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Examples of indefinite globally framed f-structures on compact Lie groups

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Abstract. We extend to the semi-Riemannian context the well-known results obtained by Blair, Ludden and Yano on toroidal principal bundles endowed with a metric globally framed f-structure. In this way we obtain examples of compact indefinite $\mathcal{S}$-manifolds. Then, we define an indefinite $\mathcal{S}$-structure on the Lie group $U(2)$ with a Lorentz left-invariant metric and, applying our results, we construct commutative diagrams involving semi-Riemannian submersions and Hopf fibrations. We also prove that $U(2)$ with such a structure is foliated by Reinhart lightlike hypersurfaces. Finally, we consider a normal indefinite globally framed f-structure on the Lie group $U(4)$ proving that it projects on $U(4)/U(3)$ in a Sasakian structure isomorphic to the standard Sasakian structure of $\mathbb{S}^7$.

1. Introduction

Studies on toroidal principal bundles have been started in the Riemannian setting by Blair, Ludden, Yano, Morimoto et al. (cf. for example [2], [4], [16]), giving the fundamental relationships between f-structures and Riemannian submersions. In particular, in [4] they constructed this kind of principal bundle, endowing the total space with a $\mathcal{K}$-structure and studying some relationships between such a structure and a Kähler structure on the base manifold. Here we are concerned with some extensions of their results to toroidal principal bundles in the semi-Riemannian context, and this involves in a natural way semi-Riemannian submersions ([17], [18]) with totally geodesic fibres.

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The present paper is organized as follows. After a brief exposition of the standard facts on indefinite Sasakian manifolds, indefinite $g.f.f$-manifolds and indefinite $\mathcal{S}$-manifolds, in Section 3 we start with an extension of the results of [4] to the semi-Riemannian case. More precisely, we consider a connected, compact smooth manifold M endowed with a normal indefinite $g.f.f$-structure and, under suitable hypotheses, we get that the projection of such structure is either an (indefinite) Kähler structure or an (indefinite) Sasakian one. On the other hand, we show that the total space of a toroidal principal bundle over a Kähler manifold, indefinite or not, may admit indefinite metrics, and the lift of the (indefinite) Kähler structure gives rise to normal $g.f.f$-structures on the total space.

In Section 4 we construct a Lorentzian $\mathcal{S}$-structure on the compact Lie group $U(2)$ having two characteristic vector fields with different causal type. We also prove that $U(2)$ with such a structure is foliated by Reinhart lightlike hypersurfaces. Then, by applying the results of Section 3, we consider three quotient manifolds of $U(2)$ and we obtain different commutative diagrams involving semi-Riemannian submersion with totally geodesic fibres.

In Section 5 an example of normal indefinite $g.f.f$-structure on the Lie group $U(4)$ is constructed, proving that it is not an indefinite $\mathcal{S}$-structure. Moreover, we project this structure in two different ways to obtain different contact structures and commutative diagrams. More precisely, the first way is carried out by using the results of Section 3, while for the second one we will use the general results of [1] and the theory of homogeneous spaces.

All manifolds, tensor fields and maps are assumed to be smooth, and all manifolds are supposed to be connected. We shall use the Einstein convention, omitting the sum symbol for repeated indexes. Later on, we shall use the symbol $\mathfrak{X}(M)$ to denote the Lie algebra of vector fields on a manifold M. Following the notations of S. Kobayashi and K. Nomizu ([14]), for the curvature tensor R we have $R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and $R(X,Y,Z,W) = g(R(Z,W,Y),X)$, for any $X,Y,Z,W \in \mathfrak{X}(M)$.

2. Preliminaries

Following [3], [5], [21], we recall some definitions. An almost contact manifold is a $(2n+1)$-dimensional manifold M endowed with an almost contact structure that is with a $(1,1)$-tensor field f of rank $2n$, a 1-form η and a vector field ξ satisfying $f^2(X) = -X + \eta(X)\xi$, $\eta(\xi) = 1$. Moreover, if g is a semi-Riemannian metric on M^{2n+1} such that, for any $X, Y \in \mathfrak{X}(M^{2n+1})$, $g(fX, fY) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$, where $\varepsilon = \pm 1$ according to the causal character of ξ, M^{2n+1} is called an indefinite almost contact metric manifold. Such a manifold is said to be an indefinite contact metric manifold if $d\eta = \Phi$, Φ being defined by $\Phi(X,Y) = g(X, fY)$. Furthermore, if the structure (f, ξ, η) is normal, i.e. $N = [f,f] + 2d\eta \otimes \xi = 0$, then the indefinite contact metric structure is called an indefinite Sasakian structure and the manifold $(M^{2n+1}, f, \xi, \eta, g)$ is called an indefinite Sasakian manifold.

In [5] we studied a generalization of these structures. In the Riemannian case such structures have been studied by Blair in [3], by Goldberg and Yano in [13]. A manifold M is called a $g.f.f$-manifold if it is endowed with a $(1,1)$-tensor field φ of constant rank, such that $\ker\varphi$ is parallelizable i.e. there exist global vector fields ξ_i, $i \in \{1, \ldots, s\}$, and 1-forms η^i, satisfying $\varphi^2 = -I + \eta^i \otimes \xi_i$ and $\eta^i(\xi_j) = \delta^i_j$.
A $g.f.f$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i)$, $i \in \{1, \ldots, s\}$, is said to be an indefinite $g.f.f$-manifold if it is given a semi-Riemannian metric g satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon_i \eta^i(X)\eta^i(Y)$$

for any vector fields X, Y, being $\varepsilon_i = \pm 1$ according to whether ξ_i is spacelike or timelike. Then, for any $i \in \{1, \ldots, s\}$ and $X \in \mathfrak{X}(M^{2n+s})$, one has $\eta^i(X) = \varepsilon_i g(X, \xi_i)$. The 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(M^{2n+s})$ and the normality condition is expressed by the vanishing of the tensor field $N = N_\varphi + 2d\eta^i \otimes \xi_i$, N_φ being the Nijenhuis torsion of φ.

An indefinite $g.f.f$-manifold is called indefinite $\mathcal{K}$-manifold if it is normal and $d\Phi = 0$ (cf. [2] for the Riemannian context). Special subclasses of indefinite $\mathcal{K}$-manifold are: the indefinite $\mathcal{S}$-manifolds which verify $d\eta^i = \Phi$, for any $i \in \{1, \ldots, s\}$, and the indefinite $\mathcal{C}$-manifolds which verify $d\eta^i = 0$, for any $i \in \{1, \ldots, s\}$.

We focus on indefinite $\mathcal{S}$-manifolds. As proved in [5], the Levi–Civita connection of an indefinite $\mathcal{S}$-manifold satisfies:

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X),$$

where $\bar{\xi} = \sum_{i=1}^{s} \xi_i$ and $\bar{\eta} = \sum_{i=1}^{s} \varepsilon_i \eta^i$. Note that, for $s = 1$, we reobtain the notion of indefinite Sasakian manifold.

We recall that $\nabla_X \xi_i = -\varepsilon_i \varphi X$ and $\ker\varphi$ is an integrable flat distribution since $\nabla_{\xi_i}\xi_j = 0$. We remark that an indefinite $\mathcal{S}$-manifold is never flat since

$K(X, \xi_i) = \varepsilon_i$ for any $X \in \mathcal{D}$ where $\mathcal{D}$ denotes the distribution $\mathrm{Im}(\varphi)$. For more details we refer to [5], where we describe three examples of non compact indefinite $\mathcal{S}$-manifolds, more precisely we construct two different indefinite $\mathcal{S}$-structures with metrics of index $\nu = 2$ on $\mathbb{R}^6$ and an indefinite $\mathcal{S}$-structure with Lorentz metric on $\mathbb{R}^4$. In this paper we will give examples of compact indefinite $\mathcal{S}$-manifolds.

Finally, (cf. [11]), given a semi-Riemannian submersion $\pi : (M, g) \to (N, g')$, we have the following link between the sectional curvatures of a non degenerate horizontal 2-plane α in T_pM spanned by orthonormal vectors $\{X, Y\}$, $p \in M$, and the 2-plane α' in $T_{\pi(p)}N$ spanned by π_*X and π_*Y:

$$K(\alpha) = K'(\alpha') - 3g(A_XY, A_XY), \tag{1}$$

where A is the O'Neill fundamental tensor field of π, also called the integrability tensor.

3. Indefinite S-manifolds and principal bundles

Every $g.f.f$-manifold is subject to the following topological condition: it has to be either non compact or compact with vanishing Euler characteristic, since it admits never vanishing vector fields. This implies that such a manifold always admits semi-Riemannian metrics, with index ν, $1 \leq \nu \leq s$. More precisely, let $(M^{2n+s}, \xi_i, \eta^i, g)$ be a (normal) metric $g.f.f$-manifold, $i \in \{1, 2, \dots, s\}$. Let p be any integer with $1 \leq p \leq s$, and define a metric $\tilde{g}$ by

$$\tilde{g}(X, Y) = g(X, Y) - 2\sum_{j=1}^{p} \eta^j(X)\eta^j(Y),$$

for any $X, Y \in \mathfrak{X}(M)$. Such a metric is semi-Riemannian with index p determined by the timelike vector fields ξ_i with $i \leq p$. Namely $\tilde{g}(\xi_i, \xi_i) = 1 - 2\sum_{j=1}^{p} \delta_i^j \delta_i^j$, therefore, $\tilde{g}(\xi_i, \xi_i) = -1$ for any $i \leq p$ and $\tilde{g}(\xi_i, \xi_i) = 1$ for any $i \geq p+1$. It is easy to verify the compatibility condition of $\tilde{g}$. Finally $\tilde{\Phi} = \Phi$ implies that, starting from an $\mathcal{S}$-manifold, one gets an indefinite $\mathcal{S}$-manifold.

In a normal $g.f.f$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ the distribution $\mathcal{F}$ spanned by the ξ_i's is completely integrable. We would like to look at a structure of smooth manifold on the quotient space $M^{2n+s}/\mathcal{F}$. By a result due to Palais [19], it is well known that for this purpose it is necessary to require the regularity condition for the distribution $\mathcal{F}$ and the compactness of its leaves. Moreover, if in addition, M^{2n+s} is connected then the map $\pi : M^{2n+s} \to M^{2n+s}/\mathcal{F}$ is a C^∞-fibration having the leaves of $\mathcal{F}$ as fibres that turn out to be all C^∞-isomorphic.

If $\mathcal{F}$ is regular and the vector fields ξ_i are regular then the normal $g.f.f$-structure and the $g.f.f$-manifold are called regular.

So, as it is required in the Riemannian case, ([4]), we assume the regularity of the normal $g.f.f$-structure and the compactness of M^{2n+s}. Thus, the maximal integral curves of each ξ_i are homeomorphic to $\mathbb{S}^1$, hence, being the ξ_i's linearly independent, the leaves of the distribution $\mathcal{F}$, i.e. the fibres of π, are homeomorphic to a torus $\mathbb{T}^s$.

The following theorem is due to Blair, Ludden and Yano:

Theorem 3.1 ([4]). *Let $(M^{2n+s}, f, \xi_i, \eta^i)$ be a connected and compact manifold with a regular normal $g.f.f$-structure. Then M^{2n+s} is the total space of a principal toroidal bundle over a complex manifold $N^{2n} = M^{2n+s}/\mathcal{F}$. Moreover, if M^{2n+s} is a $\mathcal{K}$-manifold, then N^{2n} is a Kähler manifold.*

The following result extends the above theorem to the indefinite case.

Theorem 3.2. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a normal, connected and compact regular indefinite $g.f.f$-manifold such that each ξ_i is Killing. Then M^{2n+s} is the total space of a principal toroidal bundle over a hermitian or indefinite hermitian manifold $N^{2n} = M^{2n+s}/\mathcal{F}$. Furthermore, if $(\varphi, \xi_i, \eta^i, g)$ is an indefinite $\mathcal{K}$-structure, then N^{2n} is either a Kähler manifold or an indefinite Kähler manifold.*

Proof. Applying Theorem 3.1, M^{2n+s} is the total space of a principal toroidal bundle over a complex manifold. We briefly recall the construction of the complex structure on N^{2n}(for more details [4], [11]). On the principal toroidal bundle $M^{2n+s}(N^{2n}, \mathbb{T}^s, \pi)$ we consider the Lie algebra valued connection form $\eta = (\eta^1, \eta^2, \dots, \eta^s)$. The distribution $\mathcal{D}$ is the horizontal distribution with respect to the connection form η since it is complementary to $\mathcal{F}$. Then, since the $g.f.f$-manifold is normal, we have $\mathcal{L}_{\xi_i}\varphi = 0$ and φ projects in a tensor field J on N^{2n} defined by

$$(JX)^h = \varphi(X^h),$$

where X^h is the horizontal lift of X. Being $N_J = 0$, N^{2n} is a complex manifold.

Now, since the ξ_i's are Killing, we can project the metric g to a metric $\tilde{g}$ on N^{2n}, putting

$$\tilde{g}(X, Y) \circ \pi = g(X^h, Y^h),$$

for any $X, Y \in \mathfrak{X}(N^{2n})$. Thus, π becomes a semi-Riemannian submersion and $\tilde{g}$ is a hermitian metric. We distinguish two cases determined by the index of $\tilde{g}$ which is equal to the index of the metric induced by g on the horizontal distribution. So, if $g_{|\mathcal{D}}$ is a Riemannian metric, then $(N^{2n}, J, \tilde{g})$ is a hermitian manifold, whereas if the index of $g_{|\mathcal{D}}$, necessarily even, is at least 2, then $(N^{2n}, J, \tilde{g})$ is an indefinite hermitian manifold.

Finally, if $(\varphi, \xi_i, \eta^i, g)$ is an indefinite $\mathcal{K}$-structure, then the ξ_i's are Killing and the above result applies. Thus, since the fundamental 2-form on N^{2n} verifies $\pi^*\Omega = \Phi$, then $d\Phi = 0$ implies $d\Omega = 0$ and N^{2n} is either a Kähler manifold or an indefinite Kähler manifold. $\square$

The following result has been proved by Blair in [2]. An analogous result has been found by Morimoto [16], [15] for contact structures.

Theorem 3.3 ([2])**.** *Let M^{2n+s} be the total space of a principal toroidal bundle over a Kähler manifold N^{2n} and $\gamma = (\eta^1, \eta^2, \ldots, \eta^s)$ a Lie algebra valued connection form on M^{2n+s} such that $d\eta^i = \pi^*\Omega$, for any $i \in \{1, \ldots, s\}$, where Ω is the fundamental 2-form on N^{2n} and π is the projection map. Then M^{2n+s} is an $\mathcal{S}$-manifold.*

The proof of the above theorem easily extends to the indefinite case. Namely, in the same hypotheses, allowing the base manifold $(N^{2n}, J, \tilde{g})$ to be possibly indefinite Kähler, M^{2n+s} admits indefinite $\mathcal{S}$-structures $(\varphi, \xi_i, \eta^i, g)$ such that $\mathrm{ind}(g) = \mathrm{ind}(\tilde{g}) + \mathrm{ind}(g_{|\mathbb{T}^s})$. Any such indefinite metric g is defined putting, for any $X, Y \in \mathfrak{X}(M^{2n+s})$

$$g(X,Y) = \tilde{g}(\pi_*(X), \pi_*(Y)) \circ \pi + \sum_{i=1}^{s} \varepsilon_i \eta^i(X)\eta^i(Y)$$

where $\varepsilon_i = \pm 1$ and the number of the $\varepsilon_i = -1$ depends on the index that one wants to prescribe on $\mathbb{T}^s$.

From [2] and [11, p. 134], we recall the following result.

Theorem 3.4. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a connected, compact, regular $\mathcal{S}$-manifold, then there exist commutative diagrams of the type*

$$\begin{tikzcd} M^{2n+s} \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M^{2n+1} \arrow[dl, "\pi'"] \\ & N^{2n} & \end{tikzcd}$$

where $(N^{2n}, J, \tilde{g})$ is a Kähler manifold and M^{2n+1} is a connected, compact and regular Sasakian manifold which is the total space of an $\mathbb{S}^1$-bundle over N^{2n}. Furthermore, all the maps are Riemannian submersions.

Now, we state the semi-Riemannian version of the above theorem.

Theorem 3.5. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a connected, compact and regular indefinite $\mathcal{S}$-manifold. Then, there exist commutative diagrams of the type*

$$\begin{tikzcd} M^{2n+s} \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M^{2n+1} \arrow[dl, "\pi'"] \\ & N^{2n} & \end{tikzcd}$$

where $(N^{2n}, J, \tilde{g})$ is either a Kähler or an indefinite Kähler manifold and M^{2n+1} is a connected, compact and regular Sasakian or indefinite Sasakian manifold which is the total space of an $\mathbb{S}^1$-bundle over N^{2n}. Furthermore, all the maps are semi-Riemannian submersions.

Proof. As regards the constructions of the structures, the proof goes on as in Theorem 3.4 in [2], [11]. Now we discuss the aspect related to the metrics. Assuming firstly that the index of M^{2n+s} only involves the ξ_i's, by Theorem 3.2 it is obvious that the projection $\pi : M^{2n+s} \longrightarrow N^{2n}$ is a semi-Riemannian submersion from the indefinite $\mathcal{S}$-manifold M^{2n+s} over the Kähler manifold N^{2n}.

Now, chosen a characteristic vector field, for example ξ_s, following the arguments of the same theorem and applying them to the distribution $\mathcal{F}'$ spanned by $\xi_1, \dots, \xi_{s-1}$ we have that M^{2n+s} is a principal $\mathbb{T}^{s-1}$-bundle over M^{2n+1} and we denote by τ the projection and by $\gamma = (\eta^1, \dots, \eta^{s-1})$ the connection 1-form. Being $(\varphi, \xi_i, \eta^i, g)$ an indefinite $\mathcal{S}$-structure it turns out to be projectable, then we obtain a $(1,1)$-tensor field f, a vector field ξ and a 1-form η on M^{2n+1} given by $f(X) = \tau_*(\varphi(X^h))$, $\xi = \tau_*(\xi_s)$ and $\eta(X) \circ \tau = \eta^s(X^h)$, for any $X \in \mathfrak{X}(M^{2n+1})$. Finally we put $g'(X,Y) \circ \tau = g(X^h, Y^h)$, for any $X, Y \in \mathfrak{X}(M^{2n+1})$.

Of course, (f, ξ, η, g') is a Sasakian structure if ξ_s is spacelike and it is an indefinite Sasakian structure if ξ_s is timelike, τ is a semi-Riemannian submersion. Finally, being $(M^{2n+1}, f, \xi, \eta, g')$ a regular (indefinite) Sasakian manifold it is a principal $\mathbb{S}^1$-bundle over M^{2n+1}/ξ, that is over the Kähler manifold N^{2n}.

Now, suppose that all the ξ_i's are spacelike. Then N^{2n} is indefinite Kähler. Furthermore $(M^{2n+1}, f, \xi, \eta, g')$ is an indefinite Sasakian manifold and the metrics $g, g', \tilde{g}$ have the same index, necessarily even.

Finally, it can happen that some of the ξ_i's and an even number $2p$ of vector fields orthogonal to the ξ_i's contribute to the index of g. In this case N^{2n} is indefinite Kähler of index $2p$. Moreover $(M^{2n+1}, f, \xi, \eta, g')$ is either an indefinite Sasakian manifold with $\operatorname{ind} g' = 2p$, if the chosen ξ_s is spacelike, or an indefinite Sasakian manifold with $\operatorname{ind} g' = 2p + 1$, if the chosen ξ_s is timelike. In any case it is easy to check that $\pi' \circ \tau = \pi$. $\square$

A part of the above proof allows to state the following result.

Theorem 3.6. *Let* $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ *be a normal, connected and compact regular indefinite* $g.f.f$*-manifold such that each* ξ_i *is a Killing vector field. Then, there exist commutative diagrams of the type*

$$\begin{tikzcd} M^{2n+s} \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M^{2n+1} \arrow[dl, "\pi'"] \\ & N^{2n} & \end{tikzcd} .$$

Here $(N^{2n}, J, \tilde{g})$ *is either a hermitian or an indefinite hermitian manifold. The normal, connected, compact and regular metric almost contact or indefinite almost contact manifold* M^{2n+1}*, with Killing characteristic vector field, is the total space of an* $\mathbb{S}^1$*-bundle over* N^{2n}*. Furthermore, all the maps are semi-Riemannian submersions.*

Remark 3.7. The above theorem does not rule out the possibility that the manifold M^{2n+1} can be a (indefinite) Sasakian manifold. In fact it can happen that the structure of M^{2n+1} is Sasakian. Namely, looking at the proof of the previous theorem, if $d\eta^i$ and Φ coincide on the horizontal distribution, then one gets $d\eta' = \Phi'$ and the structure on M^{2n+1} becomes a (indefinite) Sasakian structure.

4. A Lorentzian $\mathcal{S}$-structure on the Lie group $U(2)$

Let us consider the 4-dimensional manifold $U(2)$ and the Lie algebra $\mathfrak{u}(2)$. We denote by ξ_1, ξ_2, X, Y the left-invariant vector fields on $U(2)$, determined, in the same order, by the following basis of $\mathfrak{u}(2)$:

$$\begin{pmatrix} \imath & 0 \\ 0 & 0 \end{pmatrix} = \imath E_{11}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & -\imath \end{pmatrix} = -\imath E_{22},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = E_{12} - E_{21}, \qquad \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} = \imath(E_{12} + E_{21}),$$

where $(E_{ij})_{i,j\in\{1,2\}}$ is the canonical basis of $gl(2, \mathbb{C})$. Then, we get:

$$[X, Y] = 2\xi_1 + 2\xi_2, \quad [X, \xi_i] = -Y, \quad [Y, \xi_i] = X, \quad [\xi_i, \xi_j] = 0$$

for any $i, j \in \{1, 2\}$. Let us consider the left-invariant 1-forms η^1 and η^2 determined by the dual 1-forms of $\imath E_{11}$ and $-\imath E_{22}$, respectively, and a left-invariant tensor field φ such that $\varphi(X) = Y$, $\varphi(Y) = -X$ and $\varphi(\xi_1) = \varphi(\xi_2) = 0$. The manifold $U(2)$ is compact, connected, with Euler number $\chi(U(2)) = 0$, and we can define a left-invariant Lorentz metric g such that the vector fields ξ_1, ξ_2, X, Y form an orthonormal basis, that is

$$g(X,X) = 1, \quad g(X,Y) = 0, \quad g(Y,Y) = 1, \quad g(\xi_1,\xi_1) = -1, \quad g(\xi_2,\xi_2) = 1,$$
$$g(\xi_i, Y) = 0, \quad g(\xi_i, X) = 0, \quad \text{for any } i \in \{1,2\}.$$

Hence, we obtain an indefinite $g.f.f$-structure and an easy computation shows that the structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is normal, the associated Sasaki 2-form Φ verifies $\Phi = d\eta^i$ for any $i \in \{1,2\}$ so that one has a Lorentz $\mathcal{S}$-structure on $U(2)$.

Using the Koszul's formula for the Levi–Civita connection ∇, we obtain:

$$\nabla_X X = \nabla_Y Y = 0, \quad \nabla_X Y = \xi_1 + \xi_2, \quad \nabla_Y X = -\xi_1 - \xi_2. \tag{2}$$

We also have $\nabla_U \xi_i = -\varepsilon_i \varphi U$ for any $U \in \mathfrak{X}(U(2))$, where $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$. Then we find

$$\nabla_X \xi_1 = Y, \quad \nabla_X \xi_2 = -Y, \quad \nabla_Y \xi_1 = -X, \quad \nabla_Y \xi_2 = X, \tag{3}$$

and, using the symmetry of the Levi–Civita connection, we get

$$\nabla_{\xi_1} X = 2Y, \quad \nabla_{\xi_2} X = 0, \quad \nabla_{\xi_1} Y = -2X, \quad \nabla_{\xi_2} Y = 0. \tag{4}$$

So, according to the above formulas, an easy computation gives

$$R(X, \varphi X, X) = R(X, Y, X) = -4Y,$$

from which we obtain

$$K(X, \varphi X) = R(X, \varphi X, X, \varphi X) = -g(R(X, \varphi X, X), \varphi X) = 4g(Y,Y) = 4$$

that is the φ-sectional curvature is constant. Finally, $K(X, \xi_1) = K(Y, \xi_1) = -1$ and $K(X, \xi_2) = K(Y, \xi_2) = 1$.

We denote by $\mathcal{U}(2)$ the Lorentz S-manifold $(U(2), \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ and we describe lightlike hypersurfaces of $\mathcal{U}(2)$. Recall that a hypersurface M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be lightlike if the induced metric g on M is degenerate ([10]). Then, one considers the radical distribution $\mathrm{Rad}(TM)$, such that for any $p \in M$

$$\mathrm{Rad}\, T_pM = \{V \in T_pM \mid g_p(V,W) = 0 \text{ for all } W \in T_pM\} = T_pM^\perp \cap T_pM.$$

Any decomposition $T_pM = \mathrm{Rad}\, T_pM \perp S(T_pM)$ gives rise to a non-degenerate distribution $S(TM)$ on M, called a screen distribution. We recall the following theorem ([10, p. 79]).

Theorem 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a unique rank one vector subbundle $ltr(M)$ of $T\bar{M}$, with base space M, such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighbourhood $\mathcal{U} \subset M$, there exists a unique section N of $ltr(M)$ on $\mathcal{U}$ satisfying:*

$$\bar{g}(N, E) = 1, \quad \bar{g}(N, N) = 0, \quad \bar{g}(N, W) = 0 \; \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}).$$

The vector bundle $ltr(M)$ is called the lightlike transversal vector bundle of M with respect to $S(TM)$.

Now, in $\mathcal{U}(2)$ we consider the distribution $\mathcal{D}' = \operatorname{span}\{\xi_1 + \xi_2, X, Y\}$, which is involutive since $[\xi_1 + \xi_2, X] = 2Y$, $[\xi_1 + \xi_2, Y] = -2X$ and $[X, Y] = 2(\xi_1 + \xi_2)$. Thus $\mathcal{U}(2)$ is foliated by the integral submanifolds of $\mathcal{D}'$. Obviously, $\xi_1 + \xi_2$ is a lightlike left-invariant vector field, while X and Y are spacelike. Moreover, from (2), (3) and (4) it is easy to check that any integral submanifold M of $\mathcal{D}'$ is a totally geodesic lightlike hypersurface of $\mathcal{U}(2)$ such that $\operatorname{Rad}(TM) = \operatorname{span}\{\xi_1+\xi_2\}$ and $S(TM) = \operatorname{span}\{X, Y\}$. We construct a global section N of $ltr(M)$.
Being $S(TM)^{\perp} = \operatorname{span}\{\xi_1, \xi_2\}$, we choose $E = \xi_1 + \xi_2$ so that the vector field $N = \frac{1}{2}(\xi_2 - \xi_1)$ verifies the conditions in Theorem 4.1. Moreover, it is easy to check that each section of $\operatorname{Rad}(TM)$ is a Killing vector field, so that $\operatorname{Rad}(TM)$ is a Killing distribution and M is a Reinhart lightlike manifold (Theorem 5.1, p. 49 in [10]). Hence, we can state the following result.

Theorem 4.2. *The Lorentz $\mathcal{S}$-manifold $\mathcal{U}(2) = (U(2), \varphi, \xi_i, \eta^i, g)$, $i \in \{1, 2\}$, is foliated by Reinhart lightlike manifolds.*

Now we describe some applications of Theorem 3.5 involving $\mathcal{U}(2)$.

4.1. A quotient manifold of $\mathcal{U}(2)$ carrying a Sasakian structure. We denote by L_1 the involutive distribution spanned by the timelike left-invariant vector field ξ_1 on $U(2)$. We consider $U(1) = \{z \in \mathbb{C} \mid |z| = 1\} = \mathbb{S}^1$ and the canonical map

$$j : U(1) \hookrightarrow U(2) \quad \text{such that} \quad z \longmapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Identifying $U(1)$ with $j(U(1))$, $U(1)$ turns out to be a closed subgroup of $U(2)$, hence we can consider the compact homogeneous manifold $U(2)/U(1)$ and the submersion given by the canonical surjection $\pi : U(2) \to U(2)/U(1)$, ([18, p. 312]). We will show that $U(2)/L_1 = U(2)/U(1)$. Namely, being ξ_1 left-invariant, it generates a 1-parameter group $\psi : \mathbb{R} \times U(2) \to U(2)$ of diffeomorphisms of $U(2)$

such that $\psi_t(A) = A\exp_{\xi_1} t$, for any $t \in \mathbb{R}$ and $A \in U(2)$. For the exponential map $\exp_{\xi_1}$ we have

$$\exp_{\xi_1}(t) = e^{t\xi_1} = \sum_m \frac{1}{m!} t^m \xi_1^m = \begin{pmatrix} e^{\imath t} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

therefore we identify the above matrix with its element $e^{\imath t}$. According to this remark, we have $\psi_A(t) = Ae^{\imath t}$, for any $t \in \mathbb{R}$ and $A \in U(2)$, and we obtain a right action of $\mathbb{S}^1$ on $U(2)$, for which the orbit space is given by

$$\{\psi_A(z) \mid A \in U(2),\ z \in U(1)\} = \{AU(1) \mid A \in U(2)\} = U(2)/U(1).$$

We denote the compact homogeneous manifold $U(2)/U(1)$ by M_3. Being M_3 diffeomorphic to $\mathbb{S}^3$, it is connected and simply connected, and the submersion $\pi : U(2) \to M_3$ has fibres diffeomorphic to $\mathbb{S}^1$, so ξ_1 is regular, the vertical distribution is $\mathcal{V} = L_1$ and the horizontal distribution $\mathcal{H}$ is g-orthogonal to $\mathcal{V}$ and φ-invariant since $\varphi(\mathcal{H}) \subset \mathcal{H}$.

Now, as characteristic vector field of the Lorentz $\mathcal{S}$-structure on $U(2)$, ξ_1 is Killing hence the metric g is projectable to a metric G defined by $G(U,V) \circ \pi = g(U^h, V^h)$, for any $U, V \in \mathfrak{X}(M_3)$, where U^h and V^h are the horizontal lift of U and V, respectively. The metric G on M_3 is positive definite and π is a semi-Riemannian submersion with fibres $(\mathbb{S}^1, g_{|\mathbb{S}^1})$ where $g_{|\mathbb{S}^1}$ has index 1. Similarly, being $\mathcal{L}_{\xi_1}\varphi = 0$ and $\mathcal{L}_{\xi_1}\eta^2 = 0$, φ and η^2 are projectable, thus we can define a 1-form η and a $(1,1)$-tensor field φ', on M_3, putting $\eta(U) \circ \pi = \eta^2(U^h)$ and $\varphi'(U) = \pi_*(\varphi(U^h))$, for any $U \in \mathfrak{X}(M_3)$.

Moreover, by the last formula, we get $\varphi' \circ \pi_* = \pi_* \circ \varphi$. Finally, being $[\xi_2, \xi_1] = 0$, ξ_2 is a basic vector field π-related to a vector field $\xi = \pi_*(\xi_2)$ on M_3 and $\varphi'(\xi) = \pi_*(\varphi(\xi_2)) = 0$. An easy computation shows that (φ', ξ, η, G) is a Sasakian structure on M_3. Since the φ-sectional curvature of $\mathcal{U}(2)$ is constant of value 4, $K(X, \xi_2) = K(Y, \xi_2) = 1$ and $K(X, \xi_1) = K(Y, \xi_1) = -1$, using (1) and being

$$A_X Y = \frac{1}{2} v[X, Y] = \xi_1, \quad A_X \xi_2 = \frac{1}{2} v[X, \xi_2] = 0, \quad A_Y \xi_2 = \frac{1}{2} v[Y, \xi_2] = 0,$$

for the sectional curvatures of M_3 we find:

$$K'(X', Y') = 1,\ K'(X', \xi) = K(X, \xi_2) = 1,\ K'(Y', \xi) = K(Y, \xi_2) = 1,$$

where $X' = \pi_*(X)$, $Y' = \pi_*(Y)$ and $Y = \varphi X$.

Accordingly, M_3 is a Sasakian space form of sectional curvature 1 hence it is isomorphic to $\mathbb{S}^3$, with its canonical Sasaki structure, and we can consider the following diagram

$$\begin{tikzcd} U(2) \arrow[r, "\pi"] & M_3 \arrow[d, "\mathcal{P}"] \\ & \mathbb{CP}_1(4) \end{tikzcd}$$

where $\mathcal{P} : M_3 \to \mathbb{CP}_1(4)$ is the Riemannian submersion coming from the Hopf fibration and the isomorphism between M_3 and $\mathbb{S}^3$.

4.2. A quotient manifold of $\mathcal{U}(2)$ carrying an indefinite Sasakian structure. We denote by L_2 the involutive distribution spanned by the spacelike left-invariant vector field ξ_2 on $U(2)$. Again, looking at the Lie group $U(1) = \mathbb{S}^1$, we consider the map

$$\bar{j} : U(1) \hookrightarrow U(2) \quad \text{such that } z \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix},$$

which allows to regard $U(1)$ as a Lie subgroup of $U(2)$ identifying $U(1)$ and $\bar{j}(U(1))$. We will put $\overline{U(1)} = \bar{j}(U(1))$. Hence, we obtain a compact homogeneous manifold $U(2)/\overline{U(1)}$. We consider the 1-parameter group of diffeomorphisms on $U(2)$ generated by ξ_2, $\phi : \mathbb{R} \times U(2) \to U(2)$ such that, for any $A \in U(2)$ and $t \in \mathbb{R}$, $\phi_t(A) = A \ \exp_{\xi_2}(t)$. Since we have

$$\exp_{\xi_2}(t) = e^{t\xi_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-it} \end{pmatrix}, \tag{6}$$

then the orbit space $U(2)/L_2$ of the ξ_2-action is $\{A\overline{U(1)} \mid A \in U(2)\} = U(2)/\overline{U(1)}$. To simplify the notation, we denote by M_3' this compact homogeneous manifold and by $\pi' : U(2) \to M_3'$ the related smooth submersion, whose fibres are diffeomorphic to $\mathbb{S}^1$. Again, the vector field ξ_2 is regular and Killing, the vertical distribution $\mathcal{V}$ of π' and the horizontal distribution $\mathcal{H}$ are given by $\mathcal{V} = L_2$ and $\mathcal{H} = \mathrm{span}\{X, Y, \xi_1\}$. Arguing as in the above subsection, one projects the Lorentz $\mathcal{S}$-structure of $U(2)$ in a Lorentz Sasaki structure $(\varphi'', \xi', \eta', G')$ on M_3' and π' becomes a semi-Riemannian submersion with Riemannian fibres. As regards the sectional curvatures of M_3', one has

$$K(\pi'_*(X), \pi'_*(Y)) = 7, \quad K(\pi'_*X, \xi') = K(\pi'_*Y, \xi') = -1. \tag{7}$$

Theorem 4.3. *The Lorentz Sasakian manifold* $(M_3', \varphi'', \xi', \eta', G')$ *is a principal circle bundle over* $\mathbb{CP}_1(4)$ *and we obtain*

$$\begin{array}{ccc} U(2) & \xrightarrow{\pi} & M_3' \\ & & \big\downarrow \mathcal{P}' \\ & & \mathbb{CP}_1(4) \end{array}$$

where $\mathcal{P}' : M_3' \to \mathbb{CP}_1(4)$ *is a Lorentz submersion.*

PROOF. We consider the new metric $\tilde{G}'$ on M_3' defined by $\tilde{G}' = G' + 2\eta' \otimes \eta'$. This metric is Riemannian and complete since M_3' is compact. It is easy to verify that $(M_3', \varphi'', \xi', \eta', \tilde{G}')$ is a Sasakian manifold of φ''-sectional curvature constant of value 1. It follows that, as Sasakian manifold, $(M_3', \varphi'', \xi', \eta', \tilde{G}')$ is isomorphic to $\mathbb{S}^3$ with its standard Sasakian structure. Thus, the manifold M_3' is the total space of a principal circle bundle over a Kähler manifold of holomorphic curvature equal to 4, as it easily follows from (1), that is over $\mathbb{CP}_1$. □

Remark 4.4. The isomorphism described in the previous theorem can be obtained using a theorem of Takahashi. Namely, putting $\bar{\varphi} = \varphi''$, $\bar{\xi} = -\xi'$, $\bar{\eta} = -\eta'$, $\bar{G} = -G'$, one obtains $(M_3', \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{G})$ with $\mathrm{ind}(\bar{G}) = 2$, $\bar{\xi}$ spacelike and $\bar{\varphi}$-sectional curvature -7. Then, by Theorem 1 in section 4 of [21] such a structure is $\mathcal{D}$-homothetic to $\tilde{S}_0^3$, model space of constant sectional curvature 1, that is to S^3. The $\mathcal{D}$-homothetic transformation is determined by $\alpha = \frac{-7+3}{4} = -1$, so one gets

$$\tilde{\varphi} = \bar{\varphi} = \varphi'' \quad \tilde{\xi} = -\bar{\xi} = \xi', \quad \tilde{\eta} = -\bar{\eta} = \eta', \quad \tilde{g} = -\bar{G} + 2\bar{\eta} \otimes \bar{\eta} = G' + 2\eta' \otimes \eta'$$

going back to $(M_3', \varphi'', \xi', \eta', \tilde{G}')$.

4.3. A quotient manifold of $\mathcal{U}(2)$ by a torus with a Lorentzian metric. We denote by L the involutive distribution of rank 2 spanned by the left-invariant vector fields ξ_1, ξ_2 on $U(2)$. Looking at the subset $\left\{ \left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right) | z_1, z_2 \in \mathbb{S}^1 \right\}$ of $U(2)$, we obtain a compact abelian Lie subgroup of $U(2)$ that is a maximal torus. We denote it by $\mathbb{T}^2$ and notice that $\mathbb{T}^2 = U(1)\overline{U(1)} = \overline{U(1)}U(1)$. Then $U(2)/\mathbb{T}^2$ is a homogeneous manifold.
We consider the action of $\mathbb{R}^2$ on $U(2)$, $\Psi : \mathbb{R}^2 \times U(2) \to U(2)$ such that, for any $A \in U(2)$ and $(t_1, t_2) \in \mathbb{R}^2$,

$$\Psi_{(t_1,t_2)}(A) = L_A \exp\,(t_1\xi_1 + t_2\xi_2) = A\ \exp\,(t_1\xi_1) \exp\,(t_2\xi_2).$$

From (5) and (6) the orbit space $U(2)/L$ of Ψ is

$$\left\{ A \begin{pmatrix} e^{it_1} & 0 \\ 0 & e^{-it_2} \end{pmatrix} \mid A \in U(2),\ (t_1, t_2) \in \mathbb{R}^2 \right\} = U(2)/\mathbb{T}^2,$$

and we denote by $\tau : U(2) \to U(2)/\mathbb{T}^2$ the canonical submersion, whose fibres are diffeomorphic to a torus endowed with a Lorentz metric. Therefore the Lorentz $\mathcal{S}$-structure of $U(2)$ turns out to be regular.

Being $U(2)$ a compact, connected and regular Lorentz $\mathcal{S}$-manifold, applying Theorem 3.5, we obtain a Kähler structure on $U(2)/\mathbb{T}^2$ and the following commutative diagrams:

$$\begin{tikzcd} U(2) \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & M_3 \arrow[dl, "\kappa"] \\ & U(2)/\mathbb{T}^2 & \end{tikzcd} \qquad \begin{tikzcd} U(2) \arrow[rr, "\pi'"] \arrow[dr, "\tau"'] & & M'_3 \arrow[dl, "\kappa'"] \\ & U(2)/\mathbb{T}^2 & \end{tikzcd} .$$

In fact, for any $A \in U(2)$, $\tau(A) = AU(1)\overline{U(1)} = \kappa \circ \pi(A)$ and $\tau(A) = AU(1)\overline{U(1)} = A\overline{U(1)}U(1) = \kappa' \circ \pi'(A)$. Moreover, the maps are semi-Riemannian submersions.

Using (1), the holomorphic sectional curvature of $U(2)/\mathbb{T}^2$ is given by

$$K(\tau_*(X), \tau_*(Y)) = 4 + 3g(A^\tau_X Y, A^\tau_X Y) = 4,$$

where $Y = \varphi X$ and $A^\tau_X Y = \frac{1}{2}v[X,Y] = \xi_1 + \xi_2$. Hence $U(2)/\mathbb{T}^2$ is a Kähler manifold with constant holomorphic curvature 4, that is holomorphically isometric to $\mathbb{CP}_1(4)$.

Remark 4.5. We observe that, being $\mathbb{T}^2$ a maximal torus in $U(2)$, the homogeneous manifold $U(2)/\mathbb{T}^2$ is diffeomorphic to the classical flag manifold $F(2) = U(2)/U(1) \times U(1)$. One can see also [6], [7], [20] where the study of flag manifolds $F(n)$, $n \geq 2$, is carried out through the theory of tournaments.

So, up to isometries, we obtain the following theorem

Theorem 4.6. *All the following diagrams commute*

$$\begin{tikzcd} U(2) \arrow[r, "\pi"] \arrow[d, dash, "\pi'"'] \arrow[dr, "\tau"] & M_3 \arrow[d, dash, "\mathcal{P}"] \\ M'_3 \arrow[r, "\mathcal{P}'"'] & \mathbb{CP}_1 \end{tikzcd} .$$

Remark 4.7. We claim that the manifolds in the above diagram satisfy different Osserman conditions. In particular, M_3', being a Lorentz Sasaki space form by (7), turns out to be globally null Osserman with respect to the characteristic vector field. $\mathbb{CP}_1(4)$, being a Riemannian complex space form, is pointwise Osserman and M_3, being a real space form, is globally Osserman (for more details see [12]).

5. An indefinite $g.f.f$-structure on the Lie group $U(4)$

Let $U(4) = \{A \in GL(4,\mathbb{C}) \mid A\bar{A}^t = I\}$. To describe a basis of the Lie algebra $\mathfrak{u}(4)$ we again consider the four 2×2 matrices E_{ij} of the canonical basis of $gl(2,\mathbb{C})$, the following sixteen matrices as basis of $\mathfrak{u}(4)$ and the left-invariant vector fields obtained by them, using to this end the symbol $\simeq$:

$$\xi_1 \simeq: \left(\begin{array}{c|c} \imath E_{11} & 0 \\ \hline 0 & 0 \end{array}\right), \quad \xi_2 \simeq: \left(\begin{array}{c|c} -\imath E_{22} & 0 \\ \hline 0 & 0 \end{array}\right), \quad \xi_3 \simeq: \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \imath E_{11} \end{array}\right),$$

$$\xi_4 \simeq: \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\imath E_{22} \end{array}\right), \quad X_1 \simeq: \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right), \quad Y_1 \simeq: \left(\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right),$$

$$X_2 \simeq: \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X \end{array}\right), \quad Y_2 \simeq: \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Y \end{array}\right),$$

$$X_{ij} \simeq: \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline -E_{ji} & 0 \end{array}\right), \; Y_{ij} \simeq: \left(\begin{array}{c|c} 0 & \imath E_{ij} \\ \hline \imath E_{ji} & 0 \end{array}\right), i,j \in \{1,2\}$$

where $X = E_{12} - E_{21}$, $Y = \imath(E_{12} + E_{21})$ and 0 represents the 2×2 null matrix. A straightforward computation shows that the non vanishing brackets are given by

$$[\xi_1, Y_{12}] = [\xi_4, Y_{12}] = -X_{12}, [\xi_1, Y_1] = [\xi_2, Y_1] = -X_1, \; [\xi_1, X_{11}] = [X_{11}, \xi_3] = Y_{11},$$

$$[\xi_1, Y_{11}] = [Y_{11}, \xi_3] = -X_{11}, [\xi_1, X_{12}] = [\xi_4, X_{12}] = Y_{12}, [\xi_3, Y_2] = [\xi_4, Y_2] = -X_2,$$

$$[\xi_2, X_{21}] = [\xi_3, X_{21}] = -Y_{21}, [\xi_2, Y_{21}] = [\xi_3, Y_{21}] = X_{21}, \; [\xi_3, X_2] = [\xi_4, X_2] = Y_2,$$

$$[\xi_2, X_{22}] = [X_{22}, \xi_4] = -Y_{22}, [\xi_2, Y_{22}] = [Y_{22}, \xi_4] = X_{22}, \; [\xi_1, X_1] = [\xi_2, X_1] = Y_1,$$

and by the following skew-symmetric table, where we indicate the vector fields X_i and Y_i by the index i and $\underline{i}$ respectively and the vector fields X_{ij} and Y_{ij} by ij and $\underline{ij}$ respectively.

	1	$\underline{1}$	11	$\underline{11}$	12	$\underline{12}$	21	$\underline{21}$	22	$\underline{22}$	2	$\underline{2}$
1	0	$2\xi_1+2\xi_2$	-21	$-\underline{21}$	-22	$-\underline{22}$	11	$\underline{11}$	12	$\underline{12}$	0	0
$\underline{1}$	-	0	$\underline{21}$	-21	$\underline{22}$	-22	$\underline{11}$	-11	$\underline{12}$	-12	0	0
11	-	-	0	$2\xi_1-2\xi_3$	-2	$-\underline{2}$	-1	$\underline{1}$	0	0	12	$\underline{12}$
$\underline{11}$	-	-	-	0	$\underline{2}$	-2	$-\underline{1}$	-1	0	0	$\underline{12}$	-12
12	-	-	-	-	0	$2\xi_1+2\xi_4$	0	0	-1	$\underline{1}$	-11	$\underline{11}$
$\underline{12}$	-	-	-	-	-	0	0	0	$-\underline{1}$	-1	$-\underline{11}$	-11
21	-	-	-	-	-	-	0	$-2\xi_2-2\xi_3$	-2	$-\underline{2}$	22	$\underline{22}$
$\underline{21}$	-	-	-	-	-	-	-	0	$\underline{2}$	-2	$\underline{22}$	-22
22	-	-	-	-	-	-	-	-	0	$2\xi_4-2\xi_2$	-21	$\underline{21}$
$\underline{22}$	-	-	-	-	-	-	-	-	-	0	$-\underline{21}$	-21
2	-	-	-	-	-	-	-	-	-	-	0	$2\xi_3+2\xi_4$
$\underline{2}$	-	-	-	-	-	-	-	-	-	-	-	0

On the compact, connected manifold $U(4)$ we define a left-invariant metric tensor field g, $\mathrm{ind}(g) = 3$, such that the above sixteen vector fields form an orthonormal basis, namely we require

$$g(X_1, X_1) = g(Y_1, Y_1) = -1, \quad g(X_2, X_2) = g(Y_2, Y_2) = 1, \quad g(\xi_1, \xi_1) = -1,$$
$$g(\xi_2, \xi_2) = g(\xi_3, \xi_3) = g(\xi_4, \xi_4) = 1, \quad g(X_{ij}, X_{ij}) = g(Y_{ij}, Y_{ij}) = 1 \quad \forall i, j \in \{1, 2\}.$$

Let η^a denote the left-invariant dual 1-forms of ξ_a, that means $\eta^a(\xi_a) = 1$ for any $a \in \{1, 2, 3, 4\}$. We define a left-invariant $(1, 1)$-tensor field φ by setting

$$\varphi(X_i) = Y_i, \quad \varphi(X_{ij}) = Y_{ij}, \quad \varphi(Y_i) = -X_i, \quad \varphi(Y_{ij}) = -X_{ij}, \quad \varphi(\xi_a) = 0,$$

for any $i, j \in \{1, 2\}$ and $a \in \{1, 2, 3, 4\}$.

Clearly, the structure $(\varphi, \xi_a, \eta^a, g)$ yields an indefinite $g.f.f$-structure on $U(4)$ and a trivial verification shows that it is normal. Moreover, this structure does not set up an indefinite $\mathcal{S}$-structure on $U(4)$ since, for example, $d\eta^4(X_{11}, Y_{11}) = 0$ and $\Phi(X_{11}, Y_{11}) = g(X_{11}, \varphi Y_{11}) = -1$. Of course, being $3d\Phi(X_1, X_{11}, Y_{21}) = 1$, this structure does not make $U(4)$ an indefinite $\mathcal{K}$-manifold.

We denote by $\mathcal{U}(4)$ the normal indefinite $g.f.f$-manifold $(U(4), \varphi, \xi_a, \eta^a, g)$. As for the $U(2)$ case, one can check that $\xi_1, \xi_2, \xi_3, \xi_4$ are regular vector fields and they span a regular distribution $\mathcal{F}$. Besides, looking at the brackets between ξ_a

and the other vector fields, it is easy to verify that each ξ_a turns out to be a Killing vector field. Then we can apply Theorem 3.6 to $\mathcal{U}(4)$ and we get

$$\begin{tikzcd} U(4) \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M' \arrow[dl, "\pi'"] \\ & N' & \end{tikzcd}$$

where, being $\mathcal{F}_1 = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$, $M' = U(4)/\mathcal{F}_1$ is a normal indefinite almost contact metric manifold with Killing characteristic vector field $\xi = \tau_* \xi_4$ and $N' = U(4)/\mathcal{F}$ is an indefinite hermitian manifold. In this case the metrics of both manifolds M' and N' have index 2. Clearly M' have not an indefinite Sasakian structure since $d\eta^4 \neq \Phi$ on horizontal vector fields of $U(4)$.

As is the example of $U(2)$, it is allowed to consider other (indefinite) almost contact metric manifolds, taking different distribution $\mathcal{F}_2 = \operatorname{span}\{\xi_1, \xi_2, \xi_4\}$, $\mathcal{F}_3 = \operatorname{span}\{\xi_2, \xi_3, \xi_4\}$, etc., and to compare the related commutative diagrams.

Now, we will study the quotient manifold $U(4)/U(3)$ by projecting the indefinite $g.f.f$-structure of $\mathcal{U}(4)$ onto a Sasakian structure. We recall a projectability criterion described by Ako in [1]. Given a submersion $\pi : M \to N$, Ako defines the horizontal and the vertical part of any tensor field $T \in \mathcal{T}_s^r(M)$. One has: $f^H = f = f^V$ for any $f \in \mathfrak{F}(M)$, $X = X^H + X^V$ for any $X \in \mathfrak{X}(M)$. Then, in particular, for any $s \in \mathbb{N}$, $s \geq 1$, and for any $T \in \mathcal{T}_s^0(M)$ or $T \in \mathcal{T}_s^1(M)$, the horizontal and the vertical part are the tensor fields, T^H and T^V, defined putting

$$T^H(X_1, \dots, X_s) = (T(X_1^H, \dots, X_s^H))^H, \; T^V(X_1, \dots, X_s) = (T(X_1^V, \dots, X_s^V))^V,$$

for any $X_1, \dots, X_s \in \mathfrak{X}(M)$. Then, a tensor field T on M is said to be projectable if it satisfies $(\mathcal{L}_V T^H)^H = 0$, for any vertical vector field V.

We consider the canonical injection

$$j : U(3) \hookrightarrow U(4) \quad \text{such that} \quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

which allows to consider the Lie group $U(3)$ as closed Lie subgroup of $U(4)$ and its Lie algebra $\mathfrak{u}(3)$ as a subalgebra of $\mathfrak{u}(4)$. Hence, we have the compact homogeneous manifold $M^7 = U(4)/U(3)$, diffeomorphic to $\mathbb{S}^7$, and the submersion given by the canonical surjection $\pi : U(4) \to U(4)/U(3)$. The vertical distribution $\mathcal{V}$ is spanned by

$$\{\xi_1, \xi_2, \xi_3, X_1, Y_1, X_{11}, Y_{11}, X_{21}, Y_{21}\},$$

and the horizontal distribution $\mathcal{H}$, orthogonal to $\mathcal{V}$, is spanned by

$$\{\xi_4, X_{12}, Y_{12}, X_{22}, Y_{22}, X_2, Y_2\}.$$

Both the distributions $\mathcal{H}$ and $\mathcal{V}$ are φ-invariant and looking at the brackets in $\mathcal{U}(4)$, one easily checks that, for any $V \in \mathcal{V}$ and $Z \in \mathcal{H}$, $[V, Z]$ either vanishes or is a horizontal vector field.

Now, the projectability conditions of the tensor fields g and φ become

$$(\mathcal{L}_V g)(Z, W) = 0, \quad (\mathcal{L}_V \varphi)(Z) = 0,$$

for any Z and W horizontal vector fields and V vertical vector field. Long straightforward computations show that the above conditions hold, then we obtain a Riemannian metric $g' \in \mathcal{T}_2^0(M^7)$, a $(1,1)$-tensor field $f' \in \mathcal{T}_1^1(M^7)$ and π becomes a semi-Riemannian submersion with totally geodesic semi-Riemannian fibres, since it is easy to verify that the O'Neill fundamental tensor field T of π vanishes on $\mathcal{V}$.
Furthermore, being $[\xi_4, V] = 0$ for any $V \in \mathcal{V}$, ξ_4 projects in $\xi' = \pi_*(\xi_4)$ and η^4 projects in a 1-form η'. A direct computation shows that the structure (f', ξ', η', g') is a Sasakian structure on M^7 since $d\eta^4(X, Y) = \Phi(X, Y)$, for any X, Y horizontal vector fields, and this implies $d\eta' = \Phi'$. The f'-sectional curvature of $(M^7, f', \xi', \eta', g')$ is 1 since

$$K'(\pi_* X_{12}, \pi_* Y_{12}) = K(X_{12}, Y_{12}) + 3g(A_{X_{12}} Y_{12}, A_{X_{12}} Y_{12}) = 4 + 3g(\xi_1, \xi_1) = 1$$

$$K'(\pi_* X_{22}, \pi_* Y_{22}) = K(X_{22}, Y_{22}) + 3g(A_{X_{22}} Y_{22}, A_{X_{22}} Y_{22}) = -2 + 3g(-\xi_2, -\xi_2) = 1$$

$$K'(\pi_* X_2, \pi_* Y_2) = K(X_2, Y_2) + 3g(A_{X_2} Y_2, A_{X_2} Y_2) = -2 + 3g(\xi_3, \xi_3) = 1.$$

It follows that, as Sasakian manifold, $(M^7, f', \xi', \eta', g')$ is isomorphic to $\mathbb{S}^7$ with its standard Sasakian structure. Using this isomorphism, we have the following diagram

$$\begin{array}{ccc} U(4) & \longrightarrow & M^7 \\ & & \big\downarrow \mathcal{P} \\ & & \mathbb{CP}_3(4) \end{array}$$

where $\mathcal{P} : M^7 \to \mathbb{CP}_3(4)$ is the Riemannian submersion coming from the Hopf fibration and the isomorphism between M^7 and $\mathbb{S}^7$.

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