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Operators shrinking the Arveson spectrum

By JERÓNIMO ALAMINOS (Granada), JOSÉ EXTREMERA (Granada)
and ARMANDO R. VILLENA (Granada)

Abstract. Let σ and τ be representations of a locally compact abelian group G on complex Banach spaces X and Y, respectively. This paper is devoted to study whether every continuous linear operators $A : X \to Y$ with the property that $\mathrm{sp}(\tau, Ax) \subset \mathrm{sp}(\sigma, x)$ $(x \in X)$ intertwines σ and τ. We are also concerned with those operators satisfying the shrinking property only approximately.

1. Introduction and preliminaries

1.1. Introduction. In a variety of situations we find operators $A : X \to Y$ between complex Banach spaces, which have the property of *shrinking the Arveson spectrum* with respect to appropriate actions of a locally compact abelian group G on X and Y. This means that there exist representations τ_X and τ_Y of G on X and Y, respectively, such that

$$\mathrm{sp}\big(\tau_Y, Ax\big) \subset \mathrm{sp}\big(\tau_X, x\big) \quad (x \in X). \tag{1.1}$$

As a matter of fact, this works in any of the following cases:

(1) G is a locally compact abelian group and $A : L^1(G) \to L^1(G)$ has the property that

$$\mathrm{supp}\big(\widehat{Af}\big) \subset \mathrm{supp}\big(\widehat{f}\big) \quad (f \in L^1(G)).$$

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(2) H is a complex Hilbert space and $A : H \to H$ shrinks the support of the spectral measures $\mathcal{E}^S$ and $\mathcal{E}^T$ corresponding to self-adjoint operators S and T on H, *i.e.*

$$\operatorname{supp}\bigl(\mathcal{E}^T_{Ax}\bigr) \subset \operatorname{supp}\bigl(\mathcal{E}^S_x\bigr) \quad (x \in H).$$

(3) X and Y are complex Banach spaces and $A : X \to Y$ shrinks the local spectrum, *i.e.*

$$\operatorname{sp}\bigl(T, Ax\bigr) \subset \operatorname{sp}\bigl(S, x\bigr) \quad (x \in X)$$

for some invertible operators $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$.

The standard example of an operator satisfying (1.1) is one intertwining τ_X and τ_Y, *i.e.*

$$A \circ \tau_X(t) = \tau_Y(t) \circ A \quad (t \in G).$$

It turns out that in many important cases these are the only operators shrinking the Arveson spectrum. This paper grew out of a desire to determine whether every operator satisfying (1.1) necessarily intertwines the given representations. On the other hand, strongly motivated by the paper by G. R. Allan and T. J. Ransford [8], we are also addressing the problem of providing quantitative estimates for such a phenomenon.

The question of describing the operators satisfying (1.1) fits in the so-called *linear preserver problems* (we refer the reader to [18] for a full account of the theory). Particularly, the problem of describing the general form of the operators appearing in the example (3) above have some similarity with the problem of characterising the operators on $\mathcal{B}(X)$ preserving some local spectral quantities initiated by A. Bourhim and T. J. Ransford in [11] and continued by several authors (see [10] and the references given therein).

In Section 2 we show that if A satisfies (1.1) for representations τ_X and τ_Y having polynomial growth of degree α, then the following binomial identity holds

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \tau_Y^{N-n}(t) A \tau_X^n(t) = 0 \quad (t \in G) \tag{1.2}$$

for each $N > 2\alpha$. In particular, if τ_X and τ_Y are bounded representations, then $A \circ \tau_X(t) = \tau_Y(t) \circ A$ $(t \in G)$.

Section 3 is concerned with a quantitative estimate of the above mentioned result that (1.1) implies (1.2). Let K be a compact neighbourhood of the identity in G and $\varepsilon > 0$. If

$$\operatorname{sp}\bigl(\tau_Y, Ax\bigr) \subset \operatorname{sp}\bigl(\tau_X, x\bigr) U(K, \varepsilon) \quad (x \in X), \tag{1.3}$$

where

$$U(K,\varepsilon)=\bigl\{\gamma\in\widehat{G}:\gamma(K)\subset\{e^{i\theta}:|\theta|<\varepsilon\}\bigr\},$$

then

$$\left\|\sum_{n=0}^{N}(-1)^n\binom{N}{n}\tau_Y^{N-n}(t)A\tau_X^n(t)\right\|\le\phi_N(t,\varepsilon)\quad(t\in K),\tag{1.4}$$

where $\lim\limits_{\varepsilon\to 0}\phi_N(t,\varepsilon)=0$.

In order to prove this result, we give rise, in a natural way, to a bilinear map φ on the weighted Fourier algebra $A_\alpha(\mathbb{T})$ which encodes the personality of the operator A. Specifically, the map φ has the property that $\varphi(f,g)=0$ whenever $f,g\in A_\alpha(\mathbb{T})$ are such that $\operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset$. The maps enjoying the above-mentioned property were investigated in [5] in connection with the problem of describing the operators preserving zero products on Banach algebras of Lipschitz functions. It is worth pointing out that such maps have proved to be a powerful tool for analysing the operators preserving the zero products on a wide class of Banach algebras and for analysing the reflexivity of their derivation spaces as well [1], [2], [3], [4], [5], [6], [7], [19]. The key property is that such a bilinear map necessarily satisfies the identity

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}\varphi(\mathbf{z}^{N-n},\mathbf{z}^n)=0\tag{1.5}$$

for each $N>2\alpha$, and this will give (1.2). By using [8] we now obtain quantitative estimates of (1.5) which allow us to derive (1.4) from (1.3).

Finally, we emphasise that we apply all the preceding results to analyse a given operator through its local spectrum. For instance, we reprove the well-known Gelfand–Hille theorem and we provide quantitative estimates for that result in the vein of [8]. Similarly we proceed with a result by Colojoară–Foiaş (see [13, Theorem 4.5], [17, Corollary 3.4.5]). We leave it to the reader to apply the general results concerning operators shrinking the Arveson spectrum to some other contexts. It is worth mentioning some works which are in the same spirit, namely, local versions of Gelfand–Hille theorem [14], [15] and a sort of quantitative version of that theorem [16].

1.2. Preliminaries. All Banach spaces and Banach algebras which we consider throughout this paper are assumed to be complex.

Let X be a non-zero Banach space. Let X^* denote the topological dual space of X. We write $\langle\cdot,\cdot\rangle$ for the dual pairing of X and X^*. If X_* is any linear subspace of X^*, then $\sigma(X,X_*)$ stands for the coarsest topology on X for which each of the functionals in X_* are continuous. For a Banach space Y, let $\mathcal{B}(X,Y)$ denote the space of all continuous linear operators from X into Y. As usual, we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$.

We are given, along with a complex Banach space X, a linear subspace X_* of the dual X^* of X. Assume further

$$\|x\| = \sup\{|\langle x,\varphi\rangle| : \varphi \in X_*,\ \|\varphi\| \leq 1\} \quad (x \in X), \tag{1.6}$$

and

$$\text{the } \sigma(X,X_*)\text{-closed convex hull of every } \sigma(X,X_*)\text{-compact set in } X \text{ is } \sigma(X,X_*)\text{-compact.} \tag{1.7}$$

For example, if we take $X_* = X^*$, then (1.6) holds by the Hahn-Banach theorem and (1.7) follows from the Krein–Smulian theorem on weak compactness. For a second example, if X is the dual of a Banach space X_*, then (1.6) holds by definition and (1.7) holds by the Banach-Alaoglu theorem. In [9] there are some other examples.

Throughout this paper, G stands for a locally compact abelian group with dual group $\widehat{G}$. By a *representation* of G on X we mean a group homomorphism τ from G into the group of all invertible $\sigma(X,X_*)$-continuous linear operators on X such that

$$\text{the map } t \mapsto \langle \tau(t)x,\varphi\rangle \text{ is continuous for all } x \in X \text{ and } \varphi \in X_*. \tag{1.8}$$

We will restrict our attention to those representations with *polynomial growth*. This means that

$$\|\tau(kt)\| = O(|k|^\alpha) \text{ as } |k| \to \infty \quad (t \in G) \quad \text{for some } \alpha \geq 0. \tag{1.9}$$

Let us also recall that τ is said to be *bounded* if $\sup_{t\in G}\|\tau(t)\| < \infty$.

Let τ be a representation of G on X satisfying all the preceding requirements (1.6)–(1.9). Then the map $\omega : G \to \mathbb{R}$ defined by $\omega(t) = \|\tau(t)\|$ $(t \in G)$ gives a *non-quasianalytic weight* on G. The corresponding Beurling algebra consists of the Banach space $L^1(G,\omega)$ of those (equivalence classes of) Borel functions on G for which $\|f\| := \int_G |f(t)|\omega(t)dt < \infty$ endowed with convolution as multiplication. This Banach algebra can be thought of as a closed ideal of the Banach algebra $M(G,\omega)$ of all locally finite regular Borel measures μ on G for which the weighted measure $\omega\mu$ is of finite variation. Moreover, the representation τ gives rise to a norm-decreasing algebra homomorphism $\widetilde{\tau} : M(G,\omega) \to \mathcal{B}(X)$ which is defined by

$$\langle \widetilde{\tau}(\mu)x,\varphi\rangle = \int_G \langle \tau(t)x,\varphi\rangle \, d\mu(t) \quad (\mu \in M(G,\omega),\ x \in X,\ \varphi \in X_*)$$

(see [17, pp. 446–448]). For every $x \in X$, the *Arveson spectrum* of τ at x is defined as

$$\operatorname{sp}(\tau, x) = \big\{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for each } f \in L^1(G,\omega) \text{ such that } \ \widetilde{\tau}(f)x = 0\big\}.$$

The basics of the theory of the Arveson spectrum may be found in [9], [12, Section 3.2.3], and [17, Section 4.12].

Examples 1.1.

1. Let G be a locally compact abelian group and let τ be the so-called *regular representation* of G on $L^1(G)$ which is defined by

   $$\big(\tau(t)f\big)(s) = f(s-t) \quad (s, t \in G,\ f \in L^1(G)).$$

   From [17, Example 4.12.2] we deduce that

   $$\operatorname{sp}(\tau, f) = \operatorname{supp}\widehat{f} \quad (f \in L^1(G)).$$

2. In the case where τ is a strongly continuous unitary representation of $\mathbb{R}$ on a Hilbert space H, then Stone's theorem provides a projection-valued measure $\mathcal{E}$ on $\mathbb{R}$ such that

   $$\tau(t) = \int_{-\infty}^{+\infty} e^{-ist}\ d\mathcal{E}(s) \quad (t \in \mathbb{R}).$$

   In this case, if $x \in H$, then $\operatorname{sp}(\tau, x)$ is the smallest closed set C of $\mathbb{R}$ such that $\mathcal{E}(C)x = x$. See [12, Section 3.2.3]. The map $\Delta \mapsto \|\mathcal{E}(\Delta)x\|^2$ defines a measure $\mathcal{E}_x$ on the Borel subsets of $\mathbb{R}$ and $\operatorname{sp}(\tau, x) = \operatorname{supp}(\mathcal{E}_x)$.

3. Let X be a complex Banach space. Given an operator $T \in \mathcal{B}(X)$, the *local resolvent set* $\rho(T, x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T - z)f(z) = x$ for each $z \in U$. The *local spectrum* $\operatorname{sp}(T, x)$ of T at x is then defined as $\operatorname{sp}(T, x) = \mathbb{C} \setminus \rho(T, x)$. Of course, $\operatorname{sp}(T, x)$ is contained in the spectrum $\operatorname{sp}(T)$ of T for each $x \in X$. We refer the reader to [17] for a full account of the local spectral theory.

   Let $T \in \mathcal{B}(X)$ be an invertible operator with polynomial growth, in the sense that

   $$\|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

   for some $\alpha \geq 0$. Then T gives rise to a representation of $\mathbb{Z}$ on X by

   $$\tau_T(k) = T^k \quad (k \in \mathbb{Z})$$

   and [17, Example 4.12.7] shows that

   $$\operatorname{sp}(T, x) = \operatorname{sp}(\tau_T, x) \quad (x \in X).$$

2. Operators shrinking the Arveson spectrum: the seminal version

2.1. Bilinear maps on weighted Fourier algebras. Through a series of papers [1], [2], [3], [4], [5], [6], [7] we have pointed out that the analysis of the behaviour of operators satisfying certain properties may be reduced to the analysis of bilinear maps on the weighted Fourier algebra on the circle group which have the property of taking functions with disjoint support to zero. We have realised that this method still works for dealing with operators which shrink the Arveson spectrum.

For $n \in \mathbb{N}$ and $\alpha \geq 0$, let $A_\alpha(\mathbb{T}^n)$ denote the *weighted Fourier algebra* consisting of all functions $f \in C(\mathbb{T}^n)$ such that

$$\|f\|_{A_\alpha(\mathbb{T}^n)} := \sum_{k\in\mathbb{Z}^n} \left|\widehat{f}(k)\right| \big(1+|k|\big)^\alpha < \infty,$$

where for $k \in \mathbb{Z}^n$ we write $|k| = |k_1| + \cdots + |k_n|$. We abbreviate $A_0(\mathbb{T}^n)$ to $A(\mathbb{T}^n)$.

From now on, $\mathbf{z}$ stands for the function in $A(\mathbb{T})$ defined by $\mathbf{z}(z) = z$ $(z \in \mathbb{T})$. For $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ we will denote by $\mathbf{z_j}$ the function in $A(\mathbb{T}^n)$ given by $\mathbf{z_j}(z) = z_j$ $(z \in \mathbb{T}^n)$.

Given $f \in C(\mathbb{T}^m)$ and $g \in C(\mathbb{T}^n)$ we define $f \otimes g \in C(\mathbb{T}^{m+n})$ by

$$(f \otimes g)(z) = f(z_1, \dots, z_m)g(z_{m+1}, \dots, z_{m+n}) \quad (z \in \mathbb{T}^{m+n}).$$

It is straightforward to check that, if $f \in A_\alpha(\mathbb{T}^m)$ and $g \in A_\alpha(\mathbb{T}^n)$, then $f \otimes g \in A_\alpha(\mathbb{T}^{m+n})$ with

$$\|f \otimes g\|_{A_\alpha(\mathbb{T}^{m+n})} \leq \|f\|_{A_\alpha(\mathbb{T}^m)}\|g\|_{A_\alpha(\mathbb{T}^n)}.$$

We now introduce the key result. It is just the ultimate result of the abovementioned series of papers.

Theorem 2.1. [5, Theorem 2.2] *Let $\alpha \geq 0$ and let $\varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to X$ be a continuous bilinear map into some Banach space X with the property that*

$$f, g \in A_\alpha(\mathbb{T}),\ \operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset \Rightarrow \varphi(f, g) = 0. \tag{2.1}$$

Then

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \varphi(\mathbf{z}^{N-n}, \mathbf{z}^n) = 0$$

for each $N > 2\alpha$.

2.2. Operators shrinking the Arveson spectrum. Theorem 2.1 can be immediately applied for analysing the operators shrinking the Arveson spectrum.

Theorem 2.2. *Let G be a locally compact abelian group. Let X and Y be Banach spaces and let X_* and Y_* be linear subspaces of the duals X^* and Y^*, respectively, satisfying* (1.6) *and* (1.7)*. Suppose that τ_X and τ_Y are representations of G on X and Y, respectively, and that*

$$\|\tau_X(kt)\|, \|\tau_Y(kt)\| = O(|k|^\alpha) \text{ as } |k| \to \infty \quad (t \in G)$$

for some $\alpha \geq 0$. If $A \in \mathcal{B}(X, Y)$ is such that

$$\operatorname{sp}(\tau_Y, Ax) \subset \operatorname{sp}(\tau_X, x) \quad (x \in X),$$

then

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \tau_Y^{N-n}(t) A \tau_X^n(t) = 0 \quad (t \in G)$$

for each $N > 2\alpha$. Accordingly, if τ_X and τ_Y are bounded representations, then

$$\tau_Y(t)A = A\tau_X(t) \quad (t \in G).$$

Proof. Pick $t \in G$. We define a continuous linear map $\Phi_t : A_\alpha(\mathbb{T}) \to M(G, \omega)$ by

$$\Phi_t(f) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\delta_t^k \quad (f \in A_\alpha(\mathbb{T})),$$

where δ_t stands for the Dirac measure concentrated at the point t. It is immediate to check that

$$\widehat{\Phi_t(f)}(\gamma) = f(\gamma(t)) \quad (f \in A_\alpha(\mathbb{T}),\ \gamma \in \widehat{G})$$

and so

$$\operatorname{supp}\big(\widehat{\Phi_t(f)}\big) \subset \big\{\gamma \in \widehat{G} : \gamma(t) \in \operatorname{supp}(f)\big\}. \tag{2.2}$$

We now define a continuous bilinear map $\varphi_t : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to \mathcal{B}(X, Y)$ by

$$\varphi_t(f, g) = \widetilde{\tau_Y}\big(\Phi_t(f)\big) \circ A \circ \widetilde{\tau_X}\big(\Phi_t(g)\big) \quad (f, g \in A_\alpha(\mathbb{T})).$$

Let $f, g \in A_\alpha(\mathbb{T})$ and $x \in X$. On account of [17, Lemma 4.12.6] and (2.2), we have

$$\operatorname{sp}\big(\tau_Y, \varphi_t(f, g)x)\big) \subset \operatorname{supp}\big(\widehat{\Phi_t(f)}\big) \cap \operatorname{sp}\Bigg(\tau_Y, A\Big(\widetilde{\tau_X}\big(\Phi_t(g)\big)x\Big)\Bigg)$$

$$\begin{aligned}&\subset \operatorname{supp}\big(\widehat{\Phi_t(f)}\big) \cap \operatorname{sp}\Big(\tau_X, \widetilde{\tau_X}\big(\Phi_t(g)\big)x\Big)\\ &\subset \operatorname{supp}\big(\widehat{\Phi_t(f)}\big) \cap \operatorname{supp}\big(\widehat{\Phi_t(g)}\big) \cap \operatorname{sp}(\tau_X, x)\\ &\subset \big\{\gamma \in \operatorname{sp}(\tau_X, x) : \gamma(t) \in \operatorname{supp}(f) \cap \operatorname{supp}(g)\big\}.\end{aligned}$$

Accordingly, if $f, g \in A(\mathbb{T})$ are such that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, then we see that $\operatorname{sp}(\tau_Y, \varphi_t(f, g)x) = \emptyset$ and [17, Proposition 4.12.4] now yields $\varphi_t(f, g)x = 0$ for each $x \in X$.

Theorem 2.1 then gives

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \varphi_t(\mathbf{z}^{N-n}, \mathbf{z}^n) = 0.$$

for each $N > 2\alpha$. On the other hand, it is easily seen that

$$\varphi_t(\mathbf{z}^j, \mathbf{z}^k) = \tau_Y^j(t) A \tau_X^k(t) \quad (t \in G,\ j, k \in \mathbb{Z}),$$

which completes the proof. □

2.3. Determining operators through the spectrum. Let X and Y be complex Banach spaces. Given operators $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$, the *commutator* $C(T, S)$ is defined as the mapping

$$C(T, S) : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y),\ C(T, S)(A) = TA - AS, \quad (A \in \mathcal{B}(X, Y)).$$

The iterates $C(T, S)^n$ are often called *higher order commutators* and it is easily seen that

$$C(T, S)^n(A) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{n-k} A S^k, \quad (A \in \mathcal{B}(X, Y)).$$

The following special case is worth noting: if $X = Y$, and if S, T, and A are pairwise commuting operators on X, then the last formula reduces to

$$C(T, S)^n(A) = (T - S)^n A, \quad (n \in \mathbb{N}).$$

The next two corollaries are in the vein of Colojoară-Foiaş theorem (see [13, Theorem 4.5]).

Corollary 2.3. *Let X and Y be complex Banach spaces and let $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$ be invertible operators with*

$$\|S^k\|, \|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

for some $\alpha \geq 0$. If $A \in \mathcal{B}(X, Y)$ is such that

$$\mathrm{sp}(T, Ax) \subset \mathrm{sp}(S, x) \quad \forall x \in X,$$

then

$$C(T, S)^N(A) = 0$$

for each $N > 2\alpha$. Consequently,

(1) *If S and T are doubly power bounded, i.e. $\sup\{\|T^k\| : k \in \mathbb{Z}\} < \infty$, then $TA = AS$.*

(2) *If $X = Y$, and if S, T, and A are pairwise commuting operators, then $(T - S)^N A = 0$.*

Proof. It suffices to apply Theorem 2.2 by considering the representations $\tau_X = \tau_S$ and $\tau_Y = \tau_T$, which are given as in Example 1.1.3. □

Corollary 2.4. *Let X be a complex Banach space and let $S, T \in \mathcal{B}(X)$ be invertible operators with*

$$\|S^k\|, \|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

for some $\alpha \geq 0$. If

$$\mathrm{sp}(T, x) \subset \mathrm{sp}(S, x) \quad \forall x \in X,$$

then

$$C(T, S)^N(I_X) = 0$$

for each $N > 2\alpha$. Consequently,

(1) *If S and T are doubly power bounded, then $T = S$.*

(2) *If S and T commute, then $(T - S)^N = 0$.*

Proof. We only need to apply Corollary 2.3 with $Y = X$ and A being the identity operator on X. □

By applying the preceding corollary with $S = I_X$, we arrive at the classical Gelfand–Hille theorem. This theorem has been proved in a number of different ways, but for a particularly brief and elementary proof we refer the reader to [8].

Corollary 2.5. *Let X be a complex Banach space and let $T \in \mathcal{B}(X)$ invertible and such that $\mathrm{sp}(T) = \{1\}$.*

(1) *If T is doubly power bounded, then $T = I_X$.*

(2) *If $\|T^k\| = O(|k|^\alpha)$ as $|k| \to \infty$ for some $\alpha \geq 0$, then $(T - I_X)^N = 0$ for each $N > 2\alpha$.*

3. Operators shrinking the Arveson spectrum: quantitative estimates

The purpose of this section is to obtain quantitative estimates for the binomial relation provided in Theorem 2.2 in the same vein as in [8]. To this end we will restrict our attention to the binomial relation provided with the minimal N with $N > 2\alpha$.

3.1. Bilinear maps on weighted Fourier algebras: quantitative estimates. Our method for obtaining quantitative estimates of Theorem 2.2 consists in applying a quantitative version of Theorem 2.1 which will be given after several lemmas.

Throughout this paper we shall consider sets of the form

$$E_\varepsilon = \{e^{i\theta} \in \mathbb{T} : |\theta| \le \varepsilon\},$$

where $0 \le \varepsilon < \pi$. Given a nonempty closed set $\mathcal{S} \subset \mathbb{T}^n$ we define

$$I_{A_\alpha(\mathbb{T}^n)}(\mathcal{S}) = \{f \in A_\alpha(\mathbb{T}^n) : f(\mathcal{S}) = \{0\}\}.$$

Then $I_{A_\alpha(\mathbb{T}^n)}(\mathcal{S})$ is a closed ideal of $A_\alpha(\mathbb{T}^n)$.

Lemma 3.1. *Let $N \in \mathbb{Z}$ with $N \ge 0$ and $0 \le \varepsilon < \frac{\pi}{N+1}$. Then*

$$\operatorname{dist}\Big((\mathbf{z}-\mathbf{1})^{N+1}, I_{A_N(\mathbb{T})}(E_\varepsilon)\Big) \le 2\tan\left(\frac{N+1}{2}\varepsilon\right) C_1(N),$$

where, here and subsequently,

$$C_1(N) = 3^N \sum_{n=1}^{N+1} \binom{N+1}{n} n^N.$$

Proof. Our starting point is the property

$$\operatorname{dist}\Big(\mathbf{z}-\mathbf{1}, I_{A(\mathbb{T})}(E_\varepsilon)\Big) \le 2\tan(\varepsilon/2), \tag{3.1}$$

which is shown in the proof of [8, Corollary 3.3]. This gives the assertion in the lemma for $N = 0$. From now on we restrict our attention to the case $N \ge 1$.

Let $\alpha > 2\tan\left(\frac{N+1}{2}\varepsilon\right)$. On account of (3.1), there exists a function $g \in I_{A(\mathbb{T})}(E_{(N+1)\varepsilon})$ such that

$$\|\mathbf{z}-\mathbf{1}-g\|_{A(\mathbb{T})} < \alpha. \tag{3.2}$$

For $n \in \mathbb{N}$ and $f \in C(\mathbb{T})$, let $\delta_n f$ stand for the function defined by

$$(\delta_n f)(z) = f(z^n) \quad (z \in \mathbb{T}).$$

It is clear that $\|\delta_n f\|_{A(\mathbb{T})} = \|f\|_{A(\mathbb{T})}$ and that $\delta_n f \in I_{A(\mathbb{T})}(E_\varepsilon)$ $(n = 1, \ldots, N+1)$, whenever $f \in I_{A(\mathbb{T})}(E_{(N+1)\varepsilon})$. According to the preceding observation and (3.2), we have

$$\|\mathbf{z}^n - \mathbf{1} - \delta_n g\|_{A(\mathbb{T})} < \alpha \quad (n = 1, \ldots, N+1)$$

and therefore

$$\left\| \sum_{n=1}^{N+1} (-1)^{N+1-n} \binom{N+1}{n} (in)^N (\mathbf{z}^n - \mathbf{1} - \delta_n g) \right\|_{A(\mathbb{T})} < \alpha \sum_{n=1}^{N+1} \binom{N+1}{n} n^N. \tag{3.3}$$

We now observe that

$$\sum_{n=1}^{N+1} (-1)^{N+1-n} \binom{N+1}{n} (in)^j = 0 \quad (j = 1, \ldots, N), \tag{3.4}$$

because the number in the left side is nothing but the jth derivative at zero of the function $x \mapsto (e^{ix} - 1)^{N+1}$. From (3.4), the inequality (3.3) turns into

$$\|F - G\|_{A(\mathbb{T})} < \alpha \sum_{n=1}^{N+1} \binom{N+1}{n} n^N, \tag{3.5}$$

where $F, G \in A(\mathbb{T})$ are the functions given by

$$F(z) = \sum_{n=1}^{N+1} (-1)^{N+1-n} \binom{N+1}{n} (in)^N z^n \quad (z \in \mathbb{T})$$

and

$$G(z) = \sum_{n=1}^{N+1} (-1)^{N+1-n} \binom{N+1}{n} (in)^N (\delta_n g)(z) \quad (z \in \mathbb{T}).$$

Since $\delta_n g \in I_{A(\mathbb{T})}(E_\varepsilon)$ for each $n = 1, \ldots, N+1$, it follows that $G \in I_{A(\mathbb{T})}(E_\varepsilon)$.

For every $f \in C(\mathbb{T})$ with $\widehat{f}(0) = 0$, we define $Vf \in C(\mathbb{T})$ by

$$\big(Vf\big)(e^{ix}) = \int_0^x f(e^{it})\, dt \quad (x \in \mathbb{R}).$$

If $f \in A(\mathbb{T})$ and $\widehat{f}(0) = 0$, then a standard fact of the classical Fourier analysis is that

$$\big(Vf\big)(z) = \sum_{k\neq 0} \frac{\widehat{f}(k)}{ik} z^k - \sum_{k\neq 0} \frac{\widehat{f}(k)}{ik} \quad (z \in \mathbb{T}),$$

where the right side of the equality is the Fourier series of the function on the left side. Accordingly, if $f \in A_n(\mathbb{T})$ for some $n \geq 0$ and $\widehat{f}(0) = 0$, then $Vf \in A_{n+1}(\mathbb{T})$ and it is easily checked that

$$\|Vf\|_{A_{n+1}(\mathbb{T})} \leq 3\|f\|_{A_n(\mathbb{T})}. \tag{3.6}$$

It is also immediate to see that

$$f \in C(\mathbb{T}),\ \widehat{f}(0) = 0, \quad n \in \mathbb{N} \Rightarrow V\delta_n f = \frac{1}{n}\delta_n Vf \tag{3.7}$$

and that

$$f \in I_{A_n(\mathbb{T})}(E_\varepsilon),\ \widehat{f}(0) = 0 \Rightarrow Vf \in I_{A_{n+1}(\mathbb{T})}(E_\varepsilon). \tag{3.8}$$

We now intend to iterate the operator V in (3.5) to get

$$\|V^N F - V^N G\|_{A_N(\mathbb{T})} \leq \alpha C_1(N). \tag{3.9}$$

In order to do that we are required to check that

$$\widehat{V^j F}(0) = \widehat{V^j G}(0) = 0, \quad j = 0, \ldots, N-1.$$

It is a simple matter to compute $V^j F$ and (3.4) entails that $\widehat{V^j F}(0) = 0$ for $j = 0, \ldots, N-1$. On the other hand, from the fact that $\widehat{\delta_n g}(0) = \widehat{g}(0)$ for $n = 1, \ldots, N+1$, together with (3.4) and (3.8) we obtain that $\widehat{V^j G}(0) = 0$ for $j = 0, \ldots, N-1$.

Finally, we observe that $(V^N F)(z) = (z-1)^{N+1}$ $(z \in \mathbb{T})$ and that $V^N G \in I_{A_N(\mathbb{T})}(E_\varepsilon)$. Consequently, (3.9) yields

$$\operatorname{dist}\Big((\mathbf{z}-\mathbf{1})^{N+1}, I_{A_N(\mathbb{T})}(E_\varepsilon)\Big) \leq \alpha C_1(N)$$

and finally we take the limit as $\alpha \to 2\tan\big(\frac{N+1}{2}\varepsilon\big)$. □

Lemma 3.2. *Let $N \in \mathbb{Z}$ with $N \geq 0$ and $0 \leq \varepsilon < \frac{\pi}{N+1}$. Then*

$$\operatorname{dist}\Big((\mathbf{z_1}-\mathbf{z_2})^{N+1}, I_{A_N(\mathbb{T}^2)}(E^2_\varepsilon)\Big) \leq 2\tan\left(\frac{N+1}{2}\varepsilon\right) C_2(N),$$

where $E^2_\varepsilon = \{(z_1, z_2) \in \mathbb{T}^2 : z_1 z_2^{-1} \in E_\varepsilon\}$. Here and subsequently,

$$C_2(N) = C_1(N)(N+2)^N,$$

where $C_1(N)$ is the constant obtained in the previous lemma.

Proof. We define the continuous linear operator

$$\Psi : A_N(\mathbb{T}) \to A_N(\mathbb{T}^2),\ (\Psi f)(z_1, z_2) = f(z_1 z_2^{-1}) z_2^{N+1} \quad (z_1, z_2 \in \mathbb{T}).$$

If $f \in A_N(\mathbb{T})$, then

$$(\Psi f)(z_1, z_2) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k) z_1^k z_2^{N+1-k}$$

and so

$$\begin{aligned}\|\Psi f\|_{A_N(\mathbb{T}^2)} &= \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right| (1 + |k| + |N+1-k|)^N \\ &\le \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right| (2 + 2|k| + N)^N = \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right| \sum_{n=0}^{N} \binom{N}{n} 2^n (1+|k|)^n N^{N-n} \\ &\le \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right| (1+|k|)^N \sum_{n=0}^{N} \binom{N}{n} 2^n N^{N-n} = \|f\|_{A_N(\mathbb{T})} (N+2)^N.\end{aligned}$$

This shows that $\|\Psi\| \le (N+2)^N$. On the other hand, it is clear that

$$\Psi\left(I_{A_N(\mathbb{T})}(E_\varepsilon)\right) \subset I_{A_N(\mathbb{T}^2)}(E_\varepsilon^2)$$

and that

$$(\mathbf{z_1} - \mathbf{z_2})^{N+1} = \Psi(\mathbf{z} - \mathbf{1}).$$

From this we deduce that

$$\begin{aligned}&\operatorname{dist}\Big((\mathbf{z_1} - \mathbf{z_2})^{N+1}, I_{A_N(\mathbb{T}^2)}(E_\varepsilon^2)\Big) \le \operatorname{dist}\Big((\mathbf{z_1} - \mathbf{z_2})^{N+1}, \Psi\left(I_{A_N(\mathbb{T})}(E_\varepsilon)\right)\Big) \\ &= \operatorname{dist}\Big(\Psi(\mathbf{z} - \mathbf{1}), \Psi\left(I_{A_N(\mathbb{T})}(E_\varepsilon)\right)\Big) \le (N+2)^N \operatorname{dist}\Big((\mathbf{z} - \mathbf{1})^{N+1}, I_{A_N(\mathbb{T})}(E_\varepsilon)\Big)\end{aligned}$$

and finally Lemma 3.1 establishes the required inequality. □

Now, we are in a position to prove a quantitative version of Theorem 2.1.

Theorem 3.3. *Let $\alpha \in \mathbb{Z}$ with $\alpha \ge 0$, and let $\varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to X$ be a continuous bilinear map into some Banach space X with the property that*

$$f, g \in A_\alpha(\mathbb{T}), \quad \operatorname{supp}(f)\big(\operatorname{supp}(g)\big)^{-1} \cap E_\varepsilon = \emptyset \Rightarrow \varphi(f, g) = 0 \tag{3.10}$$

for some $0 \le \varepsilon < \frac{\pi}{2\alpha+1}$. Then

$$\left\| \sum_{n=0}^{N} (-1)^n \binom{N}{n} \varphi(\mathbf{z}^{N-n}, \mathbf{z}^n) \right\| \le 2 \tan\left(\frac{N}{2}\varepsilon\right) \|\varphi\| C_2(N-1) \tag{3.11}$$

for $N = 2\alpha + 1$.

Proof. The map φ gives rise to a continuous linear operator $\Phi : A_{2\alpha}(\mathbb{T}^2) \to X$ by defining

$$\Phi(f) = \sum_{k_1,k_2\in\mathbb{Z}} \widehat{f}(k_1,k_2)\varphi(\mathbf{z}^{k_1},\mathbf{z}^{k_2}) \quad (f \in A_{2\alpha}(\mathbb{T}^2)).$$

If $f \in A_{2\alpha}(\mathbb{T}^2)$, then

$$\begin{aligned}\|\Phi(f)\| &\leq \sum_{k_1,k_2\in\mathbb{Z}} \left|\widehat{f}(k_1,k_2)\right| \|\varphi(\mathbf{z}^{k_1},\mathbf{z}^{k_2})\| \leq \sum_{k_1,k_2\in\mathbb{Z}} \left|\widehat{f}(k_1,k_2)\right| \|\varphi\|(1+|k_1|)^{\alpha}(1+|k_2|)^{\alpha} \\ &\leq \|\varphi\| \sum_{k_1,k_2\in\mathbb{Z}} \left|\widehat{f}(k_1,k_2)\right| (1+|k_1|+|k_2|)^{2\alpha} = \|\varphi\|\|f\|_{A_{2\alpha}(\mathbb{T}^2)}\end{aligned}$$

and therefore $\|\Phi\| \leq \|\varphi\|$.

Furthermore, on account of the continuity of φ, we have

$$\begin{aligned}\Phi(f\otimes g) &= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{f}(k_1)\widehat{g}(k_2)\varphi(\mathbf{z}^{k_1},\mathbf{z}^{k_2}) \\ &= \varphi\left(\sum_{k_1\in\mathbb{Z}} \widehat{f}(k_1)\mathbf{z}^{k_1}, \sum_{k_2\in\mathbb{Z}} \widehat{g}(k_2)\mathbf{z}^{k_2}\right) = \varphi(f,g) \quad (f,g \in A_{2\alpha}(\mathbb{T})). \qquad (3.12)\end{aligned}$$

We claim that Φ has the following property

$$\delta > 0, \quad f \in I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta}) \Rightarrow \Phi(f) = 0. \qquad (3.13)$$

Let $\delta > 0$ and $f \in I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta})$. For every $z \in \mathbb{T}$ let $U_z = \{ze^{i\theta} : |\theta| < \delta/4\}$. By compactness, there exist $z_1,\ldots,z_m \in \mathbb{T}$ such that $\mathbb{T} = \cup_{p=1}^m U_{z_j}$. There are functions $\omega_1,\ldots,\omega_m \in C^\infty(\mathbb{T})$ with $\omega_1+\cdots+\omega_m = \mathbf{1}$ and $\mathrm{supp}(\omega_p) \subset U_{z_p}$ for $p = 1,\ldots,m$. Since

$$\sum_{p,q=1}^m \omega_p\otimes\omega_q = \mathbf{1},$$

it follows that

$$f = \sum_{p,q=1}^m f\,(\omega_p\otimes\omega_q) = \sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset} f\,(\omega_p\otimes\omega_q) + \sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon\neq\emptyset} f\,(\omega_p\otimes\omega_q).$$

Assume that $U_{z_p}U_{z_q}^{-1} \cap E_\varepsilon \neq \emptyset$ and let $z_0 \in U_{z_p}$, $w_0 \in U_{z_q}$ with $z_0w_0^{-1} \in E_\varepsilon$. If $z \in U_{z_p}$ and $w \in U_{z_q}$, then

$$zw^{-1} = (zz_0^{-1})(ww_0^{-1})(z_0w_0^{-1}) \in E_{\delta/2}E_{\delta/2}E_\varepsilon \subset E_{\varepsilon+\delta}.$$

This implies that

$$\sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon\neq\emptyset} f \quad (\omega_p\otimes\omega_q)=0$$

and so

$$f=\sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset} f \quad (\omega_p\otimes\omega_q).$$

On the other hand, we have

$$f=\sum_{j,k\in\mathbb{Z}}\widehat{f}(j,k)\mathbf{z}^j\otimes\mathbf{z}^k.$$

We thus get

$$f=\sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\widehat{f}(j,k)(\mathbf{z}^j\omega_p)\otimes(\mathbf{z}^k\omega_q).$$

On account of the continuity of Φ we arrive at

$$\Phi(f)=\sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\widehat{f}(j,k)\Phi\Big((\mathbf{z}^j\omega_p)\otimes(\mathbf{z}^k\omega_q)\Big).$$

We can apply (3.12) to get

$$\Phi(f)=\sum_{U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset}\ \sum_{j,k\in\mathbb{Z}}\widehat{f}(j,k)\varphi(\mathbf{z}^j\omega_p,\mathbf{z}^k\omega_q)=0,$$

because of (3.10) and the fact that

$$\operatorname{supp}(\mathbf{z}^j\omega_p)\big(\operatorname{supp}(\mathbf{z}^k\omega_q)\big)^{-1}\cap E_\varepsilon\subset U_{z_p}U_{z_q}^{-1}\cap E_\varepsilon=\emptyset$$

for all the terms appearing in the preceding identity for $\Phi(f)$.

Since $I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta})\subset\ker\Phi$, it follows that Φ drops to a continuous linear operator on the quotient $A_{2\alpha}(\mathbb{T}^2)/I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta})$

$$\bar{\Phi}_\delta: A_{2\alpha}(\mathbb{T}^2)/I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta})\to X,\quad \bar{\Phi}_\delta(f)=\Phi(f),\quad (f\in A_{2\alpha}(\mathbb{T}^2)),$$

with the property that $\|\bar{\Phi}_\delta\|\leq\|\Phi\|\leq\|\varphi\|$. This implies that

$$\|\Phi(f)\|\leq\|\varphi\|\operatorname{dist}_{A_{2\alpha}(\mathbb{T}^2)}\big(f,I_{A_{2\alpha}(\mathbb{T}^2)}(E^2_{\varepsilon+\delta})\big),\quad (f\in A_{2\alpha}(\mathbb{T}^2)).\tag{3.14}$$

It is a simple matter to check that

$$\Phi\left((\mathbf{z_1}-\mathbf{z_2})^N\right)=\sum_{n=0}^N\binom{N}{n}(-1)^n\varphi(\mathbf{z}^{N-n},\mathbf{z}^n).$$

According to (3.14) and Lemma 3.2, we arrive at

$$\left\|\sum_{n=0}^N\binom{N}{n}(-1)^n\varphi(\mathbf{z}^{N-n},\mathbf{z}^n)\right\|\leq 2\tan\left(\frac{N}{2}(\varepsilon+\delta)\right)\|\varphi\|C_2(N-1),$$

whenever $\delta>0$ is such that $\varepsilon+\delta<\frac{\pi}{N}$. Finally, taking $\delta\to 0$ in the preceding inequality we arrive at (3.11). $\square$

3.2. Operators shrinking the Arveson spectrum: quantitative estimates. Throughout this section we shall consider the sets of the form

$$U(K,\varepsilon)=\{\gamma\in\widehat{G}:\gamma(K)\subset E_\varepsilon\},$$

where $0\le\varepsilon<\pi$ and K is a compact neighbourhood of the identity in G. It is worth pointing out that the family consisting of all those $U(K,\varepsilon)$ is a basis of neighbourhoods of the identity in $\widehat{G}$.

Analysis similar to that in the proof of Theorem 2.2 with Theorem 2.1 replaced by Theorem 3.3 gives the following quantitative version of Theorem 2.2.

Theorem 3.4. *Let G be a locally compact abelian group. Let X and Y be Banach spaces and let X_* and Y_* be linear subspaces of the duals X^* and Y^*, respectively, satisfying* (1.6) *and* (1.7)*. Suppose that τ_X and τ_Y are representations of G on X and Y, respectively, and that*

$$\|\tau_X(kt)\|,\|\tau_Y(kt)\|=O(|k|^\alpha)\text{ as }|k|\to\infty\quad(t\in G)$$

for some $\alpha\in\mathbb{Z}$ with $\alpha\ge 0$. If $A\in\mathcal{B}(X,Y)$ is such that

$$\operatorname{sp}(\tau_Y,Ax)\subset\operatorname{sp}(\tau_X,x)U(K,\varepsilon)\quad(x\in X),$$

for some $0\le\varepsilon<\frac{\pi}{2\alpha+1}$ and some $K\subset G$ compact neighbourhood of the identity in G. Then

$$\left\|\sum_{n=0}^{N}(-1)^n\binom{N}{n}\tau_Y^{N-n}(t)A\tau_X^n(t)\right\|\le 2\tan\left(\frac{N}{2}\varepsilon\right)C_2(N-1)\eta(t)\|A\|\quad(t\in K)\tag{3.15}$$

for $N=2\alpha+1$, where

$$\eta(t)=\sup_{k\in\mathbb{Z}}\frac{\|\tau_X(kt)\|}{(1+|k|)^\alpha}\sup_{k\in\mathbb{Z}}\frac{\|\tau_Y(kt)\|}{(1+|k|)^\alpha}\quad(t\in G).$$

Accordingly, if τ_X and τ_Y are bounded representations, then

$$\|\tau_Y(t)A-A\tau_X(t)\|\le 2\tan(\varepsilon/2)\|\tau_X\|\|\tau_Y\|\|A\|\quad(t\in K).$$

Proof. This follows by the same method as in the proof of Theorem 2.2. Pick $t \in K$. We define a continuous linear map $\Phi_t : A_\alpha(\mathbb{T}) \to M(G,\omega)$ and a continuous bilinear map $\varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to \mathcal{B}(X,Y)$ by

$$\Phi_t(f) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\delta_t^k \quad (f \in A_\alpha(\mathbb{T}))$$

and

$$\varphi_t(f,g) = \widetilde{\tau_Y}\big(\Phi_t(f)\big) \circ A \circ \widetilde{\tau_X}\big(\Phi_t(g)\big) \quad (f,g \in A_\alpha(\mathbb{T})),$$

respectively. It is immediate to check that $\|\varphi_t\| \le \eta(t)\|A\|$.

By using [17, Lemma 4.12.6] and (2.2), we can prove as in the proof of Theorem 2.2 that, if $f,g \in A_\alpha(\mathbb{T})$ and $x \in X$, then

$$\mathrm{sp}\big(\tau_Y, \varphi_t(f,g)x\big)\big) \subset \big\{\gamma \in \widehat{G} : \gamma(t) \in \mathrm{supp}(f) \cap \big(\mathrm{supp}(g)E_\varepsilon\big)\big\}.$$

Accordingly, if $f,g \in A_\alpha(\mathbb{T})$ are such that $\mathrm{supp}(f)\big(\mathrm{supp}(g)\big)^{-1} \cap E_\varepsilon = \emptyset$, then we see that $\mathrm{sp}(\tau_Y, \varphi_t(f,g)x) = \emptyset$ and [17, Proposition 4.12.4] now yields $\varphi_t(f,g)x = 0$ for each $x \in X$. Theorem 3.3, completes the proof. □

3.3. Determining operators through the spectrum: quantitative estimates. We now apply Theorem 3.4 for obtaining quantitative versions of the results showed in Subsection 2.2.

Corollary 3.5. *Let X and Y be complex Banach spaces and let $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$ invertible operators with*

$$\|S^k\|, \|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

for some $\alpha \in \mathbb{Z}$ with $\alpha \ge 0$. If $A \in \mathcal{B}(X,Y)$ is such that

$$\mathrm{sp}(T,Ax) \subset \mathrm{sp}(S,x)E_\varepsilon \quad \forall x \in X,$$

for some $0 \le \varepsilon < \frac{\pi}{2\alpha+1}$, then

$$\big\|C(T,S)^N A\big\| \le 2\tan\left(\frac{N}{2}\varepsilon\right) \sup_{k\in\mathbb{Z}} \frac{\|S^k\|}{(1+|k|)^\alpha} \sup_{k\in\mathbb{Z}} \frac{\|T^k\|}{(1+|k|)^\alpha} \|A\| C_2(N-1)$$

for $N = 2\alpha+1$. Consequently,

(1) *If S and T are doubly power bounded, then*

$$\|TA - AS\| \le 2\tan(\varepsilon/2) \sup\|S^k\| \sup\|T^k\| \, \|A\|.$$

(2) *If $X = Y$, and if S, T, and A are pairwise commuting, then*

$$\left\|(T-S)^N A\right\| \le 2\tan\left(\frac{N}{2}\varepsilon\right)\sup_{k\in\mathbb{Z}}\frac{\|S^k\|}{(1+|k|)^\alpha}\sup_{k\in\mathbb{Z}}\frac{\|T^k\|}{(1+|k|)^\alpha}\|A\|C_2(N-1).$$

Proof. It suffices to apply Theorem 3.4 by considering the representations $\tau_X = \tau_S$ and $\tau_Y = \tau_T$, which are given as in Example 1.1.3. □

Corollary 3.6. *Let X be a complex Banach space and let $S, T \in \mathcal{B}(X)$ invertible operators with*

$$\|S^k\|, \|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

for some $\alpha \in \mathbb{Z}$ with $\alpha \ge 0$. If

$$\operatorname{sp}(T,x) \subset \operatorname{sp}(S,x)E_\varepsilon \quad \forall x \in X,$$

for some $0 \le \varepsilon < \frac{\pi}{2\alpha+1}$, then

$$\left\|C(T,S)^N(I_X)\right\| \le 2\tan\left(\frac{N}{2}\varepsilon\right)\sup_{k\in\mathbb{Z}}\frac{\|S^k\|}{(1+|k|)^\alpha}\sup_{k\in\mathbb{Z}}\frac{\|T^k\|}{(1+|k|)^\alpha}C_2(N-1)$$

for $N = 2\alpha+1$. Consequently,

(1) *If S and T are doubly power bounded, then*

$$\|T-S\| \le 2\tan(\varepsilon/2)\sup_{k\in\mathbb{Z}}\|S^k\|\sup_{k\in\mathbb{Z}}\|T^k\|.$$

(2) *If S and T commute, then*

$$\left\|(T-S)^N\right\| \le 2\tan\left(\frac{N}{2}\varepsilon\right)\sup_{k\in\mathbb{Z}}\frac{\|S^k\|}{(1+|k|)^\alpha}\sup_{k\in\mathbb{Z}}\frac{\|T^k\|}{(1+|k|)^\alpha}C_2(N-1).$$

Corollary 3.7. *Let X be a complex Banach space and let $T \in \mathcal{B}(X)$ invertible and such that $\operatorname{sp}(T) \subset E_\varepsilon$ for some $\varepsilon \ge 0$.*

(1) *If T is doubly power bounded and $\varepsilon < \pi$, then*

$$\|T - I_X\| \le 2\tan(\varepsilon/2)\sup_{k\in\mathbb{Z}}\|T^k\|.$$

(2) *If $\|T^k\| = O(|k|^\alpha)$ as $|k| \to \infty$ for some $\alpha \in \mathbb{Z}$ with $\alpha \ge 0$ and $\varepsilon < \frac{\pi}{2\alpha+1}$, then*

$$\left\|(T-I_X)^N\right\| \le 2\tan\left(\frac{N}{2}\varepsilon\right)\sup_{k\in\mathbb{Z}}\frac{\|T^k\|}{(1+|k|)^\alpha}C_2(N-1)$$

for $N = 2\alpha+1$.

References

[1] J. Alaminos, M. Brešar, M. Černe, J. Extremera and A. R. Villena, Zero product preserving maps on $C^1[0,1]$, *J. Math. Anal. Appl.* **347**, no. 2 (2008), 472–481.

[2] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Characterizing homomorphisms and derivations on C^*-algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **137**, no. 1 (2007), 1–7.

[3] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Maps preserving zero products, *Studia Math.* **193**, no. 2 (2009), 131–159.

[4] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Characterizing Jordan maps on C^*-algebras through zero products, *Proc. Edinb. Math. Soc.* **53**, no. 3 (2010), 543–555.

[5] J. Alaminos, J. Extremera and A. R. Villena, Zero product preserving maps on Banach algebras of Lipschitz functions, *J. Math. Anal. Appl.* **369**, no. 1 (2010), 94–100.

[6] J. Alaminos, J. Extremera and A. R. Villena, Approximately zero-product-preserving maps, *Israel J. Math.* **178**, no. 1 (2010), 1–28.

[7] J. Alaminos, J. Extremera and A. R. Villena, Hyperreflexivity of the derivation space of some group algebras, *Math. Z.* **266**, no. 3 (2010), 571–582.

[8] G. R. Allan and T. J. Ransford, Power-dominated elements in a Banach algebra, *Studia Math.* **94**, no. 1 (1989), 63–79.

[9] W. Arveson, On groups of automorphisms of operator algebras, *J. Funct. Anal.* **15**, no. 3 (1974), 217–243.

[10] A. Bourhim, Surjective linear maps preserving local spectra, *Linear Algebra Appl.* **432**, no. 1 (2010), 383–393.

[11] A. Bourhim and T. J. Ransford, Additive maps preserving local spectrum, *Integral Equat. Oper. Theory* **55**, no. 3 (2006), 377–385.

[12] O. Bratteli and D. W. Robinson, Operator algebras and quantum statistical mechanics 1. C^*- and W^*-algebras, symmetry groups, decomposition of states, Texts and Monographs in Physics, *Springer-Verlag*, *New York*, 1987, xiv+505 pp.

[13] I. Colojoară and C. Foiaş, Theory of generalized spectral operators, *Gordon and Breach*, *New York*, 1968.

[14] D. Drissi, On a theorem of Gelfand and its local generalizations, *Studia Math.* **123**, no. 2 (1997), 185–194.

[15] D. Drissi, On a generalization of Lumer-Phillips' theorem for dissipative operators in a Banach space, *Studia Math.* **130**, no. 1 (1998), 1–7.

[16] S.-Z. Huang, H. Render and M. P. H. Wolff, On the convergence of sequences of operators and the convergence of the sequence of their spectra, *Math. Scand.* **85**, no. 1 (1999), 71–78.

[17] K. B. Laursen and M. M. Neumann, An introduction to local spectral theory, Vol. 20, London Mathematical Society Monographs. New Series, *The Clarendon Press, Oxford University Press*, *New York*, 2000. xii+591 pp.

[18] L. Molnár, Selected preserver problems on algebraic structures of linear operators and on function spaces, Vol. 1895, Lecture Notes in Mathematics, *Springer-Verlag*, *Berlin*, 2007, xiv+232 pp.

[19] E. Samei, Reflexivity and hyperreflexivity of bounded n-cocycles from group algebras, *Proc. Amer. Math. Soc.* **139**, no. 1 (2011), 163–176.

JERÓNIMO ALAMINOS
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
18071, GRANADA
SPAIN

E-mail: alaminos@ugr.es

JOSÉ EXTREMERA
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
18071, GRANADA
SPAIN

E-mail: jlizana@ugr.es

ARMANDO R. VILLENA
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
18071, GRANADA
SPAIN

E-mail: avillena@ugr.es

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