Publ. Math. Debrecen
80/1-2 (2012), 143–154
DOI: 10.5486/PMD.2012.4970

General and alien solutions of a functional equation and of a functional inequality

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Abstract. The purpose of the present paper is to solve (under some assumption on the domain) the equation

$$g(x+y) - g(x) - g(y) = xf(y) + yf(x).$$

After determining the general solutions, we will investigate the so-called alien solutions. Finally, we will discuss the real solutions of the following related functional inequality:

$$g(x+y) - g(x) - g(y) \geq xf(y) + yf(x).$$

1. Introduction

In mathematics there exist several notions concerning functions that are defined through two or more identities. For example, if P and Q are rings, then the function $f : P \to Q$ is termed a *homomorphism* between P and Q if it is additive and multiplicative, i.e. if

$$f(x+y) = f(x) + f(y) \quad (x, y \in P) \tag{1.1}$$

Mathematics Subject Classification: 39B52, 39B62, 39B72.
Key words and phrases: functional equation, functional inequality, additive function, integral domain.
The research of the second author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 81402 and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund and the European Regional Development Fund.

and

$$f(xy) = f(x)f(y) \quad (x, y \in P) \tag{1.2}$$

Another example is, for instance, the notion of derivations. Let Q be a ring and P be its subring. A function $f : P \to Q$ is called a *derivation* if it is additive and

$$f(xy) = xf(y) + f(x)y \quad (x, y \in P)\,.$$

The following question naturally arises: Is it possible to characterize such type of functions via a single equation? This problem was firstly investigated by J. Dhombres in [3]. He examined the equation

$$af(xy) + bf(x)f(y) + cf(x+y) + d\,(f(x) + f(y)) = 0,$$

where the unknown function f maps a ring X to a field Y and the constants a, b, c and d belong to the center of Y.

Ten years later, in 1998 R. Ger succeed to strengthen the results of [3]. In the paper [9] he proved several statements concerning the following equation which is the sum of (1.1) and (1.2):

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y).$$

In this direction some further results can be found in Ger [10] and in Ger–Reich [11].

Similarly to the notion of homomorphisms, derivations can be characterized analogously. For example, in [12] the functional equation

$$f(x+y) - f(x) - f(y) = g(xy) - xg(y) - yg(x)$$

is solved under the assumption that the domain of the functions f and g is a commutative field and the range of these functions is a vector space over this field.

In parallel, several authors discussed various versions of the following functional inequality:

$$g(x+y) - g(x) - g(y) \geq \phi(x, y),$$

with some additional assumptions upon g and ϕ (see Baron–Kominek [1], Choczewski–Girgensohn–Kominek [2], Renardy [17], see also [6], [7], [8]). In Section 4 as a special case of ϕ we take $\phi(x, y) = f(x)y + yf(x)$ with f satisfying certain further conditions.

In this paper we will continue the above–mentioned research and we will examine the functional equation

$$g(x+y) - g(x) - g(y) = xf(y) + yf(x), \tag{$*$}$$

where the unknown functions f and g are defined on an integral domain. Firstly, we will find the general solution of equation $(*)$. After that, we will study the following problem. Let X be a ring. It is obvious that in case the function $g : X \to X$ is additive and the function $f : X \to X$ fulfills

$$xf(y) + yf(x) = 0 \quad (x, y \in X), \tag{1.3}$$

then equation $(*)$ holds. These solutions are the so-called *alien solutions* of the equation in question. We will point out that equation $(*)$ has solutions that are not alien (in the above sense). Moreover, we will also give necessary and sufficient conditions on the functions f and g to be alien solutions of the equation in question.

In last section we confine ourselves to the following functional inequality

$$g(x+y) - g(x) - g(y) \geq xf(y) + yf(x), \tag{$**$}$$

where the unknown functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy some additional technical assumptions.

Finally, let us mention that functional equations, similar to equation $(*)$ were considered by several authors. For example in Ebanks–Kannappan–Sahoo [4] the authors characterize all functions $f : \mathbb{K} \to G$ for which $f(x+y) - f(x) - f(y)$ depends only on the product xy for all $x, y \in \mathbb{K}$, where $\mathbb{K}$ is a commutative field and G is a uniquely q-divisible abelian group. In Ebanks [5] the equation

$$f(x+y) - f(x) - f(y) = g\left(H(x,y)\right)$$

is investigated, where the unknown functions f, g defined on a nonvoid interval $I \subset \mathbb{R}$ and $H\left(I \times I\right)$, respectively, satisfy some mild regularity conditions and the given function H fulfills some stronger regularity assumption. Furthermore, we also note that in Járai–Maksa–Páles [13] the authors described all Cauchy differences that can be written as a quasisum, i.e. they have dealt with the functional equation

$$f(x+y) - f(x) - f(y) = \alpha\left(\beta(x) + \beta(y)\right)$$

for the unknown function $f : I \to \mathbb{R}$, and it is solved under the supposition that the functions α and β are strictly monotonic.

Let us emphasize that in our Theorem 3.1 below *no regularity assumption* is involved. Furthermore, we will work in a quite general framework concerning the domain and the target space of the unknown functions.

2. Preliminaries

In this section we will fix the notations and the terminology that will be used subsequently. We refer the reader to the monographs of Kuczma [15] and of Shafarevich [19].

Definition 2.1. By an *integral domain* we understand a commutative unitary ring that contains no zero divisors.

The following notion will also be used in the next section.

Definition 2.2. Let n be a positive integer and G an abelian group. An element $x \in G$ is said to be *divisible by* n if there is $y \in G$ such that $x = ny$.

Lemma 2.1. *Let X be an integral domain and assume that the function $f : X \to X$ fulfills* (1.3)*. Then the function $2f : X \to X$ is identically zero.*

Proof. First let us substitute $x \to 1$ and $y \to 1$ to derive $2f(1) = 0$. Further, with the substitution $y \to 1$, the above equation yields that

$$2xf(1) + 2f(x) = 0$$

is satisfied for all $x \in X$. Due to $2f(1) = 0$, we obtain that $2f$ is identically zero on X, as claimed. □

Remark 2.1. In general it is not true that $f = 0$ in the foregoing lemma (however, under additional assumption of the divisibility by 2 postulated only for a single element $f(1)$ one can easily obtain $f = 0$). If one take $X = \mathbb{Z}_2$ and consider the mapping $f_1(x) = x$ then it is easy to check that this functions provides (nonzero) solutions of the equation. On the other hand, the maps $f_2(x) = 1$ and $f_3(x) = x + 1$ (the remaining nonzero self-mappings on X) do not solve it.

Let us also mention the following easily to verify result (the converse implication of a theorem due to Jessen–Karpf–Thorup [14, Theorem 2]).

Theorem 2.2. *Let X be an Abelian group and $f : X \to X$ an arbitrary function. Then the function $F : X \times X \to X$ defined by*

$$F(x, y) = f(x + y) - f(x) - f(x) \quad (x, y \in X)$$

is symmetric, i.e.,

$$F(x, y) = F(y, x) \quad (x, y \in X)$$

and fulfills the co–cycle equation, that is,

$$F(x + y, z) + F(x, y) = F(x, y + z) + F(y, z)$$

holds for all $x, y, z \in X$.

Finally, we will need the following two results. Recall that a map $f : X \to \mathbb{R}$ defined on an Abelian goup X is subadditive if

$$f(x+y) \le f(x) + f(y)$$

for all $x, y \in X$.

Corollary 2.1 ([6], Corollary 1). Assume that X is an Abelian group, $f : X \to \mathbb{R}$ and $\phi : X \times X \to \mathbb{R}$ satisfy

$$f(x+y) - f(x) - f(y) \ge \phi(x,y) \quad (x, y \in X), \tag{2.1}$$

$$\phi(x,-y) \ge -\phi(x,y) \quad (x, y \in X), \tag{2.2}$$

$$\begin{cases} \limsup_{n\to+\infty}, \dfrac{1}{4^n}\phi(2^n x, 2^n x) < +\infty & (x \in X), \\ \liminf_{n\to+\infty} \dfrac{1}{4^n}\phi(2^n x, 2^n y) \ge \phi(x,y) & (x, y \in X) \end{cases} \tag{2.3}$$

and

$$\phi(-x,-y) = \phi(x,y) \quad (x, y \in X). \tag{2.4}$$

Then there exists a subadditive function $A : X \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x,x) - A(x) \quad (x \in X).$$

Moreover, ϕ is biadditive and symmetric.

Corollary 2.2 ([7], Corollary 8). Assume X to be uniquely 2-divisible Abelian group and that $f : X \to \mathbb{R}$, $\phi : X \times X \to \mathbb{R}$ satisfy (2.1), (2.2),

$$\phi(2x, 2x) \le 4\phi(x,x) \quad (x \in X) \tag{2.5}$$

jointly with

$$\forall_{x\in X}\exists_{k_0\in\mathbb{N}}\forall_{k\ge k_0} f\left(\frac{x}{2^k}\right) + f\left(-\frac{x}{2^k}\right) \ge 0. \tag{2.6}$$

Then there exists an additive function $a : X \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x,x) + a(x) \quad (x \in X).$$

Moreover, ϕ is biadditive and symmetric.

3. Functional equation $(*)$

The main result in this section is the following:

Theorem 3.1. *Let X be an integral domain. Then the functions $f, g : X \to X$ fulfill functional equation $(*)$ for all $x, y \in X$, if and only if, there exist two mappings $A_1, A_2 : X \to X$ and a constant $c \in X$ such that A_1 and $2A_2$ are additive and*

$$4f(x) = 2A_1(x) + 2cx^2 \qquad (x \in X), \tag{3.1}$$

$$6g(x) = A_2(x) + 3xA_1(x) + cx^3 \qquad (x \in X). \tag{3.2}$$

Proof. The *if* part is a straightforward computation and therefore we will confine ourselves to the *only if* part.

First, observe that substitution $y \to 0$ shows that $-g(0) = xf(0)$ for each $x \in X$ which easily implies that

$$f(0) = g(0) = 0.$$

Next, apply equation $(*)$ with $y \to x$ to obtain

$$g(2x) - 2g(x) = 2xf(x) \quad (x \in X). \tag{3.3}$$

Now, let us define four new functions $f_o, f_e, g_o, g_e : X \to X$ by the following formulas:

$$f_o(x) = f(x) - f(-x), \qquad f_e(x) = f(x) + f(-x) \quad (x \in X);$$

$$g_o(x) = g(x) - g(-x), \qquad g_e(x) = g(x) + g(-x) \quad (x \in X).$$

Replace in $(*)$ x by $-x$ and y by $-y$, respectively, to arrive at

$$g(-x-y) - g(-x) - g(-y) = -xf(-y) - yf(-x) \quad (x, y \in X).$$

By adding and subtracting this equality and $(*)$ side-by-side we deduce the following two equalities:

$$g_e(x+y) - g_e(x) - g_e(y) = xf_o(y) + yf_o(x) \qquad (x, y \in X); \tag{3.4}$$

$$g_o(x+y) - g_o(x) - g_o(y) = xf_e(y) + yf_e(x) \qquad (x, y \in X). \tag{3.5}$$

On the other hand, substitution $y \to -x$ in $(*)$ leads to

$$g_e(x) = xf_o(x) \qquad (x \in X). \tag{3.6}$$

Further, substitution $x \to 2x$ and $y \to -x$ together with (3.3) gives us the equality

$$g_e(x) + 2xf_e(x) = xf(2x) \qquad (x \in X).$$

This and identity (3.6) prove that

$$f(2x) = 3f(x) + f(-x) \quad (x \in X). \tag{3.7}$$

Further, this implies the following properties of the functions f_o and f_e:

$$f_o(2x) = 2f_o(x) \quad \text{and} \quad f_e(2x) = 4f_e(x) \quad (x \in X). \tag{3.8}$$

Now, join (3.4) with (3.6) to deduce

$$(x+y)f_o(x+y) = (x+y)[f_o(x) + f_o(y)] \quad (x, y \in X),$$

which together with the fact that X is an integral domain, imply that f_o is additive. Thus there exists an additive function $A_1 : X \to X$ such that

$$f_o(x) = A_1(x) \quad (x \in X).$$

Additionally, using (3.6) we also get that

$$g_e(x) = xA_1(x) \quad (x \in X).$$

It remains to solve equation (3.5). For our convenience let us denote the Cauchy difference of g_o by C, that is, let

$$C(x, y) = g_o(x+y) - g_o(x) - g_o(y) \quad (x, y \in X).$$

Due to Theorem 2.2 the function C fulfills the co–cycle equation

$$C(x+y, z) + C(x, y) = C(x, y+z) + C(y, z) \quad (x, y, z \in X).$$

Comparing this with the right hand side of (3.5), after some rearrangements we arrive at

$$x[f_e(y+z) - f_e(y) - f_e(z)] = z[f_e(x+y) - f_e(x) - f_e(y)] \quad (x, y, z \in X).$$

Apply this for for $z \to y$ and use the second equality from (3.8) to deduce the following relation

$$2xf_e(y) = y[f_e(x+y) - f_e(x) - f_e(y)] \quad (x, y \in X).$$

If we multiply both sides of the foregoing formula by x, then the following equality:

$$2x^2 f_e(y) = xy[f_e(x+y) - f_e(x) - f_e(y)] \quad (x, y \in X)$$

can be derived. Let us observe that the right hand side of this equation is symmetric in x and y. Therefore, so is the left hand side. This implies however that

$$2x^2 f_e(y) = 2y^2 f_e(x)$$

hold for any $x \in X$. If we substitute $y \to 1$ then we see that

$$2f_e(x) = 2cx^2$$

for each $x \in X$, where $c = f_e(1)$.

To finish the proof we need to determine the function g_o. In view of the above representation of the function f_e, equation (3.5) turns into

$$2[g_o(x+y) - g_o(x) - g_o(y)] = 2cxy(x+y) \quad (x, y \in X). \tag{3.9}$$

Define the function $A_2 : X \to X$ through the formula

$$A_2(x) = 3g_o(x) - cx^3 \quad (x \in X),$$

(the constant c is the same as above). A direct calculation shows that in this case equation (3.9) yields that the function $2A_2$ is additive. Therefore

$$3g_o(x) = A_2(x) + cx^3 \quad (x \in X).$$

To conclude the proof it suffices to use the above results concerning the functions f_o, f_e, g_o and g_e jointly with the fact that

$$2f(x) = f_e(x) + f_o(x) \quad \text{and} \quad 2g(x) = g_e(x) + g_o(x) \quad (x \in X). \qquad \square$$

If we assume additionally that the ring X appearing in Theorem 3.1 is uniquely divisible by 2 and 3 then formulas (3.1) and (3.2) can be simplified. We have the following corollary.

Corollary 3.1. Let X be an integral domain which is uniquely divisible by 2 and 3 and assume that equation $(*)$ holds for $f, g : X \to X$. Then, and only then, there exist two additive mappings $A_1, A_2 : X \to X$ and a constant $c \in X$ such that

$$f(x) = A_1(x) + cx^2 \qquad (x \in X), \tag{3.10}$$

$$g(x) = A_2(x) + xA_1(x) + \frac{1}{3}cx^3 \qquad (x \in X). \tag{3.11}$$

Remark 3.1. We are grateful to Professor Maciej Sablik for a remark that the foregoing corollary can be deuced from more general lemmas from papers M. Sablik [18, Lemma 2.3] and A. Lisak, M. Sablik [16, Lemma 1]. More precisely, these general results imply that each solution of equation $(*)$ is a polynomial function of some degree. What remains to be done is to calculate the exact form of this polynomial function. However, unlike to our Theorem 3.1 both [18, Lemma 2.3] and [16, Lemma 1] require unique divisibility of the target space.

Making use of Corollary 3.1, we can easily derive a criteria on the functions f and g to be the alien solutions of equation $(*)$. By Lemma 2.1 if f and g are alien then $f = 0$ and g is additive.

Corollary 3.2. Let X be an integral domain which is uniquely divisible by 2 and 3 and consider the functions $f, g : X \to X$ and assume that equation $(*)$ holds. Then the following statements are equivalent:

(i) $f = 0$ and g is additive;

(ii) the function f is even and $f(1) = 0$;

(iii) the function g is odd and $g(2) = 2g(1)$.

Proof. The proof is a direct calculation based on Corollary 3.1. □

Finally, we investigate the case when the functions occurring in equation $(*)$ are the same. In this case we prove the following.

Corollary 3.3. Let X be an integral domain which is uniquely divisible by 2 and 3. Assume that the function $f : X \to X$ fulfills

$$f(x+y) - f(x) - f(y) = xf(y) + yf(x)$$

for any $x, y \in X$. Then and only then, the function f is identically zero.

Proof. Follows immediately from Corollary 3.1. □

4. Functional inequality $(**)$

We will apply Corollaries 2.1 and 2.2 to obtain two analogues of Theorem 3.1 for inequality $(**)$ under some additional assumptions.

Theorem 4.1. *Assume that the functions $f, g : \mathbb{R} \to \mathbb{R}$ fulfill inequality $(**)$ for each $x, y \in \mathbb{R}$. If f is odd and $f(2x) = 2f(x)$ for each $x \in \mathbb{R}$ then f is additive and there exists a subadditive mapping $A : \mathbb{R} \to \mathbb{R}$ such that*

$$g(x) = xf(x) - A(x),$$

for all $x \in \mathbb{R}$.

PROOF. Let us define

$$\phi(x,y) = xf(y) + yf(x) \quad (x,y \in \mathbb{R}).$$

One may calculate that thanks to our assumptions upon f we have

$$\phi(x,-y) = \phi(-x,y) = -\phi(x,y) \quad (x,y \in \mathbb{R})$$

and

$$\phi(2x,2y) = 4\phi(x,y) \quad (x,y \in \mathbb{R}).$$

This implies that assumptions of Corollary 2.1 are satisfied by g and ϕ. Therefore, we obtain the existence of a subadditive mapping $A : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \frac{1}{2}\phi(x,x) - A(x) \quad (x \in \mathbb{R})$$

and additionally we get that ϕ is biadditive. Using the latter assertion we may calculate that

$$\begin{aligned} xf(y+z) + (y+z)f(x) &= \phi(x,y+z) = \phi(x,y) + \phi(x,z) \\ &= xf(y) + yf(x) + xf(z) + zf(x) \end{aligned}$$

for all $x, y \in \mathbb{R}$ and this applied for $x = 1$ gives us the additivity of f. To finish the proof note that $\phi(x,x) = 2xf(x)$ for all $x \in \mathbb{R}$. □

Theorem 4.2. *Under assumptions of Theorem 4.1, if additionally for each* $x \in \mathbb{R}$ *there exists* $k_0 \in \mathbb{N}$ *such that for every* $k \geq k_0$ *we have*

$$g\left(\frac{x}{2^k}\right) + g\left(-\frac{x}{2^k}\right) \geq 0,$$

then the map $A : \mathbb{R} \to \mathbb{R}$ *postulated by Theorem 4.1 is additive.*

PROOF. Preserving notations and using some calculations from the previous proof one can check that all assumptions of Corollary 2.2 are satisfied by g and ϕ and the assertion follows from this result. □

Remark 4.1. One may easily observe that the alienation effect for inequality $(**)$ does not hold under assumptions of the foregoing two theorems, except in the trivial case $f = 0$. Indeed, assume that assertion of Theorem 4.1 holds. To get the alienation effect we expect that

$$g(x+y) - g(x) - g(y) \geq 0$$

and

$$xf(y) + yf(x) \leq 0$$

for each $x, y \in \mathbb{R}$. The second inequality applied for $y = 1$ implies that

$$f(x) \leq -f(1)x \quad (x \in \mathbb{R})$$

and this easily gives us that $f(x) = -f(1)x$ for each $x \in \mathbb{R}$ and consequently $f(1) = 0$ and thus $f = 0$.

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(Received July 28, 2010; revised July 8, 2011)