphi(t\|x-y\|). \end{aligned}$$

On the other hand, by (27), we get that

$$f(2tx+(1-2t)y) \le 2tf(x)+(1-2t)f(y)+\frac{K_{x,y}}{2^n}+\sum_{j=0}^{n-1}\frac{\varphi\big(d_{\mathbb{Z}}(2^{j+1}t)\|x-y\|\big)}{2^j}.$$

Combining these two inequalities, we obtain

$$\begin{aligned} f(tx+(1-t)y) &\le tf(x)+(1-t)f(y)+\frac{1}{2}\left(\frac{K_{x,y}}{2^n}+\sum_{j=0}^{n-1}\frac{\varphi\big(d_{\mathbb{Z}}(2^{j+1}t)\|x-y\|\big)}{2^j}\right) \\ &+\varphi(t\|x-y\|) = tf(x)+(1-t)f(y)+\frac{K_{x,y}}{2^{n+1}}+\sum_{j=0}^{n}\frac{\varphi\big(d_{\mathbb{Z}}(2^j t)\|x-y\|\big)}{2^j}. \end{aligned}$$

In the case $t \in [1/2,1]$, the proof is similar. Thus, (27) is proved for all $n \in \mathbb{N}$.

Finally, taking the limit $n \to \infty$ in (27), we get the desired inequality (25).

To see that (25) implies the φ-midconvexity of f, substitute $t=1/2$ into (25) and use the easy-to-see identity $\mathcal{T}_\varphi\big(\frac{1}{2},u\big)=\varphi(\frac{|u|}{2})$ $(u \in \mathbb{R})$. □

The optimality of the error term in (25) and the appropriate convexity properties of $\mathcal{T}_\varphi$ have recently been obtained in [23].

Theorem 27. *Let $\varphi : D^+ \to \mathbb{R}_+$ be nondecreasing with $\varphi(s)>0$ for $s>0$ and assume that $\gamma_\varphi := \sup_{0<s\in\frac{1}{2}D^+}\frac{\varphi(2s)}{\varphi(s)} < 2$. If $f : D \to \mathbb{R}$ is locally bounded from above a point of D and it is also φ-midconvex, then f is $\big(\frac{2}{2-\gamma_\varphi}\cdot\varphi\big)$-convex on D.*

PROOF. By Proposition 21 and by Theorem 26, the proof of this theorem is evident. □

Corollary 28. *Let μ be a nonnegative and nonzero finite Borel measure on $[0,1]$. Denote $\alpha := \sup D^+$ and $p_0 := \sup(\operatorname{supp}\mu)$ and assume that either $\alpha < \infty$ and μ is not concentrated at the singleton $\{1\}$ or $p_0 < 1$. Define $\varphi : D^+ \to \mathbb{R}_+$ by*

$$\varphi(s) := \int_{[0,1]} s^p d\mu(p) \quad \text{for all} \quad s \in D^+.$$

If $f : D \to \mathbb{R}$ is locally bounded from above a point of D and it is also φ-midconvex, then f is $\left(\frac{2}{2-\gamma_\varphi} \cdot \varphi\right)$-convex on D, where γ_φ is given by (22).

Corollary 29. *Let $0 \le p < 1$ and $\varepsilon \ge 0$ be arbitrary constants. If $f : D \to \mathbb{R}$ is locally bounded from above a point of D and it is also (ε, p)-midconvex, then f is $\left(\frac{2\varepsilon}{2-2^p}, p\right)$-convex on D.*

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