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On Berwald m-th root Finsler metrics

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Abstract. In this paper, we study m-th root Finsler metrics. For these metrics, we find the necessary and sufficient condition to be Berwaldian. By this result, we construct some special Berwaldian m-th root metrics. Then we prove that every m-th root Douglas metrics reduces to a Berwald metric.

1. Introduction

The Berwald metrics are very important in Finsler geometry. They were first investigated by L. Berwald. The geodesics of a Finsler metric $F(x,y)$ on a smooth manifold M are determined by the systems of second order differential equations

$$\frac{d^2x^i}{dt^2}+2G^i\left(x,\frac{dx}{dt}\right)=0, \tag{1}$$

where

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}. \tag{2}$$

The local functions $G^i=G^i(x,y)$ define a global vector field $G=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$ on $TM\backslash\{0\}$, which is called the *spray coeffcients*. By definition, F is called a *Berwald metric* if $G^i=G^i(x,y)$ are quadratic in $y\in T_xM$ at every point x, i.e.

$$G^i=\frac{1}{2}\Gamma^i_{kh}(x)y^hy^k. \tag{3}$$

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It can be shown that Berwald manifolds are modeled on a single norm space, i.e., all the tangent spaces T_xM with the induced norm F_x are linearly isometric to each other. Obviously, every Riemannian metric is a Berwald metric. In fact, the geodesics of any Berwald metric are the geodesics of some Riemannian metric [8].

A Finsler metric is said to be *locally projectively equivalent to* a Riemannian metric g if at every point x, there is a local coordinate neighborhood in which the geodesics of F coincide with that of g as point sets. In this case, the spray coefficients G^i are in the following form

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x,y)y^i. \tag{4}$$

Finsler metrics with this property are called *Douglas metrics*. Obviously, the Douglas metrics are more generalized than Berwald metrics. The local structure of Berwald metrics are shown by Z. I. Szabó [8]. The local metric structure of Douglas metrics remain unknown.

In this paper, we will discuss the following class of reversible Finsler metrics.

$$F = A^{\frac{1}{m}}, \tag{5}$$

where $A = a_{i_1i_2\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$. A Finsler metric in this form is called an *m-th root metric*. These metrics were first studied by M. Matsumoto, K. Okubo and H. Shimada ([4], [5], [6], [7]). Tensorial connections for such metrics have been studied by L. Tamássy [9]. It's easily to see that when $m = 2$, it is a Riemannian metric. When $m = 4$, it is called a *fourth root metric* [7]. The special fourth root metric in the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the *Berwald Moore metric*. This metric is singular in y.

In this paper, we consider the condition of $m > 4$. Obviously, by the definition of Finsler metric m must be even.

We prove the following.

Theorem 1.1. *Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset R^n$. Then F is a Berwald metric if and only if there exist local functions $\gamma^i_{kh} = \gamma^i_{kh}(x)$ such that*

$$\gamma^i_{kl}y^k\frac{\partial A}{\partial y^i} = \frac{\partial A}{\partial x^l}. \tag{6}$$

In this case, $G^i = \frac{1}{2}\gamma^i_{kh}y^ky^h$.

By this theorem, we can construct some non-Riemannian and non-Minkowskian Berwaldian m-th root Finsler metrics. Let us see the following examples.

Example 1.1. Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset R^n$. The spray coefficients

$$G^i = \frac{1}{2}\gamma^i_{kl} y^k y^l,$$

where $\gamma^i_{kl} = \gamma^i_{kl}(x)$ are local functions. Let

$$\gamma^i_{kl} = \begin{cases} c \text{ (constant)} & \text{if } \ i = k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By (6), we get

$$cA_1 y^1 + 0 + \cdots + 0 = A_{x^l}. \tag{7}$$

By (7), we obtain a special class of Berwaldian m-th root Finsler metrics as following.

$$F = \sqrt[m]{(y^1 y^2 \ldots y^n)^2 e^{2c(x_1 + x_2 + \cdots + x_n)}},$$

where $m = 2n$. But these metric are singular in y. It is easy to see that F is a Berwald–Moore metric when $n = 4$ and $c = 0$.

The following example is regular.

Example 1.2. Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset R^n$.

$$G^i = \frac{1}{2}\gamma^i_{kl} y^k y^l,$$

where $\gamma^i_{kl} = \gamma^i_{kl}(x)$ are local functions. Let

$$\gamma^i_{kl} = \begin{cases} \dfrac{1}{m}\dfrac{f_i^{'}(x)}{f_i(x)} & \text{if } \ i = k = l, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_i(x)$ are positive smooth functions satisfying $\frac{\partial f_i(x)}{\partial x^j} = 0$, $(i \neq j)$.

We can obtain a special solution of (6) as following.

$$A = f_1(x)(y^1)^m + f_2(x)(y^2)^m + \cdots + f_n(x)(y^n)^m.$$

Then we get a special class of Berwaldian m-th root Finsier metrics

$$F = \sqrt[m]{f_1(x)(y^1)^m + f_2(x)(y^2)^m + \cdots + f_n(x)(y^n)^m}.$$

On the other hand, one can verify it is a Berwald metric by a direct computation as following.

$$(A_{ij}) = m(m-1)\begin{pmatrix} f_1(x)(y^1)^{m-2} & 0 & \dots & 0 \\ 0 & f_2(x)(y^2)^{m-2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & f_m(x)(y^m)^{m-2} \end{pmatrix}, \quad (8)$$

$$(A^{ij}) = \frac{1}{m(m-1)}\begin{pmatrix} \frac{1}{f_1(x)}(y^1)^{2-m} & 0 & \dots & 0 \\ 0 & \frac{1}{f_2(x)}(y^2)^{2-m} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{1}{f_m(x)}(y^m)^{2-m} \end{pmatrix}, \quad (9)$$

$$A_{0j} = mf_j^{'}(x)(y^j)^m, \quad (10)$$

$$A_{x^j} = f_j^{'}(x)(y^j)^m. \quad (11)$$

Plugging (8),(9),(10),(11) into (13) yields,

$$G^i = \frac{1}{2m}\frac{f_i^{'}(x)}{f_i(x)}(y^i)^2.$$

So, G^i are quadratic in y and F is a Berwald metric.

For Douglas metrics, we prove the following

Theorem 1.2. *Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset R^n$, where $m > 4$. Assume that A is irreducible. F is a Douglas metric, if and only if it is a Berwald metric.*

When $m = 2$, it is Riemannian. Obviously, it is also Berwald. When $m = 4$, the result is same as in [3]. Therefore this theorem is an extension of the case when $m = 4$.

2. Berwald m-th root Finsler metrics

In this section, we are going to consider Berwald m-th root metric $F = A^{\frac{1}{m}}$ on an open subset $U \subset R^n$. For simplicity, we let

$$\frac{\partial A}{\partial y^i} = A_i, \quad \frac{\partial^2 A}{\partial y^i \partial y^j} = A_{ij}, \quad A_{x^k} = \frac{\partial A}{\partial x^k}, \quad A_{x^k y^i} = \frac{\partial^2 A}{\partial x^k \partial y^i},$$
$$A_{x^k} y^k = A_0, \quad A_{x^k y^i} y^k = A_{0i}.$$

Assume that $A = A(x, y) > 0$ for any $y \neq 0$. Then the Hessian $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ is given by

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij} + (2-m)A_i A_j] \tag{12}$$

First we have the following

Lemma 2.1 ([10])**.** *The spray coefficients of F is given by*

$$G^i = \frac{1}{2}(A_{0j} - A_{x^j})A^{ij}. \tag{13}$$

Proof. We claim that A_{ij} is positive definite. If there is a vector field $\xi = \{\xi^i\}$ such that

$$A_{ij}\xi^i\xi^j \leq 0,$$

then by (12)

$$g_{ij}\xi^i\xi^j = \frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij}\xi^i\xi^j + (2-m)(A_i\xi^i)^2] \leq 0. \tag{14}$$

Here we used $m \geq 2$. It is a contradiction to the positive definiteness of g_{ij}. Thus A_{ij} is positive definite.

By (12) and

$$A^{ij}A_i = \frac{1}{m-1}A^{ij}A_{il}y^l = \frac{1}{m-1}\delta^j_l y^l = \frac{1}{m-1}y^j,$$

we have

$$g^{ij} = mA^{\frac{m-2}{m}}A^{ij} + \frac{m-2}{m-1}A^{-\frac{2}{m}}y^i y^j. \tag{15}$$

Then by (2) we have

$$G^i = \frac{1}{2}(A_{0j} - A_{x^j})A^{ij}. \tag{16}$$

□

Proof of Theorem 1.1. If F is a Berwald metric, then

$$G^i = \frac{1}{2}\gamma^i_{kh}(x)y^k y^h,$$

where $\gamma^i_{kh}(x)$ are local scalar functions.

Plugging above equation into (13), we get

$$\gamma^i_{kh}y^k y^h = (A_{0j} - A_{x^j})A^{ij},$$

Contracting above equation with A_{il} and A_i respectively, we have

$$\gamma^i_{kh} y^k y^h A_{il} = A_{0l} - A_{x^l} \tag{17}$$

and

$$\gamma^i_{kh} y^k y^h A_i = A_0. \tag{18}$$

Differentiate (18) with respect to y^l yields

$$\gamma^i_{kh} y^k y^h A_{il} + 2\gamma^i_{kl} y^k A_i = A_{0l} + A_{x^l}. \tag{19}$$

(19)–(17) yields

$$\gamma^i_{kl} y^k A_i = A_{x^l}. \tag{20}$$

If (6) holds, then differentiate (6) with respect to y^h and contract with y^l yields

$$\gamma^i_{hl} y^l A_i + \gamma^i_{kl} y^k y^l A_{ih} = A_{0h}.$$

Substituting (6) into the above equation, we have

$$\gamma^i_{kl} y^k y^l A_{ih} = A_{0h} - A_{x^h}.$$

Contracting it with A^{hj}, we get

$$\gamma^j_{kl} y^k y^l = (A_{0h} - A_{x^h}) A^{hj}.$$

By (13), F is a Berwald metric. □

3. Douglas m-th root Finsler metrics

In this section, we discuss Douglas m-th root Finsler metrics. Douglas metrics are characterized by (4). From (4) it's easy to see that Douglas metrics also satisfy the following equations ([1]):

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l. \tag{21}$$

We first have the following lemma.

Lemma 3.1. *Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset R^n$, where $m > 4$. Assume that A is irreducible. If F is a Douglas metric, then it satisfies*

$$-A_0 + \Gamma^l_{kh} y^k y^h A_l = \eta A, \tag{22}$$

where η is a 1-form and $\Gamma^l_{kh} = \Gamma^l_{kh}(x)$ are scalar functions in (4).

Proof. By a direction computation , we have

$$F^2_{x^k y^i} = 4\frac{A^{\frac{2}{m}} A_i A_{x^k}}{m^2 A^2} + 2\frac{A^{\frac{2}{m}} A_{x^k y^i}}{mA} - 2\frac{A^{\frac{2}{m}} A_i A_{x^k}}{mA^2}, \tag{23}$$

$$F^2_{x^i} = 2\frac{A^{\frac{2}{m}} A_{x^i}}{mA}. \tag{24}$$

By (2), (23), (24) yields

$$\begin{aligned} G_i := g_{ij}G^j = \frac{1}{4}(F^2_{x^k y^i} - F^2_{x^i}) &= \frac{A^{\frac{2}{m}} A_0 A_i}{m^2 A^2} + \frac{A^{\frac{2}{m}} A_{0i}}{2mA} - \frac{1}{2}\frac{A^{\frac{2}{m}} A_i A_0}{mA^2} - \frac{1}{2}\frac{A^{\frac{2}{m}} A_{x^i}}{mA} \\ &= \frac{A^{\frac{2}{m}-2}}{2m^2}[(2-m)A_0 A_i + mA(A_{0i} - A_{x_i})]. \end{aligned} \tag{25}$$

By (12), and

$$A_{il}y^l = (m-1)A_i, \quad A_l y^l = mA,$$

we have

$$y_i = g_{il}y^l = \frac{A^{\frac{2}{m}}}{mA}A_{il}y^l + \left(\frac{2A^{\frac{2}{m}}}{m^2A^2} - \frac{A^{\frac{2}{m}}}{mA^2}\right)A_i A_l y^l = \frac{A_i A^{\frac{2}{m}}}{mA}. \tag{26}$$

By (21), we can obtain

$$G_i y_j - G_j y_i - \frac{1}{2}(\Gamma^l_{kh} g_{il} y_j - \Gamma^l_{kh} g_{jl} y_i) y^k y^h = 0. \tag{27}$$

Plugging (25),(26) into (27), yields

$$\frac{A^{\frac{-2(m-2)}{m}}(A_{0i} - A_{x^i} - \Gamma^l_{kh} y^k y^h A_{il})A_j}{2m^2} - \frac{A^{\frac{-2(m-2)}{m}}(A_{0j} - A_{x^j} - \Gamma^l_{kh} y^k y^h A_{jl})A_i}{2m^2} = 0,$$

i.e.

$$A_j(A_{0i} - A_{x^i} - \Gamma^l_{kh} y^k y^h A_{il}) - A_i(A_{0j} - A_{x^j} - \Gamma^l_{kh} y^k y^h A_{jl}) = 0.$$

Contracting above equation with y^i by

$$A_i y^i = mA, \quad A_{il}y^i = (m-1)A_l, \quad A_{0i}y^i = mA_0,$$

we have

$$A_j(m-1)(A_0 - \Gamma^l_{kh} y^k y^h A_l) = m(A_{0j} - A_{x^j} - \Gamma^l_{kh} y^k y^h A_{jl})A. \tag{28}$$

Since A is irreducible and $\deg(A_j) = m-1$, by (28), one can conclude that $-A_0 + \Gamma^l_{kh} y^k y^h A_l$ is divisible by A, that is, there is a 1-form η such that

$$-A_0 + \Gamma^l_{kh} y^k y^h A_l = \eta A.$$

□

Proof of Theorem 1.2. Let F be a Douglas metric. Then by (12) and (25), we have

$$\frac{A^{\frac{2}{m}}A_0A_i}{m^2A^2}+\frac{A^{\frac{2}{m}}A_{0i}}{2mA}-\frac{1}{2}\frac{A^{\frac{2}{m}}A_iA_0}{mA^2}-\frac{1}{2}\frac{A^{\frac{2}{m}}A_{x^i}}{mA} -\left\{2\frac{A^{\frac{2}{m}}A_iA_l}{m^2A^2}+\frac{A^{\frac{2}{m}}A_{il}}{mA}-\frac{A^{\frac{2}{m}}A_iA_l}{mA^2}\right\}\left\{\frac{1}{2}\Gamma^l_{kh}y^ky^h+Py^l\right\}=0. \tag{29}$$

Simplifying above equation, yields

$$(2-m)A_0A_i+mAA_{0i}-mAA_{x^i}-\{(2-m)A_iA_l+mAA_{il}\}\Gamma^l_{kh}y^ky^h-2mAA_iP=0.$$

Contracting above equation with y^i, we get

$$A_0-\Gamma^l_{kh}y^ky^hA_l=2mAP.$$

Then

$$P=-\frac{1}{2}\frac{-A_0+\Gamma^l_{kh}y^ky^hA_l}{mA}.$$

By Lemma 3.1 and above equation,

$$P=-\frac{1}{2m}\eta. \tag{30}$$

We see that $G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k-\frac{1}{2m}\eta y^i$ are quadratic in y. Therefore F is a Berwald metric.

Sufficiency is obvious. □

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