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General characterizations of anisotropic Besov spaces

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Abstract. In this paper we establish general characterizations of anisotropic Besov spaces associated with expansive matrices.

1. Introduction

Celebrated monographies of Peetre [23] and Triebel ([27], [28] and [29]) present exhaustive studies about Besov and Triebel–Lizorkin spaces. In this paper we are going to concentrate on Besov spaces but analogous properties for Triebel–Lizorkin spaces can be established.

Several equivalent ways exist to define Besov spaces. We adopt here a Littlewood–Paley type definition as starting point. Let $0 < p$, $q \leq \infty$ and $\alpha \in \mathbb{R}$. Assume that φ is a C^∞ function on $\mathbb{R}^n$ such that

$$\operatorname{supp} \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 2\},$$

and $\varphi(x) = 1$, $|x| \leq 1$. For every $j \in \mathbb{N}$, we define

$$\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \quad x \in \mathbb{R}^n,$$

and $\varphi_0 = \varphi$. A distribution $f \in S'(\mathbb{R}^n)$ is in the Besov space $B^\alpha_{p,q}$ if and only if

$$\|f\|_{B^\alpha_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \|(\hat{f}\varphi_j)\check{\ }\|_p^q \right)^{1/q} < \infty,$$

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with the usual modification when $q = \infty$. As mentioned earlier complete studies of Besov spaces $B^{\alpha}_{p,q}$ can be found in [23], [27], [28] and [29].

Recently, several authors (see [8], [12], [13], [15], [16], [17], and [22], amongst others) have investigated (diagonal)-anisotropic Besov spaces. These anisotropic Besov spaces are defined also in terms of Fourier analytical quasinorm as follows. By $a = (a_1, a_2, \dots, a_n)$ we denote an anisotropy, that is, a n-uple of positive numbers such that $a_1 + a_2 + \dots + a_n = n$. We write, for every $j \in \mathbb{Z}$,

$$2^{ja}x = (2^{ja_1}x_1, \dots, 2^{ja_n}x_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Let $0 < p,\ q \le \infty$ and $\alpha \in \mathbb{R}$. If φ is a function as above, a distribution $f \in S'(\mathbb{R}^n)$ is in the anisotropic Besov space $B^{\alpha;a}_{p,q}$ if and only if

$$\|f\|_{B^{\alpha;a}_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \|(\varphi_j \hat{f})\check{}\|_p^q \right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$, where, for every $j \in \mathbb{N}$,

$$\varphi_j(x) = \varphi(2^{-ja}x) - \varphi(2^{-(j+1)a}x), \quad x \in \mathbb{R}^n,$$

and $\varphi_0 = \varphi$. The basic result for the (diagonal)-anisotropic Besov spaces can be encountered in [20] and [25].

Bownik ([2], [3], [4], and [5]) and Ho ([18] and [21]) (see also the papers of Bownik and Ho [6], Dappa and Triebel [9], Dintelmann [10] and [11] and Ho [21]) have studied anisotropic function spaces associated with expansive matrices. In [2] general anisotropic Besov spaces are introduced. In this paper we complete that study by proving a rather general characterization of the anisotropic Besov spaces associated with expansive matrices.

We now recall some definitions and properties concerning to expansive dilations. Let $n \in \mathbb{N}$. A real $n \times n$ matrix A is an expansive matrix provided that $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the spectrum of A. A quasinorm associated with an expansive matrix A is a Borel measurable function $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying the following conditions

(i) $\rho_A(x) > 0$, $x \neq 0$,

(ii) $\rho_A(Ax) = |\det A| \rho_A(x)$, $x \in \mathbb{R}^n$,

(iii) $\rho_A(x + y) \le H(\rho_A(x) + \rho_A(y))$, $x, y \in \mathbb{R}^n$,

where $H \ge 1$.

All quasinorms associated with a fixed dilation A are equivalent ([2], Lemma 2.4). We consider in the sequel the quasinorm ρ_A defined for a fixed expansive matrix A by

$$\rho_A(x) = \sum_{k=-\infty}^{\infty} |\det A|^k \chi_{O_k}(x), \quad x \in \mathbb{R}^n,$$

where $O_k = A^k([-1,1]^n) \setminus \bigcup_{i=-\infty}^{k-1} A^i([-1,1]^n)$.

If by $|\cdot|$ we denote the Lebesgue measure, $(\mathbb{R}^n, \rho_A, |\cdot|)$ is a space of homogeneous type in the sense of COIFMANN and WEISS ([7]). For any locally integrable function f on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator $\mathcal{M}_{\rho_A}$ is defined, as usual, by

$$\mathcal{M}_{\rho_A} f(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where $\mathcal{B}$ represents the collection of $\rho_A - balls$. The following fact will be useful (see [6, (2.4)])

$$|B_{\rho_A}(0,r)| \sim r, \quad r > 0, \tag{1}$$

where $B_{\rho_A}(0,r) = \{x \in \mathbb{R}^n : \rho_A(x) < r\}$, $r > 0$. It is well known that the maximal operator M_{ρ_A} is bounded from $L_p(\mathbb{R}^n)$ into itself for every $1 < p < \infty$ and from $L_1(\mathbb{R}^n)$ into $L_{1,\infty}(\mathbb{R}^n)$.

Note that, according to [3, (2.2), p. 4], there exists $j_0 \in \mathbb{N}$ such that

$$A^{-j}[-1,1]^n \subset A^{-2}[-1/2,1/2]^n, \quad j \geq j_0,\ j \in \mathbb{N},$$

and

$$A^{-1}[-1,1]^n \bigcup [-1,1]^n \bigcup A[-1,1]^n \subset A^{j_0}[-1/2,1/2]^n.$$

Assume that $\varphi, \Phi \in S(\mathbb{R}^n)$ are such that

$$\operatorname{supp} \hat{\varphi} \subseteq \{x \in \mathbb{R}^n : |\det A|^{-j_0} \leq \rho_{A^*}(x) \leq |\det A|^{j_0}\} \tag{2}$$

$$\operatorname{supp} \hat{\Phi} \subseteq \{x \in \mathbb{R}^n : \rho_{A^*}(x) \leq |\det A|^{j_0}\} \tag{3}$$

and

$$\sup_{j \in \mathbb{N}} \{|\hat{\varphi}((A^*)^{-j}x)|, |\hat{\Phi}(x)|\} > 0,\ x \in \mathbb{R}^n. \tag{4}$$

By A^* we represent the matrix adjoint to A. Observe that if $x \neq 0$ there exists $k \in \mathbb{Z}$ such that $x \in O_k^*$, where $O_k^* = (A^*)^k([-1,1]^n) \setminus \bigcup_{i=-\infty}^{k-1} (A^*)^i([-1,1]^n)$.

Let $\alpha \in \mathbb{R}$ and $0 < p,\ q \leq \infty$. We say that a distribution $f \in S'(\mathbb{R}^n)$ is in the inhomogeneous Besov space $B_{p,q}^{\alpha,A}$ associated with A if and only if

$$\|f\|_{B_{p,q}^{\alpha,A}} = \|f * \Phi\|_p + \left(\sum_{j=1}^{\infty} (|\det A|^{j\alpha} \|f * \varphi_j\|_p)^q \right)^{1/q} < \infty.$$

Here and in the sequel, for every $j \in \mathbb{Z}$, the function φ_j is defined by

$$\varphi_j(x) = |\det A|^j \varphi(A^j x), \quad x \in \mathbb{R}^n.$$

The anisotropic Besov spaces $B^{\alpha,A}_{p,q}$ are independent of the function φ and Φ as above. These spaces were introduced in [2]. There Bownik characterized them by the magnitude of the anisotropic φ-transforms in appropriate sequence spaces. He also obtained atomic and molecular decompositions of the elements of the anisotropic Besov spaces extending isotropic results of Frazier and Jawerth [14]. Recently, the authors ([1]) have established other characterizations of the spaces $B^{\alpha,A}_{p,q}$.

In this paper our objective is to establish a $B^{\alpha,A}_{p,q}$ version of [28, Theorem 251].

We choose two functions $h_A, H_A \in S(\mathbb{R}^n)$ such that

$$\operatorname{supp} \hat{h}_A \subseteq \{x \in \mathbb{R}^n : \rho_{A^*}(x) \le |\det A|^{j_0}\} \quad \text{and} \quad \hat{h}_A(x) = 1, \ \rho_{A^*}(x) \le 1,$$

and

$$\operatorname{supp} \hat{H}_A \subseteq \{x \in \mathbb{R}^n : |\det A|^{-j_1} \le \rho_{A^*}(x) \le |\det A|^{j_1}\} \quad \text{and} \quad \hat{H}_A(x) = 1,$$
$$\frac{1}{|\det A|^{j_0}} \le \rho_{A^*}(x) \le |\det A|^{j_0},$$

where $j_1 \in \mathbb{N}$ is large enough (see [3, (2.2), p. 4]). Here A is as always an expansive matrix and in the sequel to simplify we will write h and H to refer to h_A and H_A.

Theorem 1.1. *Let $0 < q \le \infty$, $1 < p < \infty$, $\alpha > 0$ and $s_0 < \alpha < s_1$. Assume that A is an expansive matrix and that Ψ and ψ belong to $L_1(\mathbb{R}^n)$ being $\hat{\Psi}(x) \ne 0$, $\rho_{A^*}(x) \le |\det A|^{j_0}$. Suppose also that there exist C_1 and C_2 such that if $\hat{\psi}(x) \ne 0$, $C_1 \le \rho_{A^*}(x) \le C_2$, and that the following conditions hold:*

$$\sup_{t\in[0,1],\, m\in\mathbb{N}} \Big\| \Big(\frac{\hat{\psi}((A^*)^t z)\hat{H}((A^*)^{-m} z)}{\rho_{A^*}(z)^{s_0}} \Big)^{\check{}} \Big\|_1 < \infty, \tag{P1}$$

$$\sup_{m\in\mathbb{N}} \Big\| \Big(\frac{\hat{\Psi}(z)\hat{H}((A^*)^{-m} z)}{\rho_{A^*}(z)^{s_0}} \Big)^{\check{}} \Big\|_1 < \infty, \tag{P2}$$

$$\sup_{t\in[0,1]} \Big\| \Big(\frac{\hat{\psi}((A^*)^t z)\hat{h}((A^*)^{-j_0} z)}{\rho_{A^*}(z)^{s_1}} \Big)^{\check{}} \Big\|_1 < \infty, \tag{P3}$$

where j_0 is that appears in (2). Then, the quasinorms $\|\cdot\|^{(1)}_{B^{\alpha,A}_{p,q}}$ and $\|\cdot\|^{(2)}_{B^{\alpha,A}_{p,q}}$ defined by

$$\|f\|^{(1)}_{B^{\alpha,A}_{p,q}} = \|f * \Psi\|_p + \left(\int_0^1 t^{-\alpha q} \|(\hat{\psi}((A^*)^{\log_{|\det A|} t} \cdot)\hat{f}(\cdot))^{\check{}}\|_p^q \frac{dt}{t} \right)^{1/q},$$

and

$$\|f\|^{(2)}_{B^{\alpha,A}_{p,q}} = \|f*\Psi\|_p + \left(\sum_{j=1}^{\infty}(|\det A|^{j\alpha}\|f*\psi_j\|_p)^q\right)^{1/q}, \; f \in B^{\alpha,A}_{p,q},$$

(where the usual changes are made when $q=\infty$) are equivalent to $\|\cdot\|_{B^{\alpha,A}_{p,q}}$ on $B^{\alpha,A}_{p,q}$. $\boxdot$

Anisotropic Triebel–Lizorkin spaces $F^{\alpha,A}_{p,q}$ associated with expansive matrices were studied in [4], [5], [6], and [21]. A version of Theorem 1.1 for $F^{\alpha,A}_{p,q}$-spaces can be established following a similar procedure by using an anisotropic Peetre maximal function (see [28, Theorem 2.4.1] for the isotropic case).

Throughout this paper by C we always denote a positive constant that can change in each occurrence.

2. Proof of Theorem 1.1.

We will show that the quasinorms $\|\cdot\|^{(1)}_{B^{\alpha,A}_{p,q}}$ and $\|\cdot\|_{B^{\alpha,A}_{p,q}}$ are equivalent on $B^{\alpha,A}_{p,q}$. The corresponding property for $\|\cdot\|^{(2)}_{B^{\alpha,A}_{p,q}}$ can be seen in a similar way with minor changes.

For every $\alpha,\beta>0$ we will repeatedly use the following well known properties:

(i) $A^\alpha A^\beta = A^{\alpha+\beta}$;

(ii) $|\det(A^\alpha)| = |\det(A)|^\alpha$; and

(iii) $(A^*)^\alpha = (A^\alpha)^*$.

First we prove that, for every $f \in B^{\alpha,A}_{p,q}$,

$$\|f*\Psi\|_p + \left(\int_0^1 t^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|} t}\cdot)\hat{f}(\cdot))\check{}\|_p^q \frac{dt}{t}\right)^{1/q} \leq C\|f\|_{B^{\alpha,A}_{p,q}},$$

when $1 \leq q < \infty$. When $q=\infty$ we only need to make the usual changes.

We choose two functions Φ and $\varphi \in S(\mathbb{R}^n)$ such that

$$\operatorname{supp}\hat{\Phi} \subseteq \overline{B}_{\rho_{A^*}}(0,|\det A|^{j_0}),$$

$$\operatorname{supp}\hat{\varphi} \subseteq \overline{B}_{\rho_{A^*}}(0,|\det A|^{j_0}) \setminus \overline{B}_{\rho_{A^*}}(0,|\det A|^{-j_0})$$

and

$$\hat{\Phi}(y) + \sum_{k=1}^{\infty}\hat{\varphi}_k(y) = 1, \quad y \in \mathbb{R}^n.$$

Here j_0 is that appears in (2). Also we have that, for every $f \in S'(\mathbb{R}^n)$,

$$f = f * \Phi + \sum_{k=1}^{\infty} f * \varphi_k \quad \text{in } S'(\mathbb{R}^n). \tag{5}$$

To simplify we write $\varphi_0 = \Phi$ and we define $\varphi_k = 0$, when $k \in \mathbb{Z}$, $k \leq -1$.

Let $f \in B^{\alpha,A}_{p,q}$, $j \in \mathbb{N}$ and $t \in \big[|\det A|^{-j}, |\det A|^{-j+1}\big]$. We define

$$\psi_{(t)}(x) = \frac{1}{t}\psi(A^{-\log_{|\det A|} t}x), \quad x \in \mathbb{R}^n.$$

Then

$$\hat{\psi}_{(t)}(y) = \hat{\psi}((A^*)^{\log_{|\det A|} t}y), \quad y \in \mathbb{R}^n.$$

To obtain this we have used that $|det(A^\beta)| = |\det A|^\beta$, and $(A^*)^\beta = (A^\beta)^*$, for every $\beta \in \mathbb{R}$.

By (5) we can write

$$f * \psi_{(t)} = \sum_{l=-\infty}^{0} f * \psi_{(t)} * \varphi_{l+j} + \sum_{l=1}^{\infty} f * \psi_{(t)} * \varphi_{l+j} \quad \text{in } S'(\mathbb{R}^n). \tag{6}$$

Moreover, since the series in (5) also converges in $L_p(\mathbb{R}^n)$ and $\psi_{(t)} \in L_1(\mathbb{R}^n)$, the last series converges in $L_p(\mathbb{R}^n)$ and it represents, in the suitable way (that is, via some subsequence), pointwise the function $\psi_{(t)} * f$ a.e. $\mathbb{R}^n$.

We analyse the first series in the right hand side in (6). We define, for every $k \in \mathbb{Z}$ the function $\tilde{\varphi}_k$ by

$$\tilde{\varphi}_k = (\rho_{A^*}((A^*)^{-k}\cdot)^{s_1}\hat{\varphi}_k(\cdot))\check{}.$$

We can write

$$\begin{aligned}
&\Big|\sum_{l=-\infty}^{0} |\det A|^{j\alpha}(f * \psi_{(t)} * \varphi_{l+j})(x)\Big| \\
&\quad\leq \sum_{l=-\infty}^{0} |\det A|^{j\alpha}\Big|\Big(\hat{\psi}((A^*)^{\log_{|\det A|} t}z)\hat{\varphi}((A^*)^{-l-j}z)\hat{f}(z)\Big)\check{}(x)\Big| \\
&\quad\leq \sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)}\Big|\Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t}z)}{\rho_{A^*}((A^*)^{-j}z)^{s_1}}|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\tilde{\varphi}}_{j+l}(z)\Big)\check{}(x)\Big| \\
&\quad= \sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)} \\
&\qquad\times\Big|\Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t}z)}{\rho_{A^*}((A^*)^{-j}z)^{s_1}}\hat{h}((A^*)^{-j-j_0}z)|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\tilde{\varphi}}_{j+l}(z)\Big)\check{}(x)\Big|.
\end{aligned}$$

In the last equality we have used that $\hat{h}(z) = 1$, when $\rho_{A^*}(z) \leq 1$. Indeed, since

$$\operatorname{supp} \hat{\varphi} \subseteq \overline{B}_{\rho_{A^*}}(0, |\det A|^{j_0}),$$

then

$$\operatorname{supp} \hat{\tilde{\varphi}}_{j+l} = \operatorname{supp}(\rho_{A^*}((A^*)^{-(j+l)}\cdot)^{s_1} \hat{\varphi}_{j+l}) \subseteq \overline{B}_{\rho_{A^*}}(0, |\det A|^{j+l+j_0}).$$

Hence,

$$\operatorname{supp} \hat{\tilde{\varphi}}_{j+l} \subseteq \overline{B}_{\rho_{A^*}}(0, |\det A|^{j+j_0}),$$

and $\hat{h}((A^*)^{-j-j_0} z) = 1$, $z \in \operatorname{supp} \hat{\tilde{\varphi}}_{j+l}$, $l \in \mathbb{Z}$, $l \leq 0$.

For every $l \in \mathbb{Z}$, $l \leq 0$, we have

$$\begin{aligned} &\Big| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_1}} \hat{h}((A^*)^{-j-j_0} z) |\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\tilde{\varphi}}_{j+l}(z) \Big)^{\check{}}(x) \Big| \\ &\quad \leq \int_{\mathbb{R}^n} \Big| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_1}} \hat{h}((A^*)^{-j-j_0} z) \Big)^{\check{}}(y) \Big| \\ &\qquad \times \big| \big(|\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\tilde{\varphi}}_{j+l}(z) \big)^{\check{}}(x-y) \big| dy, \quad x \in \mathbb{R}^n. \end{aligned}$$

Minkowski and Young inequalities lead to

$$\begin{aligned} &\Big\| \sum_{l=-\infty}^{0} |\det A|^{j\alpha} (f * \psi_{(t)} * \varphi_{l+j}) \Big\|_p \\ &\quad \leq \sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)} \Big\| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_1}} \hat{h}((A^*)^{-j-j_0} z) \Big)^{\check{}} \Big\|_1 \\ &\qquad \times \big\| \big(|\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\tilde{\varphi}}_{j+l}(z) \big)^{\check{}} \big\|_p. \end{aligned}$$

Moreover, by making suitable changes of variables it obtains

$$\begin{aligned} &\Big\| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_1}} \hat{h}((A^*)^{-j-j_0} z) \Big)^{\check{}} \Big\|_1 \\ &\qquad = \Big\| \Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|} t} u)}{\rho_{A^*}(u)^{s_1}} \hat{h}((A^*)^{-j_0} u) \Big)^{\check{}} \Big\|_1. \end{aligned}$$

By (P3), we get

$$\begin{aligned} \sup_{|\det A|^{-j} \leq t \leq |\det A|^{-j+1}} \Big\| \sum_{l=-\infty}^{0} |\det A|^{j\alpha} (f * \psi_{(t)} * \varphi_{l+j}) \Big\|_p &\leq C \sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)} \\ &\quad \times \| (|\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\tilde{\varphi}}_{j+l}(z))^{\check{}} \|_p. \end{aligned}$$

Hence, since $s_1 > \alpha$, if $1 \leq q \leq \infty$ (with the usual changes with $q = \infty$)

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{\infty} |\det A|^{j\alpha q}\Bigg(\sup_{|\det A|^{-j}\leq t\leq |\det A|^{-j+1}} \Big\| \sum_{l=-\infty}^{0} (f * \psi_{(t)} * \varphi_{l+j})\Big\|_p\Bigg)^q\Bigg)^{1/q} \\
&\quad \leq C \sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)} \Bigg(\sum_{j=1}^{\infty} |\det A|^{\alpha(j+l)q} \|(\hat{f}\hat{\tilde{\varphi}}_{j+l})\check{\ }\|_p^q\Bigg)^{1/q} \\
&\quad \leq C\Bigg(\sum_{j=0}^{\infty} |\det A|^{\alpha jq} \|(\hat{f}\hat{\tilde{\varphi}}_j)\check{\ }\|_p^q\Bigg)^{1/q}, \qquad (7)
\end{aligned}$$

and if $0 < q < 1$,

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{\infty} |\det A|^{j\alpha q}\Bigg(\sup_{|\det A|^{-j}\leq t\leq |\det A|^{-j+1}} \Big\| \sum_{l=-\infty}^{0} (f * \psi_{(t)} * \varphi_{l+j})\Big\|_p\Bigg)^q\Bigg)^{1/q} \\
&\quad \leq C\Bigg(\sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)q} \sum_{j=1}^{\infty} |\det A|^{\alpha(j+l)q} \|(\hat{f}\hat{\tilde{\varphi}}_{j+l})\check{\ }\|_p^q\Bigg)^{1/q} \\
&\quad \leq C\Bigg(\sum_{l=-\infty}^{0} |\det A|^{l(s_1-\alpha)q} \sum_{j=0}^{\infty} |\det A|^{\alpha jq} \|(\hat{f}\hat{\tilde{\varphi}}_j)\check{\ }\|_p^q\Bigg)^{1/q} \\
&\quad \leq C\Bigg(\sum_{j=0}^{\infty} |\det A|^{\alpha jq} \|(\hat{f}\hat{\tilde{\varphi}}_j)\check{\ }\|_p^q\Bigg)^{1/q}. \qquad (8)
\end{aligned}$$

Our next objective is to see that

$$\Bigg(\sum_{j=0}^{\infty} |\det A|^{j\alpha q} \|(\hat{f}\hat{\tilde{\varphi}}_j)\check{\ }\|_p^q\Bigg)^{1/q} \leq C\Bigg(\sum_{j=0}^{\infty} |\det A|^{j\alpha q} \|(\hat{f}\hat{\varphi}_j)\check{\ }\|_p^q\Bigg)^{1/q}. \qquad (9)$$

Inequality (9) will be proved when we show that

$$\|(\hat{f}\hat{\tilde{\varphi}}_j)\check{\ }\|_p \leq C\|(\hat{f}\hat{\varphi}_j)\check{\ }\|_p \,, j \in \mathbb{N} \cup \{0\}, \qquad (10)$$

where $C > 0$ does not depend on j.

Let $j \in \mathbb{N}$. The support of $\hat{\varphi}_j$ is contained in $\Omega_j^* = \bigcup_{l=-j_0+j}^{j_0+j} O_l^*$, where

$$O_l^* = (A^*)^l[-1,1]^n \setminus \bigcup_{k=-\infty}^{l-1} (A^*)^k[-1,1]^n, \quad l \in \mathbb{Z}.$$

We can write,

$$\begin{aligned}(\hat{f}\hat{\tilde{\varphi}}_j)\check{}(x) &= (\hat{f}(z)\hat{\varphi}_j(z)\rho_{A^*}((A^*)^{-j}z)^{s_1})\check{}(x)\\
&= (\hat{f}(z)\hat{\varphi}_j(z)\chi_{\Omega_j^*}(z)\rho_{A^*}((A^*)^{-j}z)^{s_1})\check{}(x)\\
&= |\det A|^j((f*\varphi_j)\hat{}((A^*)^j z)\chi_{\Omega_j^*}((A^*)^j z)\rho_{A^*}(z)^{s_1})\check{}(A^j x)\\
&= (((f*\varphi_j)(A^{-j}u))\hat{}(z)\chi_{\Omega_0^*}(z)\rho_{A^*}(z)^{s_1})\check{}(A^j x)\\
&= \sum_{l=-j_0}^{j_0}(((f*\varphi_j)(A^{-j}u))\hat{}(z)\chi_{O_l^*}(z)\rho_{A^*}(z)^{s_1})\check{}(A^j x)\\
&= \sum_{l=-j_0}^{j_0}|\det A|^{ls_1}(((f*\varphi_j)(A^{-j}u))\hat{}(z)\chi_{O_l^*}(z))\check{}(A^j x), \quad x\in\mathbb{R}^n.\end{aligned}$$

If, for $l=-j_0,\cdots,j_0$, the function $\chi_{O_l^*}$ is an L_p-Fourier multiplier, then

$$\begin{aligned}\|(\hat{f}\hat{\tilde{\varphi}}_j)\check{}\|_p &\le C\sum_{l=-j_0}^{j_0}\|((f*\varphi_j)(A^{-j}u))\hat{}(z)\chi_{O_l^*}(z))\check{}(A^j\cdot)\|_p\\
&\le C\sum_{l=-j_0}^{j_0}|\det A|^{-\frac{j}{p}}\|(((f*\varphi_j)(A^{-j}u))\hat{}(z)\chi_{O_l^*}(z))\check{}\|_p\\
&\le C|\det A|^{-\frac{j}{p}}\|(f*\varphi_j)(A^{-j}\cdot)\|_p\le C\|f*\varphi_j\|_p = C\|(\hat{f}\hat{\varphi}_j)\check{}\|_p,\end{aligned}$$

and (10) is established, for every $j\in\mathbb{N}$.

We are going to see that $\chi_{O_1^*}$ is a L_p Fourier multiplier. In a similar way we can see that $\chi_{O_l^*}$ is a L_p Fourier multiplier, for every $l\in\mathbb{Z}$. We have that

$$O_1^* = A^*[-1,1]^n\setminus\bigcup_{l=-\infty}^{0}(A^*)^l[-1,1]^n.$$

According to [21, Lemma 1.1.1], $\lim_{l\to\infty}|(A^*)^{-l}x|=0$, uniformly in $[-1,1]^n$. Then, there exists $l_0\in\mathbb{N}$ such that

$$\bigcup_{l=-\infty}^{-l_0-1}(A^*)^l[-1,1]^n\subseteq[-1/2,1/2]^n.$$

Hence

$$O_1^* = A^*[-1,1]^n\setminus\bigcup_{l=-l_0}^{0}(A^*)^l[-1,1]^n.$$

Moreover, since $(A^*)^k[-1,1]^n$ is a polygon for every $k\in\mathbb{Z}$, O_1^* is a finite union of polygons in $\mathbb{R}^n$. Then, by [19, Corollary 3.2] (see also [26]), $\chi_{O_1^*}$ is an L_p-Fourier multiplier.

Moreover, the support of $\hat{\varphi}_0$ is contained in $\bigcup_{k=-\infty}^{j_0} O_k^*$. Then we can write

$$(\hat{\tilde{\varphi}}_0 \hat{f})\check{} = (\hat{\varphi}_0 \rho_{A^*}^{s_1} \hat{f})\check{} = \sum_{k=-j_0}^{\infty} |\det A|^{-ks_1} (\hat{\varphi}_0 \chi_{O_{-k}^*} \hat{f})\check{}.$$

As above, we have that, for every $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\|(\hat{\varphi}_0 \hat{f} \chi_{O_{-k}^*})\check{}\|_p &= \|((f * \varphi_0)\hat{} \chi_{O_{-k}^*})\check{}\|_p \\ &= \|(((f * \varphi_0)(A^k u))\hat{} \chi_{O_0^*})\check{}((A^*)^{-k} \cdot)\|_p \leq C \|f * \varphi_0\|_p.\end{aligned}$$

Hence, it follows

$$\|(\hat{\tilde{\varphi}}_0 \hat{f})\check{}\|_p \leq C \sum_{k=0}^{\infty} |\det A|^{-ks_1} \|f * \varphi_0\|_p \leq C \|f * \varphi_0\|_p.$$

By combining (7), (8) and (9) we conclude

$$\begin{aligned}&\Bigg(\sum_{j=1}^{\infty} |\det A|^{j\alpha q} \Bigg(\sup_{|\det A|^{-j} \leq t \leq |\det A|^{1-j}} \Bigg\| \sum_{l=-\infty}^{0} f * \psi_{(t)} * \varphi_{l+j} \Bigg\|_p \Bigg)^q \Bigg)^{1/q} \\ &\qquad \leq C \Bigg(\sum_{j=0}^{\infty} |\det A|^{j\alpha q} \|f * \varphi_j\|_p^q \Bigg)^{1/q}. \qquad (11)\end{aligned}$$

We now analyze the second series in the right hand side of (6). We define, for every $k \in \mathbb{Z}$, the function $\tilde{\varphi}_k$ by

$$\tilde{\varphi}_k = (\rho_{A^*}((A^*)^{-k} \cdot)^{s_0} \hat{\varphi}_k(\cdot))\check{}. \qquad (12)$$

Let $j \in \mathbb{N}$. We have that

$$\begin{aligned}&\Bigg| \sum_{l=1}^{\infty} |\det A|^{j\alpha} (f * \psi_{(t)} * \varphi_{l+j})(x) \Bigg| \\ &\quad \leq \sum_{l=1}^{\infty} |\det A|^{j\alpha} \Big| \big(\hat{\psi}((A^*)^{\log_{|\det A|} t} z) \hat{\varphi}((A^*)^{-l-j} z) \hat{f}(z) \big)\check{}(x) \Big| \\ &\quad \leq \sum_{l=1}^{\infty} |\det A|^{l(s_0 - \alpha)} \Bigg| \Bigg(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_0}} |\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\tilde{\varphi}}_{j+l}(z) \Bigg)\check{}(x) \Bigg| \\ &\quad = \sum_{l=1}^{\infty} |\det A|^{l(s_0 - \alpha)}\end{aligned}$$

$$\times \Big| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_0}} \hat{H}((A^*)^{-j-l} z) |\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\hat{\varphi}}_{j+l}(z) \Big)\check{}(x) \Big|, \quad x \in \mathbb{R}^n.$$

The last equality can be justified as follows. The support of $\hat{\hat{\varphi}}_{j+l}$ is contained in $\{z \in \mathbb{R}^n : |\det A|^{j+l-j_0} \leq \rho_{A^*}(z) \leq |\det A|^{j+l+j_0}\}$, for every $l \in \mathbb{Z}$. Then, since $\hat{H}(z) = 1$, $|\det A|^{-j_0} \leq \rho_{A^*}(z) \leq |\det A|^{j_0}$, it has

$$\hat{H}((A^*)^{-j-l} z) \hat{\hat{\varphi}}_{j+l}(z) = \hat{\hat{\varphi}}_{j+l}(z), \quad z \in \mathbb{R}^n \text{ and } l \in \mathbb{Z}.$$

By using Minkowski and Young inequality we obtain

$$\begin{aligned} &\Big\| \sum_{l=1}^{\infty} |\det A|^{j\alpha} (f * \psi_{(t)} * \varphi_{l+j}) \Big\|_p \\ &\quad \leq \sum_{l=1}^{\infty} |\det A|^{l(s_0-\alpha)} \Big\| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_0}} \hat{H}((A^*)^{-j-l} z) \Big)\check{} \Big\|_1 \\ &\quad\quad \times \| (|\det A|^{\alpha(j+l)} \hat{f}(z) \hat{\hat{\varphi}}_{j+l}(z))\check{} \|_p. \end{aligned}$$

Suitable changes of variables lead to

$$\begin{aligned} &\Big\| \Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|} t} z)}{\rho_{A^*}((A^*)^{-j} z)^{s_0}} \hat{H}((A^*)^{-j-l} z) \Big)\check{} \Big\|_1 \\ &\quad\quad = \Big\| \Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|} t} u)}{\rho_{A^*}(u)^{s_0}} \hat{H}((A^*)^{-l} u) \Big)\check{} \Big\|_1. \end{aligned}$$

By using (P1) and by proceeding as in the above case, we get

$$\begin{aligned} &\bigg(\sum_{j=1}^{\infty} |\det A|^{j\alpha q} \Big(\sup_{|\det A|^{-j} \leq t \leq |\det A|^{1-j}} \Big\| \sum_{l=1}^{\infty} f * \psi_{(t)} * \varphi_{l+j} \Big\|_p \Big)^q \bigg)^{1/q} \\ &\quad \leq C \Big(\sum_{j=1}^{\infty} |\det A|^{j\alpha q} \| (\hat{f} \hat{\hat{\varphi}}_j)\check{} \|_p^q \Big)^{1/q}. \\ &\quad \leq C \Big(\sum_{j=1}^{\infty} |\det A|^{j\alpha q} \| f * \varphi_j \|_p^q \Big)^{1/q}, \end{aligned} \tag{13}$$

because $s_0 < \alpha$.

By combining (6), (11) and (13) we conclude that

$$\Big(\int_0^1 t^{-\alpha q} \| f * \psi_{(t)} \|_p^q \frac{dt}{t} \Big)^{1/q} = \Big(\sum_{j=1}^{\infty} \int_{|\det A|^{-j}}^{|\det A|^{-j+1}} t^{-\alpha q} \| f * \psi_{(t)} \|_p^q \frac{dt}{t} \Big)^{1/q}$$

$$\leq C\Big(\sum_{j=1}^{\infty}|\det A|^{j\alpha q}\Big(\sup_{|\det A|^{-j}\leq t\leq|\det A|^{-j+1}}\|f*\psi_{(t)}\|_p\Big)^q\Big)^{1/q}$$

$$\leq C\Big(\sum_{j=1}^{\infty}|\det A|^{j\alpha q}\|f*\varphi_j\|_p^q\Big)^{1/q}. \tag{14}$$

On the other hand, we have that

$$f*\Psi=f*\Psi*\varphi_0+\sum_{l=1}^{\infty}f*\Psi*\varphi_l. \tag{15}$$

where the convergence is in $S'(\mathbb{R}^n)$ and in $L_p(\mathbb{R}^n)$, and hence, in a suitable way (via certain subsequence), pointwise in a.e. $\mathbb{R}^n$.

Since $\Psi\in L^1(\mathbb{R}^n)$, Young inequality implies that

$$\|f*\Psi*\varphi_0\|_p\leq C\|f*\varphi_0\|_p \tag{16}$$

and

$$\Big\|\sum_{j=1}^{\infty}f*\Psi*\varphi_j\Big\|_p\leq C\Big(\sum_{j=1}^{\infty}|\det A|^{j\alpha q}\|f*\varphi_j\|_p^q\Big)^{1/q}. \tag{17}$$

Then, (15), (16) and (17) imply that

$$\|f*\Psi\|_p\leq C\Big(\|f*\varphi_0\|_p+\Big(\sum_{j=1}^{\infty}|\det A|^{j\alpha q}\|f*\varphi_j\|_p^q\Big)^{1/q}\Big). \tag{18}$$

Finally, (14) and (18) lead to

$$\|f*\Psi\|_p+\Big(\int_0^1 t^{-\alpha q}\|f*\psi_{(t)}\|_p^q\frac{dt}{t}\Big)^{1/q}\leq C\|f\|_{B^{\alpha,A}_{p,q}}.$$

We are going to show that, for every $f\in B^{\alpha,A}_{p,q}$,

$$\|f\|_{B^{\alpha,A}_{p,q}}\leq C\Big(\|f*\Psi\|_p+\Big(\int_0^1 t^{-\alpha q}\|(\hat{\varphi}((A^*)^{\log_{|\det A|}t}\cdot)\hat{f}(\cdot))\check{}\|_p^q\frac{dt}{t}\Big)^{1/q}\Big),$$

for $0<q<\infty$. When $q=\infty$ we make the usual changes.

Let $f\in B^{\alpha,A}_{p,q}$ and $j\in\mathbb{N}\setminus\{0\}$. We choose a function $\Lambda\in S(\mathbb{R}^n)$ such that $\hat{\Lambda}$ has compact support and

$$\hat{\Lambda}(x)=1\,,\ \rho_{A^*}(x)\leq|\det A|^K.$$

Here $K \in \mathbb{N}$ will be specified later.

The support of $\hat{\varphi}_j$ is contained in $\overline{B}_{\rho_{A^*}}(0, |\det A|^{j+j_0}) \setminus B_{\rho_{A^*}}(0, |\det A|^{j-j_0})$. Moreover, according to [2, Lemma 2.2], we have, for certain $\xi_0, \xi_1 > 0$,

$$\begin{aligned}\rho_{A^*}((A^*)^{-j-\log_{|\det A|} t} z) &\leq C(|(A^*)^{-\log_{|\det A|} t}(A^*)^{-j} z|^{\xi_0} + 1) \\ &\leq C(\|(A^*)^{-\log_{|\det A|} t}\|^{\xi_0} |(A^*)^{-j} z|^{\xi_0} + 1) \leq C(\rho_{A^*}((A^*)^{-j} z)^{\xi_1} + 1) \\ &\leq C((|\det A|^{-j} \rho_{A^*}(z))^{\xi_1} + 1) \leq C, \ z \in \operatorname{supp} \hat{\varphi}_j \text{ and } 1 \leq t \leq |\det A|,\end{aligned}$$

and for some $\xi_2, \xi_3, \xi_4, \xi_5 > 0$,

$$\begin{aligned}&\rho_{A^*}((A^*)^{-j-\log_{|\det A|} t} z) \geq C \min\{|(A^*)^{-\log_{|\det A|} t-j} z|^{\xi_2}, |(A^*)^{-\log_{|\det A|} t-j} z|^{\xi_3}\} \\ &\quad\geq C \min\{(\|(A^*)^{\log_{|\det A|} t}\|^{-1} |(A^*)^{-j} z|)^{\xi_2}, (\|(A^*)^{\log_{|\det A|} t}\|^{-1} |(A^*)^{-j} z|)^{\xi_3}\} \\ &\quad\geq C \min\{|(A^*)^{-j} z|^{\xi_2}, |(A^*)^{-j} z|^{\xi_3}\} \geq C \min\{\rho_{A^*}((A^*)^{-j} z)^{\xi_4}, \rho_{A^*}((A^*)^{-j} z)^{\xi_5}\} \\ &\quad= C \min\{(|\det A|^{-j} \rho_{A^*}(z))^{\xi_4}, (|\det A|^{-j} \rho_{A^*}(z))^{\xi_5}\} \\ &\quad\geq C, \ z \in \operatorname{supp} \hat{\varphi}_j \text{ and } 1 \leq t \leq |\det A|.\end{aligned}$$

Then, there exist $C_1, C_2 > 0$ such that

$$C_1 \leq \rho_{A^*}((A^*)^{-j-\log_{|\det A|} t} z) \leq C_2, \ z \in \operatorname{supp} \hat{\varphi}_j \text{ and } 1 \leq t \leq |\det A|.$$

Hence, if $\hat{\psi}(x) \neq 0$, when $C_1 \leq \rho_{A^*}(x) \leq C_2$, we can write, for every $K \geq j_0$,

$$\begin{aligned}f * \varphi_j &= (\hat{f} \hat{\varphi}_j)^{\check{}} = (\hat{f}(\cdot) \hat{\varphi}_j(\cdot) \hat{\Lambda}((A^*)^{-j} \cdot))^{\check{}} \\ &= \Big(\frac{\hat{\varphi}_j(\cdot)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t} \cdot)} \hat{\psi}((A^*)^{-j-\log_{|\det A|} t} \cdot) \hat{f}(\cdot) \hat{\Lambda}((A^*)^{-j} \cdot)\Big)^{\check{}}, \\ &\qquad 1 \leq t \leq |\det A|.\end{aligned}$$

Indeed, note that $\hat{\Lambda}((A^*)^{-j} x) = 1$, provided that $\rho_{A^*}(x) \leq |\det A|^{j+K}$. Then, $\hat{\Lambda}((A^*)^{-j} x) = 1$, for every $x \in \operatorname{supp} \hat{\varphi}_j$ when $K \geq j_0$.

Interchange formula leads to

$$|(f * \varphi_j)(x)| \leq C \times$$
$$\int_{\mathbb{R}^n} \Big| \Big(\frac{\hat{\varphi}_j(z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t} z)}\Big)^{\check{}}(y) \big(\hat{\psi}((A^*)^{-j-\log_{|\det A|} t} z) \hat{f}(z) \hat{\Lambda}((A^*)^{-j} z)\big)^{\check{}}(x-y) \Big| dy,$$

for $1 \leq t \leq |\det A|$, $x \in \mathbb{R}^n$.

Let $a > 0$. By [2, Lemma 2.2] we get

$$(1 + \rho_A(y))^a \leq C(1 + |y|^{2m}), \quad y \in \mathbb{R}^n,$$

for some $m \in \mathbb{N}$. Moreover, if Δ denotes the Euclidean Laplacian operator, it follows that

$$|y|^{2m}\Big|\Big(\frac{\hat{\varphi}(u)}{\hat{\psi}((A^*)^{-\log_{|\det A|} t}u)}\Big)\check{}(y)\Big| = \Big|\Big(\Delta^m\Big(\frac{\hat{\varphi}(u)}{\hat{\psi}((A^*)^{-\log_{|\det A|} t}u)}\Big)\Big)\check{}(y)\Big| \leq C,$$

$$1 \leq t \leq |\det A| \quad \text{and} \quad y \in \mathbb{R}^n.$$

Then, by making changes of variables we obtain

$$\begin{aligned}
\Big|\Big(\frac{\hat{\varphi}_j(z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)}\Big)\check{}(y)\Big| &= \Big|\Big(\frac{\hat{\varphi}((A^*)^{-j}z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)}\Big)\check{}(y)\Big| \\
&= |\det A|^j\Big|\Big(\frac{\hat{\varphi}(z)}{\hat{\psi}((A^*)^{-\log_{|\det A|} t}z)}\Big)\check{}(A^jy)\Big| \\
&\leq C|\det A|^j(1 + \rho_A(A^jy))^{-a}, \ 1 \leq t \leq |\det A| \text{ and } y \in \mathbb{R}^n.
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
&\Big(\Big(\frac{\hat{\varphi}_j(z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)}\Big)\check{}(y)\Big(\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z)\Big)\check{}(x-y)\Big)\hat{}(u) \\
&= \Big(\frac{\hat{\varphi}_j(z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)}\Big) * \Big(e^{-ixz}\hat{\psi}(-(A^*)^{-j-\log_{|\det A|} t}z)\hat{f}(-z)\hat{\Lambda}(-(A^*)^{-j}z)\Big)(u),
\end{aligned}$$

for $1 \leq t \leq |\det A|$ and x, $u \in \mathbb{R}^n$. The support of this last function is contained in the set

$$\operatorname{supp}\hat{\varphi}_j + \operatorname{supp}\hat{\Lambda}(-(A^*)^{-j}\cdot) \subseteq \overline{B}_{\rho_{A^*}}(0, |\det A|^{j+K+L}),$$

for some $L > 0$ that is not depending on j.

Assume that $0 < r < 1$. By applying [27, Proposition 1.3.2] we get

$$\begin{aligned}
|(f * \varphi_j)(x)|^r &\leq C|\det A|^{(j+K)(1-r)} \int_{\mathbb{R}^n} \Big|\Big(\frac{\hat{\varphi}_j(z)}{\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)}\Big)\check{}(y)\Big|^r \\
&\quad \times \Big|\Big(\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z)\Big)\check{}(x-y)\Big|^r dy \\
&\leq C|\det A|^{jr+(j+K)(1-r)} \int_{\mathbb{R}^n} (1 + \rho_A(A^jy))^{-ar} \\
&\quad \times \Big|(\hat{\psi}((A^*)^{-j-\log_{|\det A|} t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}(x-y)\Big|^r dy,
\end{aligned}$$

$$1 \le t \le |\det A| \text{ and } x \in \mathbb{R}^n.$$

Then, a standard decomposition procedure by using (1) leads to

$$\begin{aligned}
&|(f*\varphi_j)(x)|^r \le C|\det A|^{jr+(j+K)(1-r)} \\
&\quad\times\Bigg(\sum_{l=0}^{\infty}\int\limits_{|\det A|^{-j+l}<\rho_A(y)\le|\det A|^{-j+l+1}}\frac{|(\hat{\psi}((A^*)^{-j-\log_{|det|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}(x-y)|^r}{(1+|\det A|^j\rho_A(y))^{ar}}dy \\
&\quad+\int\limits_{\rho_A(y)\le|\det A|^{-j}}|(\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}(x-y)|^r dy\Bigg) \\
&\quad\le C|\det A|^{jr+(j+K)(1-r)} \\
&\quad\times\Bigg(\sum_{l=0}^{\infty}\frac{1}{|\det A|^{lar}}\int\limits_{\rho_A(y)\le|\det A|^{-j+l+1}}|(\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}(x-y)|^r dy \\
&\quad+\int\limits_{\rho_A(y)\le|\det A|^{-j}}|(\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}(x-y)|^r dy\Bigg) \\
&\quad\le C|\det A|^{K(1-r)}\mathcal{M}_{\rho_A}(|\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}|^r)(x),
\end{aligned}$$

for $1 \le t \le |\det A|$ and $x \in \mathbb{R}^n$, provided that $ar > 1$.

We fix $0 < r < \min\{1, q\}$. By integrating in $t \in [1, |\det A|]$, since the Hardy–Littlewood maximal function $\mathcal{M}_{\rho_A}$ is bounded from $L_{p/r}(\mathbb{R}^n)$ into itelf ([6, Theorem 2.4]), it follows

$$\begin{aligned}
&\|(f*\varphi_j)^r\|_{p/r} \\
&\le C|\det A|^{K(1-r)}\int_1^{|\det A|}\|\mathcal{M}_{\rho_A}(|\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}|^r)\|_{p/r}dt \\
&\le C|\det A|^{K(1-r)}\int_1^{|\det A|}\|(\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}\|_p^r dt.
\end{aligned}$$

Then,

$$\begin{aligned}
&|\det A|^{j\alpha r}\|f*\varphi_j\|_p^r \\
&\le C|\det A|^{K(1-r)+j\alpha r}\int_1^{|\det A|}\|(\hat{\psi}((A^*)^{-j-\log_{|\det A|}t}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}\|_p^r\frac{dt}{t} \\
&\le C|\det A|^{K(1-r)}\int_{|\det A|^{-j-1}}^{|\det A|^{-j}}u^{-\alpha r}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z)\hat{\Lambda}((A^*)^{-j}z))\check{}\|_p^r\frac{du}{u}.
\end{aligned}$$

We conclude, by using Jensen inequality,

$$\Big(\sum_{j=1}^{\infty}(|\det A|^{j\alpha}\|f*\varphi_j\|_p)^q\Big)^{1/q}$$

$$\leq C|\det A|^{K\frac{1-r}{r}}\Bigg(\Bigg(\int_0^1 u^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z))\check{}\|_p^q\frac{du}{u}\Bigg)^{1/q}$$

$$+\Bigg(\sum_{j=1}^{\infty}\int_{|\det A|^{-j-1}}^{|\det A|^{-j}} u^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z)(1-\hat{\Lambda}((A^*)^{-j}z)))\check{}\|_p^q\frac{du}{u}\Bigg)^{1/q}\Bigg). \quad (19)$$

We now analyze the second summand in the right hand side of (19).

Let $j\in\mathbb{N}$ and $u\in(0,\infty)$. As in (6), we write

$$f*\psi_{(u)}-f*\psi_{(u)}*\Lambda_j=\sum_{l=-\infty}^{\infty}(f*\psi_{(u)}-f*\psi_{(u)}*\Lambda_j)*\varphi_{l+j}$$

$$=\sum_{l=-\infty}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{},$$

when the convergence is understood in $S'(\mathbb{R}^n)$, in $L_p(\mathbb{R}^n)$ or, in a suitable way (via certain subsequence), pointwise a.e. $\mathbb{R}^n$. Note that $\hat{\Lambda}((A^*)^{-j}z)=1$, $\rho_{A^*}(z)\leq|\det A|^{K+j}$ and $\operatorname{supp}\hat{\varphi}((A^*)^{-l-j}\cdot)\subseteq\overline{B}_{\rho_{A^*}}(0,|\det A|^{l+j+j_0})$. Then

$$f*\psi_{(u)}-f*\psi_{(u)}*\Lambda_j$$

$$=\sum_{l=K-j_0}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{},$$

and we get

$$\int_{|\det A|^{-j-1}}^{|\det A|^{-j}} u^{-\alpha q}\Bigg\|\sum_{l=K-j_0}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{}\Bigg\|_p^q\frac{du}{u}$$

$$\leq C|\det A|^{\alpha jq}\sup_{|\det A|^{-j-1}<u<|\det A|^{-j}}$$

$$\Bigg\|\sum_{l=K-j_0}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{}\Bigg\|_p^q.$$

Assume that $u\in[|\det A|^{-j-1},|\det A|^{-j}]$. By proceeding as in the paragraph in the first part of this proof concerning to the estimate of $\sum_{l=1}^{\infty}\cdot$, it obtains

$$|\det A|^{\alpha j}\Big\|\sum_{l=K-j_0}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{}\Big\|_p$$
$$\le\sum_{l=K-j_0}^{\infty}|\det A|^{l(s_0-\alpha)}\Big\|\Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{H}((A^*)^{-j-l}z)(1-\hat{\Lambda}((A^*)^{-j}z))}{\rho_{A^*}((A^*)^{-j}z)^{s_0}}\Big)\check{}\Big\|_1$$
$$\times\|(|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\tilde{\varphi}}_{j+l}(z))\check{}\|_p,$$

where $\tilde{\varphi}_j$ is defined by (12) for every $j\in\mathbb{Z}$.

Also, we have

$$\Big\|\Big(\frac{\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{H}((A^*)^{-j-l}z)(1-\hat{\Lambda}((A^*)^{-j}z))}{\rho_{A^*}((A^*)^{-j}z)^{s_0}}\Big)\check{}\Big\|_1$$
$$=\Big\|\Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|}u}z)\hat{H}((A^*)^{-l}z)(1-\hat{\Lambda}(z)}{\rho_{A^*}(z)^{s_0}}\Big)\check{}\Big\|_1$$
$$\le\Big\|\Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|}u}z)\hat{H}((A^*)^{-l}z)}{\rho_{A^*}(z)^{s_0}}\Big)\check{}\Big\|_1$$
$$+\Big\|\Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|}u}z)\hat{H}((A^*)^{-l}z)}{\rho_{A^*}(z)^{s_0}}\Big)\check{}*\Lambda\Big\|_1$$
$$\le\Big\|\Big(\frac{\hat{\psi}((A^*)^{j+\log_{|\det A|}u}z)\hat{H}((A^*)^{-l}z)}{\rho_{A^*}(z)^{s_0}}\Big)\check{}\Big\|_1(1+\|\Lambda\|_1)\,,\ l\in\mathbb{Z}.$$

Hence by (P1) we get

$$|\det A|^{\alpha j}\Big\|\sum_{l=K-j_0}^{\infty}(\hat{f}(z)\hat{\psi}((A^*)^{\log_{|\det A|}u}z)(1-\hat{\Lambda}((A^*)^{-j}z))\hat{\varphi}((A^*)^{-l-j}z))\check{}\Big\|_p$$
$$\le\sum_{l=K-j_0}^{\infty}|\det A|^{l(s_0-\alpha)}\|(|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\tilde{\varphi}}_{j+l}(z))\check{}\|_p.$$

Then,

$$\Big(\sum_{j=1}^{\infty}\Big(\int_{|\det A|^{-j-1}}^{|\det A|^{-j}}u^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z)(1-\hat{\Lambda}((A^*)^{-j}z)))\check{}\|_p^q\frac{du}{u}\Big)^{1/q}\Big)$$
$$\le C\Big(\sum_{j=1}^{\infty}\Big(\sum_{l=K-j_0}^{\infty}|\det A|^{l(s_0-\alpha)}\|(|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\tilde{\varphi}}_{j+l}(z))\check{}\|_p\Big)^q\Big)^{1/q}$$

$$\leq C\begin{cases}\displaystyle\sum_{l=K-j_0}^{\infty}|\det A|^{l(s_0-\alpha)}\Big(\sum_{j=0}^{\infty}\|(|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\hat{\varphi}}_{j+l}(z))\check{}\|_p^q\Big)^{1/q}, \\ \qquad q\geq 1, \\ \displaystyle\Big(\sum_{l=K-j_0}^{\infty}|\det A|^{l(s_0-\alpha)q}\sum_{j=0}^{\infty}\|(|\det A|^{\alpha(j+l)}\hat{f}(z)\hat{\hat{\varphi}}_{j+l}(z))\check{}\|_p^q\Big)^{1/q}, \\ \qquad 0<q<1,\end{cases}$$

$$\leq C|\det A|^{(s_0-\alpha)(K-j_0)}\Big(\sum_{j=0}^{\infty}|\det A|^{jq\alpha}\|(\hat{f}\hat{\hat{\varphi}}_j)\check{}\|_p^q\Big)^{1/q}$$

$$\leq C|\det A|^{(s_0-\alpha)(K-j_0)}\Big(\sum_{j=0}^{\infty}|\det A|^{jq\alpha}\|(\hat{f}\hat{\varphi}_j)\check{}\|_p^q\Big)^{1/q}. \quad (20)$$

Since $\hat{\Psi}(x)\neq 0$, $\rho_{A^*}(x)\leq|\det A|$, by using (P2) we obtain

$$\|f*\varphi_0\|_p\leq C|\det A|^{K\frac{1-r}{r}}\|(\hat{\Psi}\hat{f}\hat{\Lambda})\check{}\|_p$$
$$\leq C|\det A|^{K\frac{1-r}{r}}(\|f*\Psi\|_p+\|(\hat{\Psi}\hat{f}(1-\hat{\Lambda}))\check{}\|_p)$$
$$\leq C\Big(\|f*\Psi\|_p+|\det A|^{K\frac{1-r}{r}+(s_0-\alpha)(K-j_0)}\Big(\sum_{j=0}^{\infty}|\det A|^{jq\alpha}\|(\hat{f}\hat{\varphi}_j)\check{}\|_p^q\Big)^{1/q}\Big). \quad (21)$$

From (19), (20) and (21), by choosing $0<r<1$ such that $\alpha-s_0>(1-r)/r$, we get

$$\Big(\sum_{j=0}^{\infty}|\det A|^{j\alpha q}\|f*\varphi_j\|_p^q\Big)^{1/q}$$
$$\leq C\Big(|\det A|^{K\frac{1-r}{r}}\Big(\int_0^1 u^{-\alpha q}\|(\hat{\Psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z))\check{}\|_p^q\frac{du}{u}\Big)^{1/q}$$
$$+\|f*\Psi\|_p+|\det A|^{K\frac{1-r}{r}+(s_0-\alpha)(K-j_0)}\Big(\sum_{j=0}^{\infty}|\det A|^{jq\alpha}\|(\hat{f}\hat{\varphi}_j)\check{}\|_p^q\Big)^{1/q}\Big)$$
$$\leq C\Big(\|f*\Psi\|_p+\Big(\int_0^1 u^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}z)\hat{f}(z))\check{}\|_p^q\frac{du}{u}\Big)^{1/q}\Big)$$
$$+\frac{1}{2}\Big(\sum_{j=0}^{\infty}|\det A|^{jq\alpha}\|(\hat{f}\hat{\varphi}_j)\check{}\|_p^q\Big)^{1/q},$$

provided that K is large enough.

It concludes that

$$\left(\sum_{j=0}^{\infty}|\det A|^{\alpha jq}\|f*\varphi_j\|_p^q\right)^{1/q}$$
$$\leq C\left(\|f*\Psi\|_p+\left(\int_0^1 u^{-\alpha q}\|(\hat{\psi}((A^*)^{\log_{|\det A|}u}\cdot)\hat{f}(\cdot))\check{}\|_p^q\frac{du}{u}\right)^{1/q}\right).$$

Thus the proof is completed.

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