*Publ. Math. Debrecen*
**80/3-4** (2012), 255–293
DOI: 10.5486/PMD.2012.4814

# Some generalizations of the Stone Duality Theorem

By GEORGI D. DIMOV (Sofia)

**Abstract.** In this paper we obtain some extensions of the Stone Duality Theorem to the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$ of zero-dimensional locally compact Hausdorff spaces and continuous (respectively, perfect) maps. We also prove some new Stone-type duality theorems for the cofull subcategories of the category $\mathsf{BoolSp}$ determined, respectively, by the skeletal maps, quasi-open perfect maps, open maps and open perfect maps.

## Introduction

In this paper we develop further the ideas from the papers [5], [6], [3], [4] and obtain some extensions of the famous Stone Duality Theorem [18]. Recall that in 1937, M. Stone [18] proved that there exists a bijective correspondence $S$ between the class of all (up to homeomorphism) zero-dimensional locally compact Hausdorff spaces (briefly, *Boolean spaces*) and the class of all (up to isomorphism) generalized Boolean algebras (briefly, GBAs) (or, equivalently, Boolean rings with or without unit). In the class of compact Boolean spaces (briefly, *Stone spaces*) this bijection can be extended to a duality $S^t : \mathsf{Stone} \longrightarrow \mathsf{BoolAlg}$ between the category $\mathsf{Stone}$ of Stone spaces and continuous maps and the category $\mathsf{BoolAlg}$ of Boolean algebras and Boolean homomorphisms; this is the classical Stone Duality Theorem. In 1964, H. P. Doctor [7] showed that the Stone bijection $S$

---

*Mathematics Subject Classification:* 18A40, 06E15, 54D45, 54C05, 54C10, 54D80.
*Key words and phrases:* duality, contravariant adjunction, local Boolean algebra, ZLB-algebra, generalized Boolean algebra, locally compact (compact) zero-dimensional space, skeletal map, perfect map, (quasi-)open map, (dense) embedding, injective (surjective) map.
This paper is supported by the project DID 02/32/2009 "Theories of space and time: algebraical, topological and logical approaches" of the Bulgarian Ministry of Education and Science.

can be extended even to a duality between the category $\mathsf{PerfBoolSp}$ of all Boolean spaces and all perfect maps between them and the category $\mathsf{GenBoolAlg}$ of all GBAs and suitable morphisms between them. It is natural to ask whether there exists such an extension over the category $\mathsf{BoolSp}$ of all Boolean spaces and all continuous functions between them. Let me mention that it is even not easy to obtain a duality for the category $\mathsf{PerfBoolSp}$. Indeed, to every Boolean space $X$, M. Stone juxtaposed the generalized Boolean algebra $CK(X)$ of all compact open subsets of $X$ and reconstructed from it the space $X$ (up to homeomorphism). If $f : X \longrightarrow Y$ is a continuous map between two Stone spaces then its dual map $\varphi = S^t(f) : CO(Y) \longrightarrow CO(X)$ (where, for every topological space $Z$, $CO(Z)$ is the set of all clopen subsets of $Z$) is defined by the formula $\varphi(G) = f^{-1}(G)$, for every $G \in CO(Y)$. If, however, $f : X \longrightarrow Y$ is a continuous map between two Boolean spaces and at least the space $X$ is not compact then the preimages $f^{-1}(G)$ of the elements $G$ of $CK(Y)$ are not obliged to be elements of the set $CK(X)$. These preimages will belong to $CK(X)$ iff the map $f$ is perfect; then it is natural to expect that the category of GBAs and pseudolattice homomorphisms preserving zero elements (or, equivalently, the category $\mathsf{BoolRng}$ of Boolean rings and ring homomorphisms) will be the dual category of the category $\mathsf{PerfBoolSp}$ of Boolean spaces and perfect maps. However it is not the case. For example, if $X$ and $Y$ are two non-empty Boolean non-compact spaces and the 0-pseudolattice homomorphism $\varphi_0 : CK(Y) \longrightarrow CK(X)$ is defined by $\varphi_0(G) = 0(= \emptyset)$ for every $G \in CK(Y)$, then there is no function $f : X \longrightarrow Y$ such that $\varphi_0(G) = f^{-1}(G)$, for every $G \in CK(Y)$. Hence, even in the case of perfect maps, the mentioned homomorphisms are too much. In fact, as it is proved by D. Hofmann [11], the category $\mathsf{BoolRng}$ is dually equivalent to the category $\mathsf{pStone}$ of pointed Stone spaces and continuous maps preserving the fixed points. Thus, if one looks for a dual category to the category $\mathsf{PerfBoolSp}$, having GBAs as objects, then this category has to have as morphisms some subclass of the class of pseudolattice homomorphisms preserving zero elements. Such a category was described by H. P. Doctor [7] and here it is named $\mathsf{GenBoolAlg}$ (see Theorem 2.13 below where two duality functors $\Theta^t_g : \mathsf{PerfBoolSp} \longrightarrow \mathsf{GenBoolAlg}$ and $\Theta^a_g : \mathsf{GenBoolAlg} \longrightarrow \mathsf{PerfBoolSp}$ are defined). If we want to find a dual category to the category $\mathsf{BoolSp}$ then it is clear that in this case the preimages of the compact open sets are clopen sets but they are not obliged to be compact sets. In [18], M. Stone proved that clopen subsets of a Boolean space $X$ correspond to simple ideals of the GBA $CK(X)$ (i.e., those ideals of $CK(X)$ which have a complement in the frame $Idl(CK(X))$ of all ideals of $CK(X)$). Therefore one has to use the simple ideals of GBAs.

As it is proved by M. Stone, the set of all simple ideals of a GBA forms a Boolean algebra. Here we describe the objects of the desired dual category to the category $\mathsf{BoolSp}$ as pairs $(B, I)$, where $B$ is a Boolean algebra and $I$ is a dense (proper or non proper) ideal of it, satisfying a condition of completeness type; this condition is the following: for every simple ideal $J$ of $I$, the join $\bigvee_B J$ exists; it is fulfilled for every pair $(B, B)$, where $B$ is a Boolean algebra because, as it is shown by M. Stone, an ideal of a Boolean algebra is simple iff it is principal. In this way we build a category named $\mathsf{ZLBA}$ and we prove that it is dually equivalent to the category $\mathsf{BoolSp}$ (see Theorem 2.7 where two duality functors $\Theta^t_d : \mathsf{BoolSp} \longrightarrow \mathsf{ZLBA}$ and $\Theta^a_d : \mathsf{ZLBA} \longrightarrow \mathsf{BoolSp}$ are defined). The idea of the construction of the category $\mathsf{ZLBA}$ comes from the ideas and results obtained in [5]. However, the proof that the categories $\mathsf{BoolSp}$ and $\mathsf{ZLBA}$ are dually equivalent will be carried out independently from the results of [5] because this is the more economical way. In fact, we first construct a category $\mathsf{LBA}$ containing as a subcategory the category $\mathsf{ZLBA}$ and find a contravariant adjunction between the categories $\mathsf{LBA}$ and $\mathsf{BoolSp}$ which leads to the mentioned above duality between the categories $\mathsf{BoolSp}$ and $\mathsf{ZLBA}$. We define also two more categories $\mathsf{PZLBA}$ and $\mathsf{PLBA}$ which are dual to the category $\mathsf{PerfBoolSp}$ and obtain in a natural way the Doctor's category $\mathsf{GenBoolAlg}$ and a new proof of his Duality Theorem mentioned above. Finally, we define two subcategories $\mathsf{DZHLC}$ and $\mathsf{DPZHLC}$ of the category $\mathsf{DHLC}$, which was constructed in [5] as a dual category to the category $\mathsf{HLC}$ of locally compact Hausdorff spaces and continuous maps; these subcategories are dual, respectively, to the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$. We obtain also many other results. The main of them are listed below, where we describe the structure of the paper.

We start with a section containing some preliminary results. The results which we discussed above are presented in the second section. In the third section of this paper, we prove some Stone-type duality theorems for some subcategories of the category $\mathsf{BoolSp}$. These theorems are new even in the compact case (see Theorems 3.1, 3.2(b),(c), 3.4, 3.5(b), 3.6(b), 3.8). They concern the cofull subcategories $\mathsf{SkelBoolSp}$, $\mathsf{QOpenPerfBoolSp}$, $\mathsf{OpenBoolSp}$ and $\mathsf{OpenPerfBoolSp}$ of the category $\mathsf{BoolSp}$ determined, respectively, by the skeletal maps (defined by Mioduszewski and Rudolf in [15]), by the quasi-open (defined by S. Mardešic and P. Papic in [13]) perfect maps, by the open maps, and by the open perfect maps. Since the categories $\mathsf{QOpenPerfBoolSp}$ and $\mathsf{OpenPerfBoolSp}$ are cofull subcategories simultaneously of the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$, we find their images by the both functors $\Theta^t_d$ and $\Theta^t_g$ (see Corollary 3.2(b), Theorem 3.4 and Corollary 3.8). For the compact case, these theorems give the following results: (a) The category QOpenStone of compact zero-dimensional Hausdorff spaces and quasi-open maps is dually equivalent to the category CBool of Boolean algebras and complete Boolean homomorphisms (see Corollary 3.2(c)), and (b) The category OpenStone of compact zero-dimensional Hausdorff spaces and open maps is dually equivalent to the category OBool of Boolean algebras and Boolean homomorphisms $\varphi$ having lower adjoint $\psi$ (i.e., the pair $(\psi, \varphi)$ forms a Galois connection) (see Corollary 3.7(b)). Let us notice also the following result (see Theorem 3.4): the category QOpenPerfBoolSp is dually equivalent to the cofull subcategory QGBA of the category GenBoolAlg whose morphisms, in addition, preserve all meets that happen to exist. Note also that Theorem 3.1 and Corollary 3.2(b),(c) are zero-dimensional analogues of the Fedorchuk Duality Theorem [10] and its generalization presented in [4]. From the mentioned above Corollary 3.2(c) and Fedorchuk's Duality Theorem [10], we obtain, as an immediate application, the following assertion which is a special case of a much more general theorem of Monk [14]: a Boolean homomorphism can be extended to a complete homomorphism between the corresponding minimal completions iff it is a complete homomorphism.

In the fourth section we characterize the dual maps of the injective and surjective morphisms of the category BoolSp and its subcategories PerfBoolSp, OpenBoolSp. Such investigations were done by M. Stone in [18] for surjective continuous maps and for closed embeddings. Analogous results are obtained here for the homeomorphic embeddings and dense embeddings.

In the last fifth section, the connections between the dual object of a space $X \in |\mathsf{BoolSp}|$ and the dual objects of the closed, regular closed and open subsets of $X$ are found. It seems that the obtained result for regular closed subsets is new even in the compact case.

We now fix the notation.

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if $X$ is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if $f$ is a morphism of $\mathcal{C}$ with domain $X$ and codomain $Y$. We will say that a subcategory $\boldsymbol{B}$ of a category $\boldsymbol{A}$ is a *cofull subcategory* if $|\boldsymbol{B}| = |\boldsymbol{A}|$. As usual, we denote by $Id_{\mathcal{C}}$ the identity functor of $\mathcal{C}$, and by $\mathrm{id}_A$ the identity morphism of $A \in |\mathcal{C}|$.

If $f : X \longrightarrow Y$ is a function and $M \subseteq X$ then $f_{\upharpoonright M}$ is the restriction of $f$ having $M$ as a domain and $f(M)$ as a codomain.

If $(X, \tau)$ is a topological space and $M$ is a subset of $X$, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of $M$ in $(X, \tau)$ and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of $M$ in $(X, \tau)$.

The set of all clopen (= closed and open) subsets of a topological space $X$ will be denoted by $CO(X)$ and the set of all compact open subsets of $X$ by $CK(X)$.

The closed maps, as well as open maps, between topological spaces are assumed to be continuous but are not assumed to be onto.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct. Since we follow JOHNSTONE's terminology from [12], we will use the term *pseudolattice* for a poset having all finite non-empty meets and joins; the pseudolattices with a bottom will be called $\{0\}$-*pseudolattices*.

If $(A, \leq)$ is a poset and $a \in A$, we set $\downarrow_A (a) = \{b \in A \mid b \leq a\}$ (we will write even "$\downarrow (a)$" instead of "$\downarrow_A (a)$" when there is no ambiguity); if $B \subseteq A$ then we set $\downarrow (B) = \bigcup\{\downarrow (b) \mid b \in B\}$.

If $A$ is a Boolean algebra then the set of all ultrafilters of $A$ will be denoted by $Ult(A)$.

The operation "complement" in Boolean algebras will be denoted by "*".

We denote by $S^t : \mathsf{Stone} \longrightarrow \mathsf{BoolAlg}$ and $S^a : \mathsf{BoolAlg} \longrightarrow \mathsf{Stone}$ the Stone duality functors between the categories $\mathsf{Stone}$ of compact zero-dimensional Hausdorff spaces (= *Stone spaces*) and continuous maps and $\mathsf{BoolAlg}$ of Boolean algebras and Boolean homomorphisms.

The positive natural numbers will be denoted by $\mathbb{N}^+$.

## 1. Preliminaries

*1.1.* Recall that if $(A, \leq)$ is a poset and $B \subseteq A$ then $B$ is said to be a *dense subset of* $A$ if for any $a \in A \setminus \{0\}$ there exists $b \in B \setminus \{0\}$ such that $b \leq a$; when $(B, \leq_1)$ is a poset and $f : A \longrightarrow B$ is a map, then we will say that $f$ is a *dense map* if $f(A)$ is a dense subset of $B$.

*1.2.* Recall that the Stone space $S^a(A)$ of a Boolean algebra $A$ is the set $X = Ult(A)$ endowed with a topology $\mathcal{T}$ having as an open base the family $\{\lambda_A^S(a) \mid a \in A\}$, where $\lambda_A^S(a) = \{u \in X \mid a \in u\}$ for every $a \in A$; then $S^a(A) = (X, \mathcal{T})$ is a compact Hausdorff zero-dimensional space, $\lambda_A^S(A) \subseteq CO(X)$ and the map

$$\lambda_A^S : A \longrightarrow CO(X), \quad a \mapsto \lambda_A^S(a),$$

is a Boolean isomorphism.

*1.3.* Recall that a *frame* is a complete lattice $L$ satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$, for every $a \in L$ and every $S \subseteq L$.

Let $A$ be a distributive $\{0\}$-pseudolattice and $Idl(A)$ be the frame of all ideals of $A$. If $J \in Idl(A)$ then we will write $\neg_A J$ (or simply $\neg J$) for the pseudocomplement of $J$ in $Idl(A)$ (i.e., $\neg J = \bigvee\{I \in Idl(A) \mid I \wedge J = \{0\}\}$). Note that $\neg J = \{a \in A \mid (\forall b \in J)(a \wedge b = 0)\}$ (see Stone [17]). Recall that an ideal $J$ of $A$ is called *simple* (Stone [17]) if $J \vee \neg J = A$. As it is proved in [17], the set $Si(A)$ of all simple ideals of $A$ is a Boolean algebra with respect to the lattice operations in $Idl(A)$. Recall also that the regular elements of the frame $Idl(A)$ (i.e., those $J \in Idl(A)$ for which $\neg\neg J = J$) are called *normal ideals* (Stone [17]).

*1.4.* Let us recall the notion of *lower adjoint* for posets. Let $\varphi : A \longrightarrow B$ be an order-preserving map between posets. If $\psi : B \longrightarrow A$ is an order-preserving map satisfying the following condition

($\Lambda$) for all $a \in A$ and all $b \in B$, $b \leq \varphi(a)$ iff $\psi(b) \leq a$

(i.e., the pair $(\psi, \varphi)$ forms a Galois connection between posets $B$ and $A$) then we will say that $\psi$ is a *lower adjoint* of $\varphi$. It is easy to see that condition ($\Lambda$) is equivalent to the following condition:

($\Lambda'$) $\forall a \in A$ and $\forall b \in B$, $\psi(\varphi(a)) \leq a$ and $\varphi(\psi(b)) \geq b$.

Note that if $\varphi : A \longrightarrow B$ is an (order-preserving) map between posets, $A$ has all meets and $\varphi$ preserves them then, by the Adjoint Functor Theorem (see, e.g., [12]), $\varphi$ has a lower (or *left*) adjoint which will be denoted by $\varphi_\Lambda$.

*1.5.* Recall that:

(a) a map is *perfect* if it is compact (i.e., point inverses are compact sets) and closed;

(b) a continuous map $f : X \longrightarrow Y$ is called *quasi-open* ([13]) if for every non-empty open subset $U$ of $X$, $\mathrm{int}(f(U)) \neq \emptyset$ holds;

(c) a function $f : X \longrightarrow Y$ is called *skeletal* ([15]) if

$$\mathrm{int}(f^{-1}(\mathrm{cl}(V))) \subseteq \mathrm{cl}(f^{-1}(V)) \tag{1}$$

for every open subset $V$ of $Y$. It is well known that a function $f : X \longrightarrow Y$ is skeletal iff $\mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$ for every non-empty open subset $U$ of $X$.

*1.6.* Recall that a subset $F$ of a topological space $(X, \tau)$ is called *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$. Clearly, $F$ is regular closed ff it is the closure of an open set.

For any topological space $(X, \tau)$, the collection $RC(X, \tau)$ (we will often write simply $RC(X)$) of all regular closed subsets of $(X, \tau)$ becomes a complete Boolean algebra $(RC(X, \tau), 0, 1, \wedge, \vee, ^*)$ under the following operations:

$$1 = X,\ 0 = \emptyset,\ F^* = \mathrm{cl}(X \setminus F),\ F \vee G = F \cup G,\ F \wedge G = \mathrm{cl}(\mathrm{int}(F \cap G)).$$

The infinite operations are given by the following formulas: $\bigvee\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\bigcup\{F_\gamma \mid \gamma \in \Gamma\})$, and $\bigwedge\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\mathrm{int}(\bigcap\{F_\gamma \mid \gamma \in \Gamma\}))$.

A subset $U$ of a topological space $(X, \tau)$ is called *regular open* if the set $X \setminus U$ is regular closed.

For all notions and notation not defined here see [1], [12], [9], [8], [16].

## 2. Generalizations of the Stone Duality

In this section we obtain some generalizations of the Stone Duality Theorem [18]. A category $\mathsf{LBA}$ is constructed and a contravariant adjunction between it and the category $\mathsf{BoolSp}$ of *Boolean spaces* (= zero-dimensional locally compact Hausdorff spaces) and continuous maps is obtained. The fixed objects of this adjunction give us a duality between the category $\mathsf{BoolSp}$ and the subcategory $\mathsf{ZLBA}$ of the category $\mathsf{LBA}$. As it was already mentioned, H. P. Doctor [7] introduced a category $\mathsf{GenBoolAlg}$ and proved that it is dual to the category $\mathsf{PerfBoolSp}$ of Boolean spaces and perfect maps. Here two new categories $\mathsf{PZLBA}$ and $\mathsf{PLBA}$ dual to the category $\mathsf{PerfBoolSp}$ are described and a new proof of the Doctor Duality Theorem is given. The restrictions of the obtained duality functors to the category $\mathsf{Stone}$ coincide with the Stone duality functor $S^t : \mathsf{Stone} \longrightarrow \mathsf{BoolAlg}$. We describe as well two subcategories $\mathsf{DZHLC}$ and $\mathsf{DPZHLC}$ of the category $\mathsf{DHLC}$, constructed in [5] as a dual category to the category $\mathsf{HLC}$ of locally compact Hausdorff spaces and continuous maps, which are dual, respectively, to the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$.

*Definition 2.1.* A pair $(A, I)$, where $A$ is a Boolean algebra and $I$ is an ideal of $A$ (possibly non proper) which is dense in $A$ (shortly, dense ideal), is called a *local Boolean algebra* (abbreviated as LBA). An LBA $(A, I)$ is called a *prime local Boolean algebra* (abbreviated as PLBA) if $I = A$ or $I$ is a prime ideal of $A$. Two LBAs $(A, I)$ and $(B, J)$ are said to be *LBA-isomorphic* (or, simply, *isomorphic*) if there exists a Boolean isomorphism $\varphi : A \longrightarrow B$ such that $\varphi(I) = J$.

Let $\mathsf{LBA}$ be the category whose objects are all LBAs and whose morphisms are all functions $\varphi : (A, I) \longrightarrow (B, J)$ between the objects of $\mathsf{LBA}$ such that $\varphi : A \longrightarrow B$ is a Boolean homomorphism satisfying the following condition:

(LBA) For every $b \in J$ there exists $a \in I$ such that $b \leq \varphi(a)$;

let the composition between the morphisms of $\mathsf{LBA}$ be the usual composition between functions, and the $\mathsf{LBA}$-identities be the identity functions.

*Remark 2.1.* Note that by Stone's result about the existence of a bijective correspondence between the ideals and open sets, any LBA $(A, I)$ determines a pair $(X, L)$ (we will write $(X, L) = p(A, I)$), where $X = S^a(A)$ (and hence $X$ is a Stone space) and $L = \bigcup\{\lambda_A^S(a) \mid a \in I\}$ (and thus $L$ is an open subset of $X$). Moreover, since $I$ is dense in $A$, it is easy to see that $L$ is dense in $X$ (see, e.g., Lemma 2.3 below). Therefore, $X$ is a 0-dimensional compactification of $L$. Clearly, by the results of M. Stone, $X$ is the one-point compactification of $L$ iff $I$ is a prime ideal iff $(A, I)$ is a PLBA and $I \neq A$. It can be shown that the category $\mathsf{LBA}$ is dually equivalent to the category of pairs $(X, L)$, where $X$ is a Stone space and $L$ is a dense open subset of $X$, having as morphisms the continuous maps between pairs (this assertion was noted by the referee).

*Remark 2.2.* Note that two LBAs $(A, I)$ and $(B, J)$ are $\mathsf{LBA}$-isomorphic iff they are LBA-isomorphic. Indeed, let $\varphi : (A, I) \longrightarrow (B, J)$ be an $\mathsf{LBA}$-isomorphism. Then, obviously, $\varphi : A \longrightarrow B$ is a Boolean isomorphism. We have to show that $\varphi(I) = J$. Let $\psi \in \mathsf{LBA}((B, J), (A, I))$ be such that $\varphi \circ \psi = \mathrm{id}_B$ and $\psi \circ \varphi = \mathrm{id}_A$. Let $a \in I$. Then, by condition (LBA), there exists $b \in J$ such that $a \leq \psi(b)$. Thus $\varphi(a) \leq b$; this implies that $\varphi(a) \in J$. So, $\varphi(I) \subseteq J$. Analogously, we get that $\psi(J) \subseteq I$. Let $b \in J$. Then $a = \psi(b) \in I$ and $\varphi(a) = b$. Hence, $\varphi(I) = J$. Therefore, $(A, I)$ and $(B, J)$ are LBA-isomorphic. The converse implication is obvious.

*Remark 2.3.* Note that a prime (= maximal) ideal $I$ of a Boolean algebra $A$ is a dense subset of $A$ iff $I$ is a non-principal ideal of $A$. For proving this, observe first that if $I$ is a prime ideal, $a \in A \setminus \{1\}$ and $I \leq a$ then $a \in I$. (Indeed, if $a \notin I$ then $a^* \in I$ and hence $a^* \leq a$, i.e., $a = 1$.) Let now $I$ be dense in $A$. Suppose that $I =\downarrow (a)$ for some $a \in A \setminus \{1\}$. Then $a^* \neq 0$. There exists $b \in I \setminus \{0\}$ such that $b \leq a^*$. Since $b \leq a$, we get that $b = 0$, a contradiction. Hence, $I$ is a non-principal ideal. Conversely, let $I$ be a non-principal ideal and $b \in A \setminus \{0\}$. Suppose that $b \wedge a = 0$, for every $a \in I$. Then $I \leq b^*$. Hence $I =\downarrow (b^*)$, a contradiction. Thus, there exists $a \in I$ such that $a \wedge b \neq 0$. Then $a \wedge b \in I \setminus \{0\}$ and $a \wedge b \leq b$. Therefore, $I$ is a dense subset of $A$.

Recall that a distributive $\{0\}$-pseudolattice $A$ is called a *generalized Boolean algebra* (briefly, *GBA*) if it satisfies the following condition:

(GBA) for every $a \in A$ and every $b, c \in A$ such that $b \leq a \leq c$ there exists $x \in A$ with $a \wedge x = b$ and $a \vee x = c$ (i.e., $x$ is the *relative complement of $a$ in the interval* $[b, c]$).

**Fact 2.1.** (a) *A distributive $\{0\}$-pseudolattice $A$ is a generalized Boolean algebra iff every principal ideal of $A$ is simple.*

(b) *If $A$ is a generalized Boolean algebra then the correspondence*

$$e_A : A \longrightarrow Si(A), \quad a \mapsto \downarrow (a),$$

*is a dense $\{0\}$-pseudolattice embedding of $A$ in the Boolean algebra $Si(A)$ and the pair $(Si(A), e_A(A))$ is an LBA.*

(c) (M. Stone [17]) *An ideal of a Boolean algebra is simple iff it is principal.*

Proof. (a) ($\Rightarrow$) Let $A$ be a generalized Boolean algebra and $a \in A$. We have to prove that $\downarrow (a) \vee \neg(\downarrow (a)) = A$. Let $b \in A$. Then $c = a \wedge b \in [0, b]$. Hence there exists $d \in A$ such that $d \wedge c = 0$ and $d \vee c = b$. Thus $d \leq b$, i.e., $d \wedge b = d$. Therefore, $d \wedge a = d \wedge b \wedge a = d \wedge c = 0$. We obtain that $d \in \neg(\downarrow (a))$, $c \in \downarrow (a)$ and $c \vee d = b$. So, $\downarrow (a) \vee \neg(\downarrow (a)) = A$.

($\Leftarrow$) Let $a, b, c \in A$ and $a \in [b, c]$. Since $\downarrow (a) \vee \neg(\downarrow (a)) = A$, we get that there exists $y \in \neg(\downarrow (a))$ such that $c = a \vee y$. Set $x = y \vee b$. Then $x \wedge a = (y \vee b) \wedge a = b \wedge a = b$ and $x \vee a = y \vee b \vee a = y \vee a = c$. So, $A$ is a generalized Boolean algebra.

(b) By (a), for every $a \in A$, $\downarrow (a) \in Si(A)$. Further, it is easy to see that $e_A$ is a $\{0\}$-pseudolattice embedding and $I = e_A(A)$ is dense in $Si(A)$. Let us show that $I$ is an ideal of $Si(A)$. Since $I$ is closed under finite joins, it is enough to prove that $I$ is a lower set. Let $J \in Si(A)$, $a \in A$ and $J \subseteq \downarrow (a)$. We need to show that $J$ is a principal ideal of $A$. Since $J \in Si(A)$, there exist $b \in J$ and $c \in \neg J$ such that $a = b \vee c$. We will prove that $J = \downarrow (b)$. Note first that if $b' \in J$ and $a = b' \vee c$ then $b = b'$. Indeed, we have that $b' = a \wedge b' = (b \vee c) \wedge b' = b \wedge b'$ and $b = a \wedge b = (b' \vee c) \wedge b = b \wedge b'$; thus $b = b'$. Let now $d \in J$. Then $d \leq a$ and hence $a = a \vee d = (b \vee d) \vee c$. Since $b \vee d \in J$, we get that $b \vee d = b$, i.e., $d \leq b$. So, $J = \downarrow (b)$, and hence $J \in I$. Thus $(Si(A), e_A(A))$ is an LBA.

(c) Let $B$ be a Boolean algebra and $J \in Si(B)$. Then there exist $a \in J$ and $b \in \neg J$ such that $1 = a \vee b$. Now we obtain, as in the proof of (b), that $J = \downarrow (a)$. So, every simple ideal of $B$ is principal. Thus, using (a), we complete the proof. □

*Notation 2.1.* Let $I$ be a proper ideal of a Boolean algebra $A$. We set

$$B_A(I) = I \cup \{a^* \mid a \in I\}.$$

When there is no ambiguity, we will often write "$B(I)$" instead of "$B_A(I)$".

It is clear that $B_A(I)$ is a Boolean subalgebra of $A$ and $I$ is a prime ideal of $B_A(I)$ (see, e.g., [8]).

**Fact 2.2.** *Let $(A, I)$ be an LBA. Then:*

(a) *$I$ is a generalized Boolean algebra;*

(b) *If $(B,J)$ is a PLBA and there exists a poset-isomorphism $\psi : J \longrightarrow I$ then $\psi$ can be uniquely extended to a Boolean embedding $\varphi : B \longrightarrow A$ (and $\varphi(B) = B_A(I)$); in particular, if $(A,I)$ is also a PLBA then $\varphi$ is a Boolean isomorphism and an isomorphism between LBAs $(A,I)$ and $(B,J)$;*

(c) *There exists a bijective correspondence between the class of all (up to isomorphism) generalized Boolean algebras and the class of all (up to isomorphism) PLBAs.*

Proof. (a) Obviously, for every $a \in I$, $\neg_I(\downarrow(a)) = I \cap \downarrow_A (a^*)$; then, clearly, $\downarrow(a) \vee \neg_I(\downarrow(a)) = I$. Now apply Fact 2.1(a).

(b) By [16, Theorem 12.5], $\psi$ can be uniquely extended to a Boolean isomorphism $\psi' : B \longrightarrow B_A(I)$. Now, define $\varphi : B \longrightarrow A$ by $\varphi(b) = \psi'(b)$, for every $b \in B$.

(c) For every PLBA $(A,I)$, set $f(A,I) = I$. Then, by (a), $I$ is a generalized Boolean algebra. Conversely, if $I$ is a generalized Boolean algebra then there exists a dense embedding $e : I \longrightarrow Si(I)$ (see Fact 2.1(b)). Thus, setting $g(I) = (B_{Si(I)}(e(I)), e(I))$, we get that $g(I)$ is a PLBA. Now, using (b), we obtain that for every PLBA $(A,I)$, $g(f(A,I))$ is isomorphic to $(A,I)$. Finally, it is clear that for every generalized Boolean algebra $I$, $f(g(I))$ is isomorphic to $I$. $\square$

We will need a simple lemma.

**Lemma 2.3.** *Let $A$ be a Boolean algebra, $M \subseteq A$, $X = S^a(A)$ and*

$$L_M = \{u \in X \mid u \cap M \neq \emptyset\}$$

*(sometimes we will write $L_M^A$ instead of $L_M$). Then:*

(a) *$L_M = \bigcup\{\lambda_A^S(a) \mid a \in M\}$;*

(b) *$L_M$ is an open subset of $X$ and hence the subspace $L_M$ of $X$ is a zero-dimensional locally compact Hausdorff space; $L_M \neq \emptyset$ iff $M \not\subseteq \{0\}$;*

(c) *$\lambda_A^S(M) \subseteq CK(L_M)$;*

(d) *If $M$ is dense in $A$ then $L_M$ is dense in $X$;*

(e) *If $M$ is a lower set and $L_M$ is dense in $X$ then $M$ is dense in $A$;*

(f) *If $L_M$ is dense in $X$ then the map*

$$\lambda_{(A,M)} : A \longrightarrow CO(L_M), \quad a \mapsto L_M \cap \lambda_A^S(a), \tag{2}$$

*is a Boolean monomorphism;*

(g) *If $M$ is an ideal of $A$ then $\lambda_A^S(M) = CK(L_M)$ and hence $\lambda_A^S(M)(= \{\lambda_A^S(a) \mid a \in M\})$ is a base of $L_M$;*

(h) *If $(A, M)$ is an LBA then $\lambda_{(A,M)} : A \longrightarrow CO(L_M)$ is a dense Boolean embedding;*

(i) *If $M_1, M_2 \subseteq A$ then $L_{M_1} = L_{M_2}$ iff the ideals of $A$ generated by $M_1$ and $M_2$ coincide.*

PROOF. Assertions (a)–(c) and (i) are obvious, and (h) follows from (b), (d), (f), (g).

(d) It is enough to prove that $\lambda_A^S(a) \cap L_M \neq \emptyset$, for every $a \in A \setminus \{0\}$. So, let $a \in A \setminus \{0\}$. Then there exists $b \in M \setminus \{0\}$ such that $b \leq a$. There exists $u \in X$ such that $b \in u$. Then $a \in u$. Thus $u \in \lambda_A^S(a) \cap L_M$.

(e) Let $M$ be a lower set and $L_M$ be dense in $X$. Let $a \in A \setminus \{0\}$. Then, by (b), $\lambda_A^S(a) \cap L_M$ is an open non-empty subset of $X$. Hence, there exists $b \in A \setminus \{0\}$ such that $\lambda_A^S(b) \subseteq L_M \cap \lambda_A^S(a)$. Let $u \in \lambda_A^S(b)$. Then there exists $c \in u \cap M$. Since $M$ is a lower set, we get that $b \wedge c \in u \cap M$. Thus $b \wedge c \neq 0$, $b \wedge c \in M$ and $b \wedge c \leq a$ (because, by the Stone Duality Theorem, $b \leq a$). Therefore, $M$ is dense in $A$.

(f) Since, by the Stone Duality Theorem, the map $\lambda_A^S : A \longrightarrow CO(X)$, $a \mapsto \lambda_A^S(a)$, is a Boolean isomorphism, it is clear that the map $\lambda_{(A,M)}$ is a Boolean homomorphism. Further, since $L_M$ is dense in $X$, we have that if $a \in A \setminus \{0\}$ then $\lambda_{(A,M)}(a) \neq \emptyset$. Therefore, $\lambda_{(A,M)}$ is a Boolean monomorphism.

(g) Let $M$ be an ideal of $A$ and $U \in CK(L_M)$. For every $u \in U$ there exists $b_u \in M \cap u$, and hence $u \in \lambda_A^S(b_u) \subseteq L_M$. Thus $U \subseteq \bigcup\{\lambda_A^S(b_u) \mid u \in U\}$. Since $U$ is compact, there exist $\{u_i \in U \mid i = 1, \ldots, n\}$, where $n$ is some natural number, such that $U \subseteq \bigcup\{\lambda_A^S(b_{u_i}) \mid i = 1, \ldots, n\}$. Let $b_0 = \bigvee\{b_{u_i} \mid i = 1, \ldots, n\}$. Then $b_0 \in M$ and $\lambda_A^S(b_0) = \bigcup\{\lambda_A^S(b_{u_i}) \mid i = 1, \ldots, n\} \supseteq U$. Now, for every $u \in U$ there exists $a_u \in A$ such that $u \in \lambda_A^S(a_u) \subseteq U$ and thus $\lambda_A^S(a_u) \subseteq \lambda_A^S(b_0)$. Therefore, for every $u \in U$, $a_u \leq b_0$, and hence, $a_u \in M$. Using again the compactness of $U$, we get that there exists $a_0 \in M$ such that $U = \lambda_A^S(a_0)$. So, $\lambda_A^S(M) \supseteq CK(L_M)$. This fact together with (c) imply that $\lambda_A^S(M) = CK(L_M)$. □

*Notation 2.2.* Let $X$ be a topological space. For every $x \in X$, we set

$$u_x^{CO(X)} = \{F \in CO(X) \mid x \in F\}.$$

When there is no ambiguity, we will write "$u_x^C$" instead of "$u_x^{CO(X)}$".

Recall that a *contravariant adjunction* $(T, S, \varepsilon, \eta) : \mathcal{A} \longrightarrow \mathcal{B}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of two contravariant functors $T : \mathcal{A} \longrightarrow \mathcal{B}$ and $S : \mathcal{B} \longrightarrow \mathcal{A}$ and two natural transformations $\eta : Id_{\mathcal{B}} \longrightarrow T \circ S$ and $\varepsilon : Id_{\mathcal{A}} \longrightarrow S \circ T$ such that $T(\varepsilon_A) \circ \eta_{TA} = \mathrm{id}_{TA}$ and $S(\eta_B) \circ \varepsilon_{SB} = \mathrm{id}_{SB}$, for all $A \in |\mathcal{A}|$ and $B \in |\mathcal{B}|$. The pair $(T, S)$ is a duality iff $\eta$ and $\varepsilon$ are natural isomorphisms.

**Theorem 2.4.** *There exists a contravariant adjunction* $(\Theta^a, \Theta^t, \lambda^C, t^C) : \mathsf{LBA} \longrightarrow \mathsf{BoolSp}$, *where* $\mathsf{BoolSp}$ *is the category of locally compact zero-dimensional Hausdorff spaces and continuous maps.*

Proof. We will first define two contravariant functors $\Theta^a : \mathsf{LBA} \longrightarrow \mathsf{BoolSp}$ and $\Theta^t : \mathsf{BoolSp} \longrightarrow \mathsf{LBA}$.

Let $X \in |\mathsf{BoolSp}|$. Define

$$\Theta^t(X) = (CO(X), CK(X)).$$

Obviously, $\Theta^t(X)$ is an LBA.

Let $f \in \mathsf{BoolSp}(X, Y)$. Define $\Theta^t(f) : \Theta^t(Y) \longrightarrow \Theta^t(X)$ by the formula

$$\Theta^t(f)(G) = f^{-1}(G), \quad \forall G \in CO(Y). \tag{3}$$

Set $\varphi_f = \Theta^t(f)$. Clearly, $\varphi_f$ is a Boolean homomorphism between $CO(Y)$ and $CO(X)$. If $F \in CK(X)$ then $f(F)$ is a compact subset of $Y$. Since $CK(Y)$ is an open base of the space $Y$ and $CK(Y)$ is closed under finite unions, we get that there exists $G \in CK(Y)$ such that $f(F) \subseteq G$. Then $F \subseteq f^{-1}(G) = \varphi_f(G)$. So, $\varphi_f$ satisfies condition (LBA). Therefore $\varphi_f$ is an $\mathsf{LBA}$-morphism, i.e., $\Theta^t(f)$ is well-defined.

Now we get easily that $\Theta^t$ is a contravariant functor.

For every LBA $(A, I)$, set

$$\Theta^a(A, I) = L_I^A \tag{4}$$

(see Lemma 2.3 for the notation $L_I^A$). Then Lemma 2.3 implies that $L = \Theta^a(A, I)$ is a zero-dimensional locally compact Hausdorff space and $\lambda_{(A,I)}(I)$ is an open base of $L$ (see (2) for the notation $\lambda_{(A,I)}$). So, $\Theta^a(A, I) \in |\mathsf{BoolSp}|$.

Let $\varphi \in \mathsf{LBA}((A, I), (B, J))$. We define the map

$$\Theta^a(\varphi) : \Theta^a(B, J) \longrightarrow \Theta^a(A, I)$$

by the formula

$$\Theta^a(\varphi)(u') = \varphi^{-1}(u'), \quad \forall u' \in \Theta^a(B, J). \tag{5}$$

Set $f_\varphi = \Theta^a(\varphi)$, $L = \Theta^a(A, I)$ and $M = \Theta^a(B, J)$.

By Lemma 2.3, if $(A', I')$ is a LBA then the set $\Theta^a(A', I')$ consists of all *bounded ultrafilters in* $(A', I')$ (i.e., those ultrafilters $u$ in $A'$ for which $u \cap I' \neq \emptyset$). Since any $\mathsf{LBA}$-morphism is a Boolean homomorphism, we get that the inverse image of an ultrafilter is an ultrafilter.

So, let $u' \in M$. Then $u'$ is a bounded ultrafilter in $(B, J)$. Set $u = f_\varphi(u')$. Then, as we have seen, $u$ is an ultrafilter in $A$. We have to show that $u$ is bounded in $(A, I)$. Indeed, since $u'$ is bounded, there exists $b \in u' \cap J$. By condition (LBA), there exists $a \in I$ such that $\varphi(a) \geq b$. Then $\varphi(a) \in u'$, and hence, $a \in u$. Thus $a \in u \cap I$. Therefore, $f_\varphi : M \longrightarrow L$.

We will show that $f_\varphi$ is a continuous function. Let $u' \in M$ and $u = f_\varphi(u')$. Let $a \in A$ and $u \in \lambda_{(A,I)}(a) (= \mathrm{int}(\lambda_{(A,I)}(a)))$. Then $a \in u$. Hence $\varphi(a) \in u'$, i.e., $u' \in \lambda_{B,J}(\varphi(a))$. We will prove that

$$f_\varphi(\lambda_{(B,J)}(\varphi(a))) \subseteq \lambda_{(A,I)}(a). \tag{6}$$

Indeed, let $v' \in \lambda_{(B,J)}(\varphi(a))$. Then $\varphi(a) \in v'$. Thus $a \in f_\varphi(v')$, i.e., $f_\varphi(v') \in \lambda_{(B,J)}(a)$. So, (6) is proved. Since $\{\lambda_{(A,I)}(a) \mid a \in A\}$ is an open base of $L$, we get that $f_\varphi$ is a continuous function. So,

$$\Theta^a(\varphi) \in \mathsf{BoolSp}(\Theta^a(B, J), \Theta^a(A, I)).$$

Now it becomes obvious that $\Theta^a$ is a contravariant functor.

Let $X \in |\mathsf{BoolSp}|$. Then it is easy to see that for every $x \in X$, $u_x^C$ is an ultrafilter in $CO(X)$ and hence, using the fact that $u_x^C$ contains always elements of $CK(X)$, we get that $u_x^C \in \Theta^a(CO(X), CK(X))$. We will show that the map

$$t_X^C : X \longrightarrow \Theta^a(\Theta^t(X)), \quad x \mapsto u_x^C, \tag{7}$$

is a homeomorphism. Set $L = \Theta^a(\Theta^t(X))$ and $A = CO(X)$, $I = CK(X)$. We will prove that $t_X^C$ is a continuous map. Let $x \in X$, $F \in I$ and $u_x^C \in \lambda_{(A,I)}(F)$. Then $F \in u_x^C$ and hence, $x \in F$. It is enough to show that $t_X^C(F) \subseteq \lambda_{(A,I)}(F)$. Let $y \in F$. Then $F \in u_y^C = t_X^C(y)$. Hence $t_X^C(y) \in \lambda_{(A,I)}(F)$. So, $t_X^C(F) \subseteq \lambda_{(A,I)}(F)$. Since $\lambda_{(A,I)}(I)$ is an open base of $L$, we get that $t_X^C$ is a continuous map. Let us show that $t_X^C$ is a bijection. Let $u \in L$. Then $u$ is a bounded ultrafilter in $(A, I)$. Hence, there exists $F \in u \cap I$. Since $F$ is compact, we get that $\bigcap u \neq \emptyset$. Suppose that $x, y \in \bigcap u$ and $x \neq y$. Then there exist $F_x, F_y \in I$ such that $x \in F_x$, $y \in F_y$ and $F_x \cap F_y = \emptyset$. Since, clearly, $F_x, F_y \in u$, we get a contradiction. So, $\bigcap u = \{x\}$ for some $x \in X$. It is clear now that $u = u_x^C$, i.e., $u = t_X^C(x)$ and $u \neq t_X^C(y)$, for $y \in X \setminus \{x\}$. So, $t_X^C$ is a bijection. For showing that $(t_X^C)^{-1}$ is a continuous function, let $u_x^C \in L$. Then $(t_X^C)^{-1}(u_x^C) = x$. Let $F \in I$ and $x \in F$. Then $F \in u_x^C$ and thus $u_x^C \in \lambda_{(A,I)}(F)$. We will prove that $(t_X^C)^{-1}(\lambda_{(A,I)}(F)) \subseteq F$. Since $I$ is a base of $X$, this will imply that $(t_X^C)^{-1}$ is a continuous function. So, let $y \in (t_X^C)^{-1}(\lambda_{(A,I)}(F))$. Then $t_X^C(y) \in \lambda_{(A,I)}(F)$, i.e., $F \in u_y^C$. Then $y \in F$. Therefore, $t_X^C$ is a homeomorphism.

We will show that

$$t^C : Id_{\mathsf{BoolSp}} \longrightarrow \Theta^a \circ \Theta^t,$$

defined by $t^C(X) = t_X^C$, $\forall X \in |\mathsf{BoolSp}|$, is a natural isomorphism.

Let $f \in \mathsf{BoolSp}(X, Y)$ and $\hat{f} = \Theta^a(\Theta^t(f))$. We have to show that $\hat{f} \circ t_X^C = t_Y^C \circ f$. Let $x \in X$. Then $\hat{f}(t_X^C(x)) = \hat{f}(u_x^{CO(X)})$ and $(t_Y^C \circ f)(x) = u_{f(x)}^{CO(Y)}$. Set $y = f(x)$, $u_x = u_x^{CO(X)}$ and $u_y = u_{f(x)}^{CO(Y)}$. We will prove that

$$\hat{f}(u_x) = u_y.$$

Let $\varphi = \Theta^t(f)$. Then $\hat{f} = \Theta^a(\varphi)(= f_\varphi)$. Hence, $\hat{f}(u_x) = \varphi^{-1}(u_x) = \{G \in CO(Y) \mid \varphi(G) \in u_x\} = \{G \in CO(Y) \mid x \in \varphi(G)\} = \{G \in CO(Y) \mid x \in f^{-1}(G)\} = \{G \in CO(Y) \mid f(x) \in G\} = u_y$. So, $t^C$ is a natural isomorphism.

Let $(A, I)$ be an LBA and $L = \Theta^a(A, I)$. Then, by Lemma 2.3(h), $\lambda_{(A,I)} : A \longrightarrow CO(L)$ is a dense Boolean embedding. Also, by Lemma 2.3(g), $\lambda_{(A,I)}(I) = CK(L)$. We denote by $\lambda^C_{(A,I)}$ the map $\lambda^C_{(A,I)} : (A, I) \longrightarrow (CO(L), CK(L))$, where $\lambda^C_{(A,I)}(a) = \lambda_{(A,I)}(a)$, for every $a \in A$; we will write sometimes "$\lambda_A^C$" instead of "$\lambda^C_{(A,I)}$". Note that

$$\lambda^C_{(A,I)} : (A, I) \longrightarrow \Theta^t(\Theta^a(A, I)).$$

We will prove that

$$\lambda^C : Id_{\mathsf{LBA}} \longrightarrow \Theta^t \circ \Theta^a, \text{ where } \lambda^C(A, I) = \lambda^C_{(A,I)}, \quad \forall (A, I) \in |\mathsf{LBA}|,$$

is a natural transformation.

Let $\varphi \in \mathsf{LBA}((A, I), (B, J))$ and $\hat{\varphi} = \Theta^t(\Theta^a(\varphi))$. We have to prove that $\lambda_B^C \circ \varphi = \hat{\varphi} \circ \lambda_A^C$. Set $f = \Theta^a(\varphi)$ and $M = \Theta^a(B, J)$. Then $\hat{\varphi} = \Theta^t(f)(= \varphi_f)$. Let $a \in A$. Then $\hat{\varphi}(\lambda_A^C(a)) = f^{-1}(\lambda_A^C(a)) = \{u \in M \mid f(u) \in \lambda_A^C(a)\} = \{u \in M \mid a \in f(u)\} = \{u \in M \mid a \in \varphi^{-1}(u)\} = \{u \in M \mid \varphi(a) \in u\} = \lambda_B^C(\varphi(a))$. So, $\lambda^C$ is a natural transformation.

Let us show that $\Theta^t(t_X^C) \circ \lambda^C_{\Theta^t(X)} = \mathrm{id}_{\Theta^t(X)}$, for every $X \in |\mathsf{BoolSp}|$. Indeed, let $X \in |\mathsf{BoolSp}|$ and $Y = \Theta^a(\Theta^t(X))$. Then $\Theta^t(t_X^C) : \Theta^t(Y) \longrightarrow \Theta^t(X)$, $G \mapsto (t_X^C)^{-1}(G)$, for every $G \in \Theta^t(Y) = (CO(Y), CK(Y))$. Let $F \in CO(X)$. Then

$(\Theta^t(t^C_X) \circ \lambda^C_{\Theta^t(X)})(F) = (t^C_X)^{-1}(\lambda^C_{\Theta^t(X)}(F)) = H$. We have to show that $F = H$. Since $t^C_X(H) = \lambda^C_{\Theta^t(X)}(F)$, we get that $\{u^C_x \mid x \in H\} = \{u \in Y \mid F \in u\}$. Thus $x \in H \iff F \in u^C_x \iff x \in F$. Therefore, $F = H$.

Finally, we will prove that $\Theta^a(\lambda^C_{(A,I)}) \circ t^C_{\Theta^a(A,I)} = \mathrm{id}_{\Theta^a(A,I)}$ for every $(A,I) \in |\mathsf{LBA}|$. So, let $(A,I) \in |\mathsf{LBA}|$ and $X = \Theta^a(A,I)$. We have that

$$f = \Theta^a(\lambda^C_{(A,I)}) : \Theta^a(CO(X), CK(X)) \longrightarrow X$$

is defined by

$$u \mapsto (\lambda^C_{(A,I)})^{-1}(u),$$

for every bounded ultrafilter $u$ in $(CO(X), CK(X))$. Let $x \in X$. Then $f(t^C_X(x)) = f(u^C_x) = (\lambda^C_{(A,I)})^{-1}(u^C_x) = y$. We have to show that $x = y$. Indeed, for every $a \in A$, we get that $a \in y \iff a \in (\lambda^C_{(A,I)})^{-1}(u^C_x) \iff \lambda^C_{(A,I)}(a) \in u^C_x \iff x \in \lambda^C_{(A,I)}(a) \iff a \in x$. Therefore, $x = y$.

We have proved that $(\Theta^a, \Theta^t, \lambda^C, t^C)$ is a contravariant adjunction between the categories $\mathsf{LBA}$ and $\mathsf{BoolSp}$. Moreover, we have even shown that $t^C$ is a natural isomorphism. Hence $\Theta^t$ is a full and faithful contravariant functor and, thus, it reflects isomorphisms. □

*Definition 2.2.* An LBA $(A,I)$ is called a *ZLB-algebra* (briefly, *ZLBA*) if, for every $J \in Si(I)$, the join $\bigvee_A J (= \bigvee_A \{a \mid a \in J\})$ exists.

Let $\mathsf{ZLBA}$ be the full subcategory of the category $\mathsf{LBA}$ having as objects all ZLBAs.

*Example 2.1.* Let $B$ be a Boolean algebra. Then the pair $(B,B)$ is a ZLBA. This follows from Fact 2.1(c).

*Remark 2.4.* Note that if $A$ and $B$ are Boolean algebras then any Boolean homomorphism $\varphi : A \longrightarrow B$ is a $\mathsf{ZLBA}$-morphism between the ZLBAs $(A,A)$ and $(B,B)$. Hence, the full subcategory $\boldsymbol{B}$ of the category $\mathsf{ZLBA}$ whose objects are all ZLBAs of the form $(A,A)$ is isomorphic (it can be even said that it coincides) with the category $\mathsf{BoolAlg}$ of Boolean algebras and Boolean homomorphisms.

We will need the following result of M. Stone [18]:

**Proposition 2.5** (M. Stone [18, Theorem 5(3)])**.** *Let $X \in |\mathsf{BoolSp}|$. Then the map $\Sigma : Si(CK(X)) \longrightarrow CO(X)$, $J \mapsto \bigvee_{RC(X)} J$, is a Boolean isomorphism.*

Proof. For completeness of our exposition, we will verify this fact. Let $J \in Si(CK(X))$. Set $U = \bigcup\{F \mid F \in J\}$ and $V = \bigcup\{G \mid G \in \neg J\}$. Obviously, $U$ and $V$ are disjoint open subsets of $X$. We will show that $U \cup V = X$. Indeed, let

$x \in X$. Then there exists $H \in CK(X)$ such that $x \in H$. Since $J \vee \neg J = CK(X)$, we get that there exist $F \in J$ and $G \in \neg J$ such that $H = F \cup G$. Thus $x \in F$ or $x \in G$, and hence, $x \in U$ or $x \in V$. So, $U$ is a clopen subset of $X$. Thus $U \in CO(X)$ and $U = \bigvee_{RC(X)} J = \bigvee_{CO(X)} J$. Conversely, it is easy to see that if $U \in CO(X)$ then $J = \{F \in CK(X) \mid F \subseteq U\} \in Si(CK(X))$. This implies easily that $\Sigma$ is a Boolean isomorphism. $\square$

**Proposition 2.6.** *Let $(A, I)$ be an LBA and $L = \Theta^a(A, I)$. Then $(A, I)$ is a ZLBA iff $\lambda_{(A,I)}(A) = CO(L)$ (see (2) for the notation $\lambda_{(A,I)}$).*

Proof. Let $(A, I)$ be a ZLBA. We will prove that $\lambda_{(A,I)}(A) = CO(L)$. Let $U \in CO(L)$ and $J' = \{F \in CK(L) \mid F \subseteq U\}$. Then $J'$ is a simple ideal of $CK(L)$ and $\bigvee_{RC(L)} J' = U$. Since the restriction $\varphi : I \longrightarrow CK(L)$ of $\lambda_{(A,I)}$ is a $\{0\}$-pseudolattice isomorphism, we get that $J = \varphi^{-1}(J')$ is a simple ideal of $I$. Set $b_J = \bigvee_A J$ and $C = \lambda_{(A,I)}(A)$ (note that the join $\bigvee_A J$ exists because $(A, I)$ is a ZLBA). Now, the restriction $\psi : A \longrightarrow C$ of $\lambda_{(A,I)}$ is a Boolean isomorphism, and hence $\lambda_{(A,I)}(b_J) = \psi(b_J) = \psi(\bigvee_A J) = \bigvee_C \psi(J) = \bigvee_C J'$. The fact that $C$ is a dense Boolean subalgebra of the Boolean algebra $CO(L)$, and hence of $RC(L)$, implies that $C$ is a regular subalgebra of $RC(L)$. Thus $\bigvee_C J' = \bigvee_{RC(L)} J' = U$. Therefore, $\lambda_{(A,I)}(b_J) = U$. So, we have proved that $\lambda_{(A,I)}(A) = CO(L)$.

Let now $(A, I)$ be an LBA and $\lambda_{(A,I)}(A) = CO(L)$. Set, for short, $\psi = \lambda_{(A,I)}$. Then the map $\psi : A \longrightarrow CO(L)$ is a Boolean isomorphism. Let $J \in Si(I)$. Since the restriction of $\psi$ to $I$ is a 0-pseudolattice isomorphism between $I$ and $CK(L)$, we get that $\psi(J) \in Si(CK(L))$. Then, by the proof of Proposition 2.5, $U = \bigcup\{F \mid F \in \psi(J)\}(= \bigcup\{\psi(a) \mid a \in J\})$ is a clopen subset of $L$. Therefore, the join $\bigvee_{CO(L)}\{\psi(a) \mid a \in J\}$ exists. Since $\psi^{-1} : CO(L) \longrightarrow A$ is a Boolean isomorphism, we obtain that $\psi^{-1}(U) = \psi^{-1}(\bigvee_{CO(L)}\{\psi(a) \mid a \in J\}) = \bigvee_A\{\psi^{-1}(\psi(a)) \mid a \in J\} = \bigvee_A\{a \mid a \in J\}$. Hence, the join $\bigvee_A J$ exists. Thus, $(A, I)$ is a ZLBA. $\square$

*Remark 2.5.* Note that, by Lemma 2.3(h), if $(A, I)$ is an LBA then $\lambda_{(A,I)}(A)$ is isomorphic to $A$ and is a Boolean subalgebra of $CO(L)$, where $L = \Theta^a(A, I)$ (see (4) for the notation $\Theta^a$). If $(A, I)$ is an LBA and $(X, L) = p(A, I)$ (see Remark 2.1 for this notation) then, by Lemma 2.3(a), $L = \Theta^a(A, I))$; thus, by Proposition 2.6, $(A, I)$ is a ZLBA iff $A$ is mapped isomorphically by $\lambda_{(A,I)}$ to $CO(L)$; since the Banaschewski compactification $\beta_0 L$ of $L$ (see [2] and [8, Theorem 13.1]) is constructed as $S^a(CO(L))$ (i.e., it is the Stonification of $L$), we get that $(A, I)$ is a ZLBA iff $p(A, I) = (\beta_0 L, L)$, where $L$ is defined as in Remark 2.1 (i.e., $L = \bigcup\{\lambda_A^S(a) \mid a \in I\}$).

**Theorem 2.7.** *The categories* $\mathsf{BoolSp}$ *and* $\mathsf{ZLBA}$ *are dually equivalent.*

Proof. In Theorem 2.4, we constructed a contravariant adjunction

$$(\Theta^a, \Theta^t, \lambda^C, t^C)$$

between the categories $\mathsf{LBA}$ and $\mathsf{BoolSp}$, where $t^C$ was even a natural isomorphism. Let us check that the functor $\Theta^t$ is in fact a functor from the category $\mathsf{BoolSp}$ to the category $\mathsf{ZLBA}$. Indeed, let $X \in |\mathsf{BoolSp}|$. Then $\Theta^t(X) = (CK(X), CO(X))$. As it follows from Proposition 2.5, for every $J \in Si(CK(X))$, $\bigvee_{CO(X)} J$ exists. Hence, $\Theta^t(X) \in |\mathsf{ZLBA}|$. So, the restriction

$$\Theta^t_d : \mathsf{BoolSp} \longrightarrow \mathsf{ZLBA}$$

of the contravariant functor $\Theta^t : \mathsf{BoolSp} \longrightarrow \mathsf{LBA}$ is well-defined. Further, by Proposition 2.6, the natural transformation $\lambda^C$ becomes a natural isomorphism exactly on the subcategory $\mathsf{ZLBA}$ of the category $\mathsf{LBA}$. We will denote by

$$\Theta^a_d : \mathsf{ZLBA} \longrightarrow \mathsf{BoolSp}$$

the restriction of the contravariant functor $\Theta^a$ to the category $\mathsf{ZLBA}$. All this shows that there is a duality between the categories $\mathsf{BoolSp}$ and $\mathsf{ZLBA}$. $\square$

**Corollary 2.8** (The Stone Duality Theorem [18]). *The categories* $\mathsf{BoolAlg}$ *and* $\mathsf{Stone}$ *are dually equivalent.*

Proof. Obviously, the restriction of the contravariant functor $\Theta^a_d$ to the subcategory $\boldsymbol{B}$ of the category $\mathsf{ZLBA}$ (see Remark 2.4 for the notation $\boldsymbol{B}$) produces a duality between the categories $\boldsymbol{B}$ and $\mathsf{Stone}$. $\square$

**Corollary 2.9.** *For every ZLBA $(A, I)$, the map $\Sigma_{(A,I)} : Si(I) \longrightarrow A$, where $\Sigma_{(A,I)}(J) = \bigvee_A \{a \mid a \in J\}$ for every $J \in Si(I)$, is a Boolean isomorphism.*

Proof. Let $L = \Theta^a_d(A, I)$ (see the proof of Theorem 2.7 for the notation $\Theta^a_d$). Then, as it was shown in the proof of Theorem 2.7, the map $\lambda^C_A : (A, I) \longrightarrow (CO(L), CK(L))$, where $\lambda^C_A(a) = \lambda_{(A,I)}(a)$ for every $a \in A$, is a $\mathsf{ZLBA}$-isomorphism. By Proposition 2.5, the map

$$\Sigma = \Sigma_{(CO(L),CK(L))} : Si(CK(L)) \longrightarrow CO(L), \quad J \mapsto \bigvee_{CO(L)} J,$$

is a Boolean isomorphism. Define a map $\lambda'_A : Si(I) \longrightarrow Si(CK(L))$ by the formula $\lambda'_A(J) = \lambda^C_A(J)$, for every $J \in Si(I)$. Then, obviously, $\lambda'_A$ is a Boolean isomorphism and $\Sigma_{(A,I)} = (\lambda^C_A)^{-1} \circ \Sigma \circ \lambda'_A$. Thus $\Sigma_{(A,I)}$ is a Boolean isomorphism. $\square$

*Definition 2.3.* Let $\mathsf{PZLBA}$ be the cofull subcategory of the category $\mathsf{ZLBA}$ whose morphisms $\varphi : (A, I) \longrightarrow (B, J)$ satisfy the following additional condition: (PLBA) $\varphi(I) \subseteq J$.

**Theorem 2.10.** *The category* $\mathsf{PerfBoolSp}$ *of all locally compact Hausdorff zero-dimensional spaces and all perfect maps between them is dually equivalent to the category* $\mathsf{PZLBA}$.

Proof. Let $f \in \mathsf{PerfBoolSp}(X, Y)$. Then, as we have seen in the proof of Theorem 2.7, $\Theta^t_d(f) : \Theta^t_d(Y) \longrightarrow \Theta^t_d(X)$ is defined by the formula $\Theta^t_d(f)(G) = f^{-1}(G), \forall G \in CO(Y)$. Set $\varphi_f = \Theta^t_d(f)$. Since $f$ is a perfect map, we have that for any $K \in CK(Y)$, $\varphi_f(K) = f^{-1}(K) \in CK(X)$. Hence, $\varphi_f$ satisfies condition (PLBA). Thus, $\varphi_f$ is a $\mathsf{PZLBA}$-morphism. So, the restriction $\Theta^t_p$ of the duality functor $\Theta^t_d$ to the subcategory $\mathsf{PerfBoolSp}$ of the category $\mathsf{BoolSp}$ is a contravariant functor from $\mathsf{PerfBoolSp}$ to $\mathsf{PZLBA}$.

Let $\varphi \in \mathsf{PZLBA}((A, I), (B, J))$. The map $\Theta^a_d(\varphi) : \Theta^a_d(B, J) \longrightarrow \Theta^a_d(A, I)$ was defined in Theorem 2.7 by the formula $\Theta^a_d(\varphi)(u') = \varphi^{-1}(u')$, $\forall u' \in \Theta^a_d(B, J)$. Set $f_\varphi = \Theta^a_d(\varphi)$, $L = \Theta^a_d(A, I)$ and $M = \Theta^a_d(B, J)$.

Let $a \in I$. We will show that $f^{-1}_\varphi(\lambda_{(A,I)}(a))$ is compact. We have, by (PLBA), that $\varphi(a) \in J$. Let us prove that

$$\lambda_{(B,J)}(\varphi(a)) = f^{-1}_\varphi(\lambda_{(A,I)}(a)). \tag{8}$$

Let $u' \in f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Then $u = f_\varphi(u') \in \lambda_{(A,I)}(a)$, i.e., $a \in u$. Thus $\varphi(a) \in u'$, and hence $u' \in \lambda_{(B,J)}(\varphi(a))$. Therefore, $\lambda_{(B,J)}(\varphi(a)) \supseteq f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Now, (6) implies that $\lambda_{(B,J)}(\varphi(a)) = f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Since $\lambda_{(B,J)}(\varphi(a))$ is compact, we get that $f^{-1}_\varphi(\lambda_{(A,I)}(a))$ is compact. Let now $K$ be a compact subset of $L$. Since $\lambda_{(A,I)}(I)$ is an open base of $L$ and $\lambda_{(A,I)}(I)$ is closed under finite unions, we get that there exists $a \in I$ such that $K \subseteq \lambda_{(A,I)}(a)$. Then $f^{-1}_\varphi(K) \subseteq f^{-1}_\varphi(\lambda_{(A,I)}(a))$, and hence, as a closed subset of a compact set, $f^{-1}_\varphi(K)$ is compact. This implies that $f_\varphi$ is a perfect map (see, e.g.,[9]). Therefore, the restriction $\Theta^a_p$ of the duality functor $\Theta^a_d$ to the subcategory $\mathsf{PZLBA}$ of the category $\mathsf{ZLBA}$ is a contravariant functor from $\mathsf{PZLBA}$ to $\mathsf{PerfBoolSp}$. The rest follows from Theorem 2.7. □

The above theorem can be stated in a better form. We will do this now.

*Definition 2.4.* Let $\mathsf{PLBA}$ be the subcategory of the category $\mathsf{LBA}$ whose objects are all PLBAs and whose morphisms are all $\mathsf{LBA}$-morphisms

$$\varphi : (A, I) \longrightarrow (B, J)$$

between the objects of $\mathsf{PLBA}$ satisfying condition (PLBA).

*Remark 2.6.* It is obvious that PLBA is indeed a category. Note also that any Boolean homomorphism $\varphi : A \longrightarrow B$ is a PLBA-morphism between the PLBAs $(A, A)$ and $(B, B)$. Hence, the full subcategory $\boldsymbol{B}$ of the category PLBA whose objects are all PLBAs of the form $(A, A)$ is isomorphic (it can be even said that it coincides) with the category BoolAlg of Boolean algebras and Boolean homomorphisms.

**Theorem 2.11.** *The categories* PerfBoolSp *and* PLBA *are dually equivalent.*

Proof. In virtue of Theorem 2.10, it is enough to show that the categories PLBA and PZLBA are equivalent.

Let $(B, I)$ be a ZLBA. Set $A = B_B(I)$ (see Notation 2.1 for this notation). Then, obviously, $(A, I)$ is a PLBA. Set $E^z(B, I) = (A, I)$.

If $\varphi \in \mathsf{PZLBA}((B_1, I_1), (B_2, I_2))$ then let $E^z(\varphi)$ be the restriction of $\varphi$ to $E^z(B_1, I_1)$. Then, clearly, $E^z(\varphi) \in \mathsf{PLBA}(E^z(B_1, I_1), E^z(B_2, I_2))$. It is evident that $E^z$ is a (covariant) functor from PZLBA to PLBA.

Let $(A, I)$ be a PLBA. Then, by Fact 2.2(a), $I$ is a generalized Boolean algebra. Hence, according to Fact 2.1(b), the map $e_I : I \longrightarrow Si(I)$, where $e_I(a) = \downarrow(a)$, is a dense embedding of $I$ in the Boolean algebra $Si(I)$ and the pair $(Si(I), e_I(I))$ is an LBA. Set $I' = e_I(I)$ and $E^p(A, I) = (Si(I), I')$. Then, for every $J \in Si(I)$, $\bigvee_{Si(I)} e_I(J) = \bigvee_{Si(I)} \{\downarrow(a) \mid a \in J\} = J$. This implies that $(Si(I), I') \in |\mathsf{PZLBA}|$.

Let $\varphi \in \mathsf{PLBA}((A_1, I_1), (A_2, I_2))$. Let the map $\varphi' = E^p(\varphi)$ be defined by the formula $\varphi'(J_1) = \bigcup\{\downarrow(\varphi(a)) \mid a \in J_1\}$, for every $J_1 \in Si(I_1)$. We will show that $\varphi'$ is a PZLBA-morphism between $E^p(A_1, I_1)$ and $E^p(A_2, I_2)$. Obviously, $\varphi'(\{0\}) = \{0\}$ and, thanks to conditions (LBA) and (PLBA), $\varphi'(I_1) = I_2$. Let $J_1 \in Si(I_1)$. Set $J_2 = \varphi'(J_1)$. Then condition (PLBA) and the fact that $\varphi$ is a homomorphism imply that $J_2$ is an ideal of $I_2$. Let us show that $J_2 \vee \neg J_2 = I_2$. Indeed, let $a_2 \in I_2$. Then condition (LBA) implies that there exists $a_1 \in I_1$ such that $a_2 \leq \varphi(a_1)$. Since $J_1 \vee \neg J_1 = I_1$, there exist $a_1' \in J_1$ and $a_1'' \in \neg J_1$ such that $a_1 = a_1' \vee a_1''$. Then $a_2 = (\varphi(a_1') \wedge a_2) \vee (\varphi(a_1'') \wedge a_2)$. Obviously, $(\varphi(a_1') \wedge a_2) \in J_2$. We will prove that $(\varphi(a_1'') \wedge a_2) \in \neg J_2$. It is enough to show that $\varphi(a_1'') \in \neg J_2$. Let $b_2 \in J_2$. Then, by the definition of $J_2$, there exists $b_1 \in J_1$ such that $b_2 \leq \varphi(b_1)$. Since $b_1 \wedge a_1'' = 0$, we get that $\varphi(b_1) \wedge \varphi(a_1'') = 0$. Thus $\varphi(a_1'') \wedge b_2 = 0$. Therefore, $\varphi(a_1'') \in \neg J_2$. So, $J_2 \in Si(I_2)$. Note that this implies that $\varphi'(J_1) = \bigvee_{Si(I_2)} \{\downarrow(\varphi(a)) \mid a \in J_1\}$. The above arguments show also that $\varphi'(\neg J_1) \subseteq \neg\varphi'(J_1)$, for every $J_1 \in Si(I_1)$. In fact, there is an equality here, i.e., $\varphi'(\neg J_1) = \neg\varphi'(J_1)$. Indeed, let $b_2 \in \neg\varphi'(J_1)$. Then $b_2 \wedge a_2 = 0$, for every $a_2 \in \varphi'(J_1)$. By condition (LBA), there exists $a_1 \in I_1$ such that $b_2 \leq \varphi(a_1)$. We have again that there exist $a_1' \in J_1$ and $a_1'' \in \neg J_1$ such that $a_1 = a_1' \vee a_1''$. Then $b_2 = (\varphi(a_1') \wedge b_2) \vee (\varphi(a_1'') \wedge b_2) = \varphi(a_1'') \wedge b_2$. Thus, $b_2 \leq \varphi(a_1'')$. This shows that $b_2 \in \varphi'(\neg J_1)$. Further, if $J, J' \in Si(I_1)$ then $\varphi'(J) \wedge \varphi'(J') = \varphi'(J) \cap \varphi'(J') = \bigcup\{\downarrow (a) \wedge \downarrow (b) \mid a \in J, b \in J'\} = \bigcup\{\downarrow (a) \mid a \in J \cap J'\} = \varphi'(J \cap J') = \varphi'(J \wedge J')$. Therefore, $\varphi' : Si(I_1) \longrightarrow Si(I_2)$ is a Boolean homomorphism. Since, for every $a \in I_1$, $\varphi'(\downarrow (a)) = \downarrow (\varphi(a))$, we have that $e_{I_2} \circ \varphi_{|I_1} = \varphi' \circ e_{I_1}$. This shows that $\varphi' \in \mathsf{PZLBA}(E^p(A_1, I_1), E^p(A_2, I_2))$. Now one can easily see that $E^p$ is a (covariant) functor between the categories $\mathsf{PLBA}$ and $\mathsf{PZLBA}$.

Finally, we have to verify that the compositions $E^p \circ E^z$ and $E^z \circ E^p$ are naturally isomorphic to the corresponding identity functors.

Let us start with the composition $E^z \circ E^p$.

Let $(A, I)$ be a PLBA. Then, as we have seen above, the map $e_I : I \longrightarrow Si(I)$, where $e_I(a) = \downarrow (a)$, is a dense embedding of $I$ in the Boolean algebra $Si(I)$ and the pair $(Si(I), e_I(I))$ is an LBA. Now Fact 2.2(b) implies that the map $(e_I)_{\upharpoonright I} : I \longrightarrow e_I(I)$ can be extended to a Boolean isomorphism $e_{(A,I)} : A \longrightarrow B_{Si(I)}(e_I(I))$. (Note that $A = I \cup I^*$ and $B_{Si(I)}(e_I(I)) = e_I(I) \cup (e_I(I))^*$, so that the map $e_{(A,I)}$ is defined by the following formula: for every $a \in I$, $e_{(A,I)}(a^*) = (e_I(a))^*$.) Set $I' = e_I(I)$ and $A' = e_{(A,I)}(A)$. Then the map $e_{(A,I)} : (A, I) \longrightarrow (A', I')$ is a $\mathsf{PLBA}$-isomorphism. Note that $(A', I') = (E^z \circ E^p)(A, I)$. Hence, $e_{(A,I)} : (A, I) \longrightarrow (E^z \circ E^p)(A, I)$ is a $\mathsf{PLBA}$-isomorphism. We will show that $e : Id_{\mathsf{PLBA}} \longrightarrow E^z \circ E^p$, defined by $e(A, I) = e_{(A,I)}$ for every $(A, I) \in |\mathsf{PLBA}|$, is the required natural isomorphism. Indeed, if $\varphi \in \mathsf{PLBA}((A, I), (B, J))$ and $\varphi' = (E^z \circ E^p)(\varphi)$ then we have to prove that $e_{(B,J)} \circ \varphi = \varphi' \circ e_{(A,I)}$. Clearly, for doing this it is enough to show that $e_J \circ (\varphi_{|I}) = (\varphi')_{|e_I(I)} \circ e_I$. Since this is obvious, we obtain that the functors $Id_{\mathsf{PLBA}}$ and $E^z \circ E^p$ are naturally isomorphic.

Let us proceed with the composition $E^p \circ E^z$. Let $(B, I)$ be a ZLBA. Then, by Corollary 2.9, the map $\Sigma_{(B,I)} : Si(I) \longrightarrow B$, where $\Sigma_{(B,I)}(J) = \bigvee_B\{a \mid a \in J\}$ for every $J \in Si(I)$, is a Boolean isomorphism. We will show that $s : Id_{\mathsf{PZLBA}} \longrightarrow E^p \circ E^z$, defined by $s(B, I) = (\Sigma_{(B,I)})^{-1}$ for every $(B, I) \in |\mathsf{PZLBA}|$, is the required natural isomorphism. Indeed, if $\varphi \in \mathsf{PZLBA}((A, I), (B, J))$ and $\varphi' = (E^p \circ E^z)(\varphi)$ then we have to prove that $\Sigma_{(B,J)} \circ \varphi' = \varphi \circ \Sigma_{(A,I)}$. Let $I_1 \in Si(I)$. Then $(\varphi \circ \Sigma_{(A,I)})(I_1) = \varphi(\bigvee_A I_1)$ and $(\Sigma_{(B,J)} \circ \varphi')(I_1) = \Sigma_{(B,J)}(\varphi'(I_1)) = \Sigma_{(B,J)}(\bigvee_{Si(J)}\{\downarrow (\varphi(a)) \mid a \in I_1\}) = \bigvee_B\{\Sigma_{(B,J)}(\downarrow (\varphi(a))) \mid a \in I_1\} = \bigvee_B \varphi(I_1)$. So, we have to prove that $\varphi(\bigvee_A I_1) = \bigvee_B \varphi(I_1)$. Set $b = \varphi(\bigvee_A I_1)$ and $c = \bigvee_B \varphi(I_1)$. Since $a \leq \bigvee_A I_1$, for every $a \in I_1$, we have that $\varphi(a) \leq b$ for every $a \in I_1$. Hence $c \leq b$. We will now prove that $b \leq c$. Since $J$ is dense in $B$, we get that $b = \bigvee_B\{d \in J \mid d \leq b\}$. By condition (LBA), for every $d \in J$ there exists $e_d \in I$ such that $d \leq \varphi(e_d)$. So, let $d \in J$ and $d \leq b$. Since $I_1 \vee \neg I_1 = I$, there exist $e_d^1 \in I_1$ and $e_d^2 \in \neg I_1$ such that $e_d = e_d^1 \vee e_d^2$. Now we obtain that $d \leq \varphi(e_d) \wedge b = \varphi(e_d \wedge \bigvee_A I_1) = \varphi(\bigvee_A \{e_d \wedge a \mid a \in I_1\}) = \varphi(\bigvee_A \{e_d^1 \wedge a \mid a \in I_1\}) = \varphi(e_d^1 \wedge \bigvee_A I_1) \leq \varphi(e_d^1) \leq c$. Thus $b = \bigvee_B \{d \in J \mid d \leq b\} \leq c$. So, the functors $Id_{\mathsf{PZLBA}}$ and $E^p \circ E^z$ are naturally isomorphic. □

**Corollary 2.12.** *There exists a bijective correspondence between the classes of all (up to* PLBA*-isomorphism) PLBAs, all (up to* ZLBA*-isomorphism) ZLBAs and all (up to homeomorphism) locally compact zero-dimensional Hausdorff spaces.*

We can even express Theorem 2.11 in a more simple form; in this way we will obtain a new proof of the Doctor Duality Theorem [7].

*Definition 2.5* ([7]). Let GenBoolAlg be the category whose objects are all generalized Boolean algebras and whose morphisms are all $\{0\}$-pseudolattice homomorphisms $\varphi : I \longrightarrow J$ between its objects satisfying condition (LBA) (i.e., $\forall b \in J\ \exists a \in I$ such that $b \leq \varphi(a)$).

**Theorem 2.13** ([7])**.** *The categories* PerfBoolSp *and* GenBoolAlg *are dually equivalent.*

Proof. By virtue of Theorem 2.11, it is enough to show that the categories PLBA and GenBoolAlg are equivalent.

Define a functor $E^l : \mathsf{PLBA} \longrightarrow \mathsf{GenBoolAlg}$ by setting $E^l(A, I) = I$, for every $(A, I) \in |\mathsf{PLBA}|$, and for every $\varphi \in \mathsf{PLBA}((A, I), (B, J))$, put $E^l(\varphi) = \varphi_{|I} : I \longrightarrow J$. Using Fact 2.2(a) and condition (PLBA), we get that $E^l$ is a well-defined functor.

Define a functor $E^g : \mathsf{GenBoolAlg} \longrightarrow \mathsf{PLBA}$ by setting

$$E^g(I) = (B_{Si(I)}(e_I(I)), e_I(I))$$

for every $I \in |\mathsf{GenBoolAlg}|$ (see Fact 2.1(b) and Notation 2.1 for the notation), and for every $\varphi \in \mathsf{GenBoolAlg}(I, J)$ define $E^g(\varphi) : B_{Si(I)}(e_I(I)) \longrightarrow B_{Si(J)}(e_J(J))$ to be the obvious extension of the map $\varphi_e : e_I(I) \longrightarrow e_J(J)$ defined by

$$\varphi_e(\downarrow(a)) = \downarrow(\varphi(a)).$$

Then, using Facts 2.1(a) and 2.2(b), it is easy to see that $E^g$ is a well-defined functor.

Finally, it is almost obvious that the compositions $E^g \circ E^l$ and $E^l \circ E^g$ are naturally isomorphic to the corresponding identity functors. So, the functors $\Theta_g^t = E^l \circ E^z \circ \Theta_p^t : \mathsf{PerfBoolSp} \longrightarrow \mathsf{GenBoolAlg}$ and $\Theta_g^a = \Theta_p^a \circ E^p \circ E^g :$

$\mathsf{GenBoolAlg} \longrightarrow \mathsf{PerfBoolSp}$ (see Theorems 2.11 and 2.10 for the notation) are the desired duality functors. Note that $\Theta_g^t(X) = CK(X)$, for every $X \in |\mathsf{PerfBoolSp}|$, and if $f \in \mathsf{PerfBoolSp}(X,Y)$ then $\varphi = \Theta_g^t(f) : CK(Y) \longrightarrow CK(X)$ is defined by the formula $\varphi(G) = f^{-1}(G)$, for every $G \in CK(Y)$. $\square$

The definition of the functor $\Theta_g^t$ is very simple but that of $\Theta_g^a$ is more complicated. We will recall the definition of the contravariant functor

$$\Theta_s^a : \mathsf{GenBoolAlg} \longrightarrow \mathsf{PerfBoolSp}$$

of H. P. Doctor [7] where the original Stone construction (see [18]) of the dual space of a GBA is used. Then the pair $(\Theta_g^t, \Theta_s^a)$ will be a duality between the categories $\mathsf{GenBoolAlg}$ and $\mathsf{PerfBoolSp}$. This will imply that $\Theta_g^a$ and $\Theta_s^a$ are naturally isomorphic; hence, we will obtain that the spaces $\Theta_g^a(I)$ and $\Theta_s^a(I)$ are homeomorphic for any GBA $I$ (the last assertion can be proved directly as well).

Let $I$ be a GBA. Set $\Theta_s^a(I)$ to be the set $X$ of all prime ideals of $I$ endowed with a topology $\mathcal{O}$ having as an open base the set $\{\gamma_I(b) \mid b \in I\}$ where, for every $b \in I$, $\gamma_I(b) = \{i \in X \mid b \notin i\}$ (see M. Stone [18]). Then, as it is proved in [18], $(X, \mathcal{O})$ is a Boolean space and $\gamma_I : I \longrightarrow CK(X, \mathcal{O})$, $b \mapsto \gamma_I(b)$, is a 0-pseudolattice isomorphism and hence, a $\mathsf{GenBoolAlg}$-isomorphism. If $\varphi \in \mathsf{GenBoolAlg}(I,J)$ then set $X = \Theta_s^a(I)$, $Y = \Theta_s^a(J)$ and define a map $f = f_\varphi : Y \longrightarrow X$ by the formula $f(j) = \varphi^{-1}(j)$, for every $j \in Y$. Since $\varphi$ is a $\mathsf{GenBoolAlg}$-morphism, we get that this definition is correct and, for every $b \in I$,

$$f_\varphi^{-1}(\gamma_I(b)) = \gamma_J(\varphi(b)). \tag{9}$$

This implies easily that $f$ is a perfect map. It becomes now clear that $\Theta_s^a$ is a contravariant functor, and also, it is not difficult to show that the pair $(\Theta_g^t, \Theta_s^a)$ is a duality between the categories $\mathsf{GenBoolAlg}$ and $\mathsf{PerfBoolSp}$ (see [7]).

**Corollary 2.14** (M. Stone [18])**.** *There exists a bijective correspondence between the class of all (up to 0-pseudolattice isomorphism) generalized Boolean algebras and all (up to homeomorphism) locally compact zero-dimensional Hausdorff spaces.*

Note that in [17], M. Stone proved that there exists a bijective correspondence between generalized Boolean algebras and Boolean rings (with or without unit).

In [5], a category $\mathsf{DHLC}$ dual to the category $\mathsf{HLC}$ of locally compact Hausdorff spaces and continuous maps was described. Obviously, the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$ are subcategories of the category $\mathsf{HLC}$. In the next theorem we will find the subcategories of the category $\mathsf{DHLC}$ which are dual to the categories $\mathsf{BoolSp}$ and $\mathsf{PerfBoolSp}$. All notions and notation used in it can be found in [5]. We do not recall them here because they are used only in this theorem.

*Definition 2.6.* Let $\mathsf{DZHLC}$ (resp., $\mathsf{DPZHLC}$) be the full subcategory of the category $\mathsf{DHLC}$ (resp., $\mathsf{D_1PHLC}$) having as objects all CLCAs $(A,\rho,\mathbb{B})$ such that if $a,b\in\mathbb{B}$ and $a\ll_\rho b$ then there exists $c\in\mathbb{B}$ with $c\ll_\rho c$ and $a\le c\le b$.

**Theorem 2.15.** *The following categories are dually equivalent:*

(a) $\mathsf{BoolSp}$ *and* $\mathsf{DZHLC}$*;*

(b) $\mathsf{PerfBoolSp}$ *and* $\mathsf{DPZHLC}$.

Proof. We will show that the contravariant functors $\Lambda^t_z=(\Lambda^t)_{|\mathsf{BoolSp}}$ and $\Lambda^a_z=(\Lambda^a)_{|\mathsf{DZHLC}}$ are the required duality functors (see [5, the text immediately after Theorem 3.12] for $\Lambda^t$ and $\Lambda^a$) for the first pair of categories. Indeed, if $X\in|\mathsf{BoolSp}|$ then

$$\Lambda^t(X)=(RC(X),\rho_X,CR(X))$$

(see [5, 2.3 and 2.15] for the notation) and, obviously, $(RC(X),\rho_X,CR(X))\in|\mathsf{DZHLC}|$. Conversely, if $(A,\rho,\mathbb{B})\in|\mathsf{DZHLC}|$ then $X=\Lambda^a(A,\rho,\mathbb{B})$ is a locally compact Hausdorff space. For proving that $X$ is a zero-dimensional space, let $x\in X$ and $U$ be an open neighborhood of $x$. Then there exist open sets $V,W$ in $X$ such that $x\in V\subseteq \mathrm{cl}(V)\subseteq W\subseteq\mathrm{cl}(W)\subseteq U$ and $\mathrm{cl}(V)$, $\mathrm{cl}(W)$ are compacts. Then there exist $a,b\in\mathbb{B}$ such that $\lambda^g_A(a)=\mathrm{cl}(V)$ and $\lambda^g_A(b)=\mathrm{cl}(W)$ (see [5, (16)] for the notation $\lambda^g_A$) . Obviously, $a\ll_\rho b$. Thus, there exists $c\in\mathbb{B}$ such that $c\ll_\rho c$ and $a\le c\le b$. Then $F=\lambda^g_A(c)$ is a clopen subset of $X$ and $x\in F\subseteq U$. So, $X$ is zero-dimensional. Now, all follows from [5, Theorem 3.12].

The restrictions of the obtained above duality functors to the categories of the second pair give, according to [3, Theorem 3.10], the desired second duality. □

## 3. Some other Stone-type Duality Theorems

Recall that a homomorphism $\varphi$ between two Boolean algebras is called *complete* if it preserves all joins (and, consequently, all meets) that happen to exist; this means that if $\{a_i\}$ is a family of elements in the domain of $\varphi$ with join $a$, then the family $\{\varphi(a_i)\}$ has a join and that join is equal to $\varphi(a)$.

*Definition 3.1.* We will denote by $\mathsf{SkelBoolSp}$ the category of all zero-dimensional locally compact Hausdorff spaces and all skeletal maps between them (see 1.5 for the definition of a skeletal map).

Let $\mathsf{SZLBA}$ be the cofull subcategory of the category $\mathsf{ZLBA}$ whose morphisms are, in addition, complete homomorphisms.

**Theorem 3.1.** *The categories* $\mathsf{SkelBoolSp}$ *and* $\mathsf{SZLBA}$ *are dually equivalent.*

Proof. Having in mind Theorem 2.7, it is enough to prove that if $f$ is a morphism of the category $\mathsf{SkelBoolSp}$ then $\Theta_d^t(f)$ is complete, and if $\varphi$ is a $\mathsf{SZLBA}$-morphism then $\Theta_d^a(\varphi)$ is a skeletal map.

So, let $f \in \mathsf{SkelBoolSp}(X,Y)$ and $\varphi = \Theta_d^t(f)$. Then

$$\varphi : (CO(Y), CK(Y)) \longrightarrow (CO(X), CK(X))$$

and $\varphi(G) = f^{-1}(G)$, for all $G \in CO(Y)$. Let $\{G_\gamma \mid \gamma \in \Gamma\} \subseteq CO(Y)$ and let this family have a join $G$ in $CO(Y)$. Set $W = \bigcup\{G_\gamma \mid \gamma \in \Gamma\}$. Since $Y$ is zero-dimensional, we get easily that $G = \mathrm{cl}(W)$. Thus $\varphi(G) \supseteq \mathrm{cl}(\bigcup\{\varphi(G_\gamma) \mid \gamma \in \Gamma\}) = F$. Let $x \in f^{-1}(G)(= \varphi(G))$. Then $f(x) \in G$ and there exists a neighborhood $U$ of $x$ such that $f(U) \subseteq G$. Suppose that $x \notin F$. Then there exists a neighborhood $V$ of $x$ such that $V \subseteq U$ and $V \cap f^{-1}(G_\gamma) = \emptyset$ for all $\gamma \in \Gamma$. Thus $f(V) \cap W = \emptyset$. Then $\mathrm{cl}(f(V)) \cap W = \emptyset$. Since $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(f(U)) \subseteq G = \mathrm{cl}(W)$, we get that $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(W) \setminus W (= \mathrm{Fr}(W))$. This leads to a contradiction because $f$ is skeletal and thus $\mathrm{int}(\mathrm{cl}(f(V))) \neq \emptyset$ (see 1.5). So, $\varphi(G) = f^{-1}(G) = F$. Since $\varphi(G)$ is clopen, we get that $\varphi(G)$ is the join of the family $\{\varphi(G_\gamma) \mid \gamma \in \Gamma\}$ in $CO(X)$. Therefore, $\varphi$ is complete.

Let now $\varphi \in \mathsf{SZLBA}((A,I),(B,J))$ and $f = \Theta_d^a(\varphi)$. Set $X = \Theta_d^a(A,I)$ and $Y = \Theta_d^a(B,J)$. Then $f : Y \longrightarrow X$. Since $CK(Y)$ is an open base of $Y$, for proving that $f$ is skeletal it is enough to show that for every $G \in CK(Y) \setminus \{\emptyset\}$, $\mathrm{int}(f(G)) \neq \emptyset$. So, let $G \in CK(Y) \setminus \{\emptyset\}$. Then there exists $b \in J \setminus \{0\}$ such that $G = \lambda_{(B,J)}(b)$. Suppose that $\bigwedge\{c \in A \mid b \leq \varphi(c)\} = 0$. Then, using the completeness of $\varphi$, we get that $0 = \varphi(0) = \bigwedge\{\varphi(c) \mid c \in A, b \leq \varphi(c)\} \geq b$. Since $b \neq 0$, we get a contradiction. Hence there exists $a \in A \setminus \{0\}$ such that $a \leq c$ for all $c \in A$ for which $b \leq \varphi(c)$. We will prove that $\lambda_{(A,I)}(a) \subseteq f(\lambda_{(B,J)}(b))(= f(G))$. This will imply that $\mathrm{int}(f(G)) \neq \emptyset$. Let $u \in \lambda_{(A,I)}(a)$. Then $a \in u$. Suppose that there exists $c \in u$ such that $b \wedge \varphi(c) = 0$. Then $b \leq \varphi(c^*)$. Thus $a \leq c^*$, i.e., $a \wedge c = 0$. Since $a, c \in u$, we get a contradiction. Therefore, the set $\{b\} \cup \varphi(u)$ is a filter-base. Hence there exists an ultrafilter $v$ in $B$ such that $\{b\} \cup \varphi(u) \subseteq v$. Then $b \in v$ and $u \subseteq \varphi^{-1}(v)$. Thus $u = \varphi^{-1}(v)$, i.e., $f(v) = u$. So, $u \in f(\lambda_{(B,J)}(b))$. $\square$

*Remarks 3.1.* Note that in the definition of the category $\mathsf{SZLBA}$ the requirement that the morphisms $\varphi : (A,I) \longrightarrow (B,J)$ are complete can be replaced by the following condition:

(SkeZLBA) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, the proof of the above theorem shows the sufficiency of this condition and its necessity can be established as follows. Let $f \in \mathsf{SkelBoolSp}(X, Y)$ and $\varphi = \Theta^t(f)$. Then $\varphi : (CO(Y), CK(Y)) \longrightarrow (CO(X), CK(X))$ and $\varphi(G) = f^{-1}(G)$, for all $G \in CO(Y)$. Let $F \in CK(X) \setminus \{\emptyset\}$. Then $\mathrm{int}(f(F)) \neq \emptyset$. Hence there exists $G \in CK(Y) \setminus \{\emptyset\}$ such that $G \subseteq \mathrm{int}(f(F))$. Let $H \in CO(Y)$ and $F \subseteq f^{-1}(H)$. Then $G \subseteq \mathrm{int}(f(F)) \subseteq f(F) \subseteq H$. So, condition (SkeZLBA) is satisfied.

Moreover, condition (SkeZLBA) can be replaced by the following one:

(CEP) For every $b \in B \setminus \{0\}$ there exists $a \in A \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, if $b \in B \setminus \{0\}$ then, by the density of $J$ in $B$, there exists $b_1 \in I \setminus \{0\}$ such that $b_1 \leq b$. Now, applying (SkeZLBA) for $b_1$, we get an $a \in I \setminus \{0\}$ which satisfies also the requirements of (CEP) about $b$. Conversely, if $b \in J \setminus \{0\}$ then, by (CEP), there exists $a \in A \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$; but, by condition (LBA) (see Definition 2.1), there exists $a_1 \in I$ such that $b \leq \varphi(a_1)$; thus $a \leq a_1$; since $I$ is an ideal, we get that $a \in I$; so, condition (SkeZLBA) is satisfied.

The assertion (c) of the next corollary is a zero-dimensional analogue of the Fedorchuk Duality Theorem [10].

**Corollary 3.2.** (a) *Let $f$ be a* $\mathsf{PerfBoolSp}$*-morphism. Then $f$ is a quasi-open map iff $\Theta^t(f)$ is complete. In particular, if $f$ is a* $\mathsf{Stone}$*-morphism then $f$ is a quasi-open map iff $S^t(f)$ is complete.*

(b) *The cofull subcategory* $\mathsf{QOpenPerfBoolSp}$ *of the category* $\mathsf{PerfBoolSp}$ *(see Theorem 2.10) whose morphisms are, in addition, quasi-open maps, is dually equivalent to the cofull subcategory* $\mathsf{QPZLBA}$ *of the category* $\mathsf{PZLBA}$ *whose morphisms are, in addition, complete homomorphisms;*

(c) *The category* $\mathsf{QOpenStone}$ *of compact zero-dimensional Hausdorff spaces and quasi-open maps is dually equivalent to the category* $\mathsf{CBool}$ *of Boolean algebras and complete Boolean homomorphisms.*

PROOF. The assertion (a) follows from the proof of Theorem 3.1 and [4, Corollary 2.5]. The assertions (b) and (c) follow from (a) and Theorem 3.1. $\square$

The last corollary together with Fedorchuk's Duality Theorem [10] imply the following assertion in which the equivalence $(a) \iff (b)$ is a special case of a much more general theorem due to MONK [14].

**Corollary 3.3.** *Let $\varphi \in \mathsf{BoolAlg}(A, B)$ and $A', B'$ be minimal completions of $A$ and $B$ respectively. We can suppose that $A \subseteq A'$ and $B \subseteq B'$. Then the following conditions are equivalent:*

(a) *$\varphi$ can be extended to a complete homomorphism $\psi : A' \longrightarrow B'$;*

(b) *$\varphi$ is a complete homomorphism;*

(c) *$\varphi$ satisfies condition (CEP) (see Remark 3.1 above).*

Proof. (a)$\Rightarrow$(b) This is obvious.

(b)$\Rightarrow$(c) This was already established in the proof of Theorem 3.1 (see also Remark 3.1).

(c)$\Rightarrow$(a) Obviously, $\varphi \in \mathsf{ZLBA}((A, A), (B, B))$. Then, by Remark 3.1 and Theorem 3.1, condition (CEP) implies that $f = \Theta^a_d(\varphi)(= S^a(\varphi))$ is a skeletal map. Since $f$ is closed, we get that $f$ is a quasi-open map between $Y = \Theta^a_d(B, B)(= S^a(B))$ and $X = \Theta^a_d(A, A)(= S^a(A))$. Now, by Fedorchuk's Duality Theorem [10], the map $\psi : RC(X) \longrightarrow RC(Y)$, $F \mapsto \mathrm{cl}(f^{-1}(\mathrm{int}(F)))$, is a complete homomorphism. Obviously, for every $F \in CO(X)$, $\psi(F) = f^{-1}(F) = \varphi'(F)$ (here $\varphi' = \Theta^t_d(\Theta^a_d(\varphi))$). Then the existence of a natural isomorphism between the composition $\Theta^t_d \circ \Theta^a_d$ and the identity functor (see Theorem 2.7), and the fact that $RC(X)$ and $RC(Y)$ are minimal completions of, respectively, $A$ and $B$, imply our assertion. $\square$

Now, using Theorem 2.13, we will present in a simpler form the result established in Corollary 3.2(b).

**Theorem 3.4.** *The category $\mathsf{QOpenPerfBoolSp}$ is dually equivalent to the cofull subcategory $\mathsf{QGBA}$ of the category $\mathsf{GenBoolAlg}$ whose morphisms, in addition, preserve all meets that happen to exist.*

Proof. Having in mind Theorem 2.13 and Corollary 3.2(b), it is enough to show that the functor $E^a$ (see Theorem 2.13) maps $\mathsf{QPZLBA}$ to $\mathsf{QGBA}$ and the functor $E^b$ (see again Theorem 2.13) maps $\mathsf{QGBA}$ to $\mathsf{QPZLBA}$ because with this we will obtain that the categories $\mathsf{QPZLBA}$ and $\mathsf{QGBA}$ are equivalent. Obviously, if $\varphi' : (A, I) \longrightarrow (B, J)$ is a $\mathsf{QPZLBA}$-morphism then $\varphi = E^a(\varphi') = (\varphi')_{|I} : I \longrightarrow J$ preserves all meets in $I$ that happen to exist (indeed, since $I$ is an ideal of $A$, every meet in $I$ of elements of $I$ is also a meet in $A$). Conversely, let $\varphi : I \longrightarrow J$ be a $\mathsf{QGBA}$-morphism. We will show that $\varphi$ satisfies the following condition:

(QGBPL) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in I)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, let $b \in J \setminus \{0\}$. Suppose that $\bigwedge_I \{c \in I \mid b \leq \varphi(c)\} = 0$. Then, using the completeness of $\varphi$, we get that $0 = \varphi(0) = \bigwedge\{\varphi(c) \mid c \in I, b \leq \varphi(c)\} \geq b$. Since

$b \neq 0$, we get a contradiction. Hence there exists $a \in I \setminus \{0\}$ such that $a \leq c$ for all $c \in I$ for which $b \leq \varphi(c)$.

Let $\varphi' = E^b(\varphi)$. We will show that the map $\varphi'$ satisfies condition (SkeZLBA). We have that $\varphi' : (SI(I), e_I(I)) \longrightarrow (Si(J), e_J(J))$. Let $J_1 \in e_J(J) \setminus \{0\}$. Then there exists $b \in J \setminus \{0\}$ such that $J_1 = \downarrow(b)$. By (QGBPL), there exists $a \in I \setminus \{0\}$ such that $(\forall c \in I)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$. Let $I_1 \in Si(I)$ and $J_1 \subseteq \varphi'(I_1)$. Then, by the definition of the map $\varphi'$ (see Theorem 2.13), we have that $\downarrow(b) \subseteq \bigcup \{\downarrow(\varphi(c)) \mid c \in I_1\}$. Thus there exists $c \in I_1$ such that $b \leq \varphi(c)$. Since $c \in I$, we get that $a \leq c$. Therefore, $\downarrow(a) \subseteq I_1$. So, the map $\varphi'$ satisfies condition (SkeZLBA). Now Remark 3.1 implies that $\varphi'$ is a complete homomorphism. Thus $\varphi'$ is a $\mathsf{QPZLBA}$-morphism. $\square$

*Remark 3.2.* The proof of Theorem 3.4 shows that in the definition of the category $\mathsf{QPZLBA}$ the requirement that its morphisms $\varphi : I \longrightarrow J$ preserve all meets that happen to exist can be replaced by the condition (QGBPL) introduced above.

**Theorem 3.5.** (a) *Let $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $\Theta^t(Y) = (B, J)$. Then the map $f$ is open iff there exists a map $\psi : I \longrightarrow J$ which satisfies the following conditions:*

(OZL1) *For every $b \in J$ and every $a \in I$, $(a \wedge \varphi(b) = 0) \rightarrow (\psi(a) \wedge b = 0)$, and*

(OZL2) *For every $a \in I$, $\varphi(\psi(a)) \geq a$*

*(such a map $\psi$ will be called a* lower pre-adjoint *of $\varphi$).*

(b) *The cofull subcategory $\mathsf{OpenBoolSp}$ of the category $\mathsf{BoolSp}$ whose morphisms are open maps is dually equivalent to the cofull subcategory $\mathsf{OZLBA}$ of the category $\mathsf{ZLBA}$ whose morphisms have, in addition, lower pre-adjoints.*

Proof. (a) Let $f \in \mathsf{BoolSp}(X, Y)$ be an open map. For every $F \in CK(X)$ $(= I)$ set $\psi(F) = f(F)$. Then, clearly, $\psi(F) \in CK(Y)(= J)$ and $\varphi(\psi(F)) = f^{-1}(f(F)) \supseteq F$. Hence, condition (OZL2) is satisfied. Let $F \in CK(X)$, $G \in CK(Y)$ and $F \wedge \varphi(G) = 0$. Then $F \cap f^{-1}(G) = \emptyset$. Thus $f(F) \cap G = \emptyset$, i.e., $\psi(F) \wedge G = 0$. Therefore, condition (OZL1) is satisfied as well.

Let now $\varphi$ has a lower pre-adjoint. We will show that $f' = \Theta^a(\varphi)$ is an open map. This will imply that $f$ is open. Let $X' = \Theta^a(\Theta^t(X))$ and $Y' = \Theta^a(\Theta^t(Y))$. Since $\lambda_{(A,I)}(I)$ is an open base of $X'$, it is enough to show that $f'(\lambda_{(A,I)}(a))$ is an open set, for every $a \in I$. So, let $a \in I$. We will prove that $f'(\lambda_{(A,I)}(a)) = \lambda_{(B,J)}(\psi(a))$. Let $u \in \lambda_{(A,I)}(a)$. Then $a \in u$. Let $v = f'(u)$, i.e., $v = \varphi^{-1}(u)$. By (OZL2), $\varphi(\psi(a)) \geq a$ and hence $\varphi(\psi(a)) \in u$. Thus $\psi(a) \in \varphi^{-1}(u) = v$, i.e., $f'(u) \in \lambda_{(B,J)}(\psi(a))$. Therefore $f'(\lambda_{(A,I)}(a)) \subseteq \lambda_{(B,J)}(\psi(a))$. Conversely, let $v \in \lambda_{(B,J)}(\psi(a))$. Then $\psi(a) \in v$. Suppose that there exists $b \in v$ such that $a \wedge \varphi(b) = 0$. Since $v$ is a bounded ultrafilter, there exists $b_0 \in v \cap J$. Then $b_1 = b \wedge b_0 \in J \cap v$ and $a \wedge \varphi(b_1) = 0$. Now, condition (OZL1) implies that $\psi(a) \wedge b_1 = 0$, which is a contradiction. Hence, the set $\{a\} \cup \varphi(v)$ is a filter-base. Thus there exists an ultrafilter $u \supseteq \{a\} \cup \varphi(v)$. Then $a \in u \cap I$ and $v \subseteq \varphi^{-1}(u)$. Therefore, $v = \varphi^{-1}(u) = f(u)$. This shows that $f'(\lambda_{(A,I)}(a)) \supseteq \lambda_{(B,J)}(\psi(a))$. Hence, $f'$ is an open map.

(b) It follows from (a) and Theorem 2.7. □

*Remarks 3.3.* Note that condition (OZL2) implies condition (LBA). Indeed, in the notation of Theorem 3.5, if $a \in I$ then $b = \psi(a) \in J$ and $\varphi(b) \geq a$. Further, condition (OZL1) implies that (again in the notation of Theorem 3.5) $\psi(0) = 0$. Indeed, $0 \wedge \varphi(\psi(0)) = 0$ implies that $\psi(0) \wedge \psi(0) = 0$, i.e., that $\psi(0) = 0$.

**Theorem 3.6.** (a) *Let* $f \in \mathsf{PerfBoolSp}(X,Y)$, $(A,I) = \Theta^t(X)$, $(B,J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$. *Then the map* $f$ *is open iff* $\varphi : B \longrightarrow A$ *has a lower adjoint* $\psi : A \longrightarrow B$.

(b) *The cofull subcategory* OpenPerfBoolSp *of the category* PerfBoolSp *whose morphisms are, in addition, open maps is dually equivalent to the cofull subcategory* OPZLBA *of the category* PZLBA *whose morphisms have, in addition, lower adjoints.*

Proof. (a) Let $f \in \mathsf{PerfBoolSp}(X,Y)$ and $f$ is open. Then set $\psi(F) = f(F)$, for every $F \in CO(X)(= A)$. Then, since $f$ is open and closed map, $\psi : A \longrightarrow B(= CO(Y))$. Obviously, $\varphi(\psi(F)) = f^{-1}(f(F)) \supseteq F$ for every $F \in CO(X)$ and $\psi(\varphi(G)) = f(f^{-1}(G)) \subseteq G$ for every $G \in CO(Y)$. Hence $\psi$ is a lower adjoint of $\varphi$. Conversely, let $\varphi$ has a lower adjoint $\psi$. Then $\psi(I) \subseteq J$. Indeed, let $a \in I$. Then, by condition (LBA), there exists $b \in J$ such that $a \leq \varphi(b)$. Then $\psi(a) \leq \psi(\varphi(b)) \leq b \in J$. Thus, $\psi(a) \in J$. Further, condition (OZL2) is clearly fulfilled as well as condition (OZL1) (see, e.g., [4, Fact 1.22(a)]). So, $(\psi)_{|I}$ is a lower pre-adjoint of $\varphi$. Then, by Theorem 3.5(a), $f : X \longrightarrow Y$ is an open map.

(b) It follows from (a) and Theorem 2.10. □

**Corollary 3.7.** (a) *Let* $f \in \mathsf{Stone}(X,Y)$, $\varphi = S^t(f)$, $A = S^t(X)$ *and* $B = S^t(Y)$. *Then the map* $f$ *is open iff* $\varphi : B \longrightarrow A$ *has a lower adjoint* $\psi : A \longrightarrow B$.

(b) *The category* OpenStone *of compact zero-dimensional Hausdorff spaces and open maps is dually equivalent to the category* OBool *of Boolean algebras and Boolean homomorphisms having lower adjoints.*

Proof. It follows immediately from Theorem 3.6. □

*Definition 3.2.* Let $\varphi \in \mathsf{GenBoolAlg}(J, I)$. If $\psi : I \longrightarrow J$ is a map which satisfies conditions (OZL1) and (OZL2) (see Theorem 3.5) then $\psi$ is called a *lower preadjoint* of $\varphi$.

Let $\mathsf{OGBA}$ be the cofull subcategory of the category $\mathsf{GenBoolAlg}$ whose morphisms have, in addition, lower preadjoints.

**Corollary 3.8.** *The categories* $\mathsf{OpenPerfBoolSp}$ *and* $\mathsf{OGBA}$ *are dually equivalent.*

Proof. It follows from Theorems 2.13, 3.5 and 3.6. Indeed, it is enough to show that the categories $\mathsf{OGBA}$ and $\mathsf{OPZLBA}$ are equivalent. From the proof of Theorem 3.6, it follows that if $\varphi'$ is an $\mathsf{OPZLBA}$-morphism then $\varphi = E^a(\varphi')$ has a lower preadjoint. Conversely, if $\varphi$ is an $\mathsf{OGBA}$-morphism then $\varphi' = E^b(\varphi)$ can be regarded as an extension of $\varphi$. This implies immediately that $\varphi'$ has a lower pre-adjoint. Now, Theorem 3.5 implies that $f = \Theta^a(\varphi')$ is an open map. Thus, by Theorem 3.6, $\varphi'$ has a lower adjoint. $\square$

## 4. Characterization of the dual maps of embeddings, surjections and injections

In this section we will investigate the following problem: characterize the dual morphisms of the injective and surjective morphisms of the category $\mathsf{BoolSp}$ and its subcategories $\mathsf{PerfBoolSp}$, $\mathsf{OpenBoolSp}$. Such a problem was regarded by M. Stone in [18] for surjective continuous maps and for closed embeddings (i.e., for injective morphisms of the category $\mathsf{PerfBoolSp}$). An analogous problem will be investigated for the homeomorphic embeddings and dense embeddings.

We start with a simple observation.

**Proposition 4.1.** *Let* $f \in \mathsf{BoolSp}(X, Y)$, $(A, I) = \Theta^t(X)$, $(B, J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$. *Then* $\varphi$ *is an injection* $\iff$ $\varphi_{|J}$ *is an injection* $\iff$ $\mathrm{cl}_Y(f(X)) = Y$.

Proof. We have that $\varphi : CO(Y) \longrightarrow CO(X)$, $(A, I) = (CO(X), CK(X))$ and $(B, J) = (CO(Y), CK(Y))$.

Obviously, if $\varphi$ is an injection then $\varphi_{|J}$ is an injection.

Let $\varphi_{|J}$ be an injection, $G \in CK(Y)$ and $G \neq \emptyset$. Then $\varphi(G) \neq \emptyset$, i.e., $f^{-1}(G) \neq \emptyset$. This means that $f(X) \cap G \neq \emptyset$. Thus $\mathrm{cl}(f(X)) = Y$.

Finally, let $\mathrm{cl}(f(X)) = Y$, $G \in CO(Y)$ and $G \neq \emptyset$. Then $G \cap f(X) \neq \emptyset$ and thus $\varphi(G) = f^{-1}(G) \neq \emptyset$. So, $\varphi$ is an injection. $\square$

**Proposition 4.2.** *Let* $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$, $(B, J) = \Theta^t(Y)$ *and* $\varphi(B) \supseteq I$ *(or* $\varphi(J) \supseteq I$*). Then* $f$ *is an injection.*

Proof. Suppose that there exist $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then there exists $U \in CK(X)$ such that $x \in U \subseteq X \setminus \{y\}$. There exists $V \in CO(Y)$ (or, respectively, $V \in CK(Y)$) with $\varphi(V) = U$, i.e., $f^{-1}(V) = U$. Then $f(U) = f(X) \cap V$ and $f^{-1}(f(U)) = f^{-1}(V) = U$. Since $f(y) = f(x) \in f(U)$, we get that $y \in U$, a contradiction. Thus, $f$ is an injection. $\square$

**Theorem 4.3.** *Let $f \in \mathsf{BoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ and $(B,J) = \Theta^t(Y)$. Then $f$ is an injection iff $\varphi : (B,J) \longrightarrow (A,I)$ satisfies the following condition:*

(InZLC) *For any $a, b \in I$ such that $a \wedge b = 0$ there exists $a', b' \in J$ with $a' \wedge b' = 0$, $\varphi(a') \geq a$ and $\varphi(b') \geq b$.*

Proof. Let $f : X \longrightarrow Y$ be an injection. We have that

$$\varphi : CO(Y) \longrightarrow CO(X), \quad G \mapsto f^{-1}(G),$$

$(A,I) = (CO(X), CK(X))$ and $(B,J) = (CO(Y), CK(Y))$. Let $F_1, F_2 \in CK(X)$ and $F_1 \cap F_2 = \emptyset$. Since $f$ is a continuous injection, we get that $f(F_1)$ and $f(F_2)$ are disjoint compact subsets of $Y$. Using the fact that $CK(Y)$ is a base of $Y$, we get that there exist disjoint $G_1, G_2 \in CK(Y)$ such that $f(F_i) \subseteq G_i$, $i = 1, 2$. Then $F_i \subseteq f^{-1}(G_i)$, i.e., $F_i \subseteq \varphi(G_i)$, $i = 1, 2$. Hence, $\varphi$ satisfies condition (InZLC).

Let now $\varphi$ satisfies condition (InZLC). We will prove that $f$ is an injection. Let $x, y \in X$ and $x \neq y$. Then there exist disjoint $F_x, F_y \in CK(X)$ such that $x \in F_x$ and $y \in F_y$. Now, by condition (InZLC), there exist $G_x, G_y \in CK(Y)$ such that $G_x \cap G_y = \emptyset$, $f^{-1}(G_x) \supseteq F_x$ and $f^{-1}(G_y) \supseteq F_y$. Then $f(x) \in G_x$ and $f(y) \in G_y$. Thus $f(x) \neq f(y)$. $\square$

**Corollary 4.4.** *The cofull subcategory $\mathsf{InjBoolSp}$ of the category $\mathsf{BoolSp}$ whose morphisms are, in addition, injective maps, is dually equivalent to the cofull subcategory $\mathsf{DInjBoolSp}$ of the category $\mathsf{ZLBA}$ whose morphism satisfy, in addition, condition (InZLC).*

Proof. It follows from Theorems 4.3 and 2.7. $\square$

*In the sequel, we will not formulate corollaries like that because they follow directly from the respective characterization of injectivity or surjectivity and the corresponding duality theorems.*

*Remark 4.1.* Let us show how Theorem 4.3 implies Proposition 4.2. Let $\varphi(B) \supseteq I$. Then $\varphi(J) \supseteq I$. Indeed, let $a \in I$; then, by condition (LBA), there exists $b_1 \in J$ such that $\varphi(b_1) \geq a$; since there exists $b_2 \in B$ with $\varphi(b_2) = a$, we get that $\varphi(b_1 \wedge b_2) = a$ and $b_1 \wedge b_2 \in J$. Hence, $\varphi(J) \supseteq I$. Let now $a, b \in I$

and $a \wedge b = 0$. There exist $a_1, b_1 \in J$ such that $\varphi(a_1) = a$ and $\varphi(b_1) = b$. Then $\varphi(a_1 \wedge b_1^*) = a \wedge b^* = a$, $a_1 \wedge b_1^* \in J$ and $(a_1 \wedge b_1^*) \wedge b_1 = 0$. Therefore, $\varphi$ satisfies condition (InZLC).

In the next theorem we will assume that the ideals and prime ideals could be non-proper.

**Theorem 4.5.** *Let $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then the following conditions are equivalent:*

(a) *$f$ is a surjection;*

(b) *$\varphi : B \longrightarrow A$ is an injection and for every bounded ultrafilter $v$ in $(B, J)$ there exists $a \in I$ such that $a \wedge \varphi(v) \neq 0$ (i.e., $a \wedge \varphi(b) \neq 0$ for any $b \in v$);*

(c) *$\varphi : B \longrightarrow A$ is an injection and for every prime ideal $J_1$ of $J$, we have that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$ implies $J_1 = J$ (where $I_{\varphi(b)} = \{a \in I \mid a \leq \varphi(b)\}$);*

(d) *$\varphi : B \longrightarrow A$ is an injection and for every ideal $J_1$ of $J$, $[(\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I) \rightarrow (J_1 = J)]$.*

Proof. (a)⇒(b) Let $f(X) = Y$. Then, by Proposition 4.1, $\varphi$ is an injection. Further, by (7), the bounded ultrafilters in $(B, J) = (CO(Y), CK(Y))$ are of the form $u_y^C$ (see Notation 2.2 for this notation) and analogously for $(A, I)$. So, let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. This implies that $\varphi(u_y^C) \subseteq u_x^C$. There exists $F \in CK(X) \cap u_x^C$. Then $F \cap f^{-1}(G) \neq \emptyset$, for every $G \in u_y^C$, i.e., $F \wedge \varphi(u_y^C) \neq 0$.

(b)⇒(c) Let $J_1$ be a prime ideal of $J$. Let $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. Suppose that $J_1 \neq J$. Then $v_1 = \{b \in B \mid b \wedge (J \setminus J_1) \neq 0\}$ is a bounded ultrafilter in $(B, J)$ and $v_1 \cap J = J \setminus J_1$. This follows from the more general result [5, Proposition 3.6] but, for completeness of our exposition, we will supply it with a new proof. So, it is clear that $J \setminus J_1$ is a filter in $J$, and hence $J \setminus J_1 \subseteq v_1$; also, $J \setminus J_1 \neq \emptyset$ and $v_1$ is an upper set. We will show that $v_1 \cap J = J \setminus J_1$. Since $J \setminus J_1 \subseteq v_1$, it is enough to prove that $v_1 \cap J_1 = \emptyset$. Let $d \in J_1$. There exists $e \in J \setminus J_1$. If $d^* \wedge e \in J_1$ then $e = (e \wedge d) \vee (e \wedge d^*) \in J_1$, a contradiction. Hence $c = d^* \wedge e \in J \setminus J_1$ and $d \wedge c = 0$. Therefore, $d \notin v_1$. So, $v_1 \cap J_1 = \emptyset$ and thus $v_1 \cap J = J \setminus J_1$. Further, if $b_1 \in v_1$ and $b_2 \in J \setminus J_1$ then $b_1 \wedge b_2 \in J \setminus J_1$. Indeed, if $b = b_1 \wedge b_2 \in J_1$ then $b \notin v_1$ and hence there exists $c \in J \setminus J_1$ such that $b \wedge c = 0$, i.e., $b_1 \wedge (b_2 \wedge c) = 0$; since $c \wedge b_2 \in J \setminus J_1$, we get a contradiction. Let now $b_1, b_2 \in v_1$. We will show that $b_1 \wedge b_2 \in v_1$ and this will imply that $v_1$ is a filter in $B$. Let $c \in J \setminus J_1$. Then $b_1 \wedge c, b_2 \wedge c \in J \setminus J_1$ and thus $(b_1 \wedge c) \wedge (b_2 \wedge c) \in J \setminus J_1$; hence $(b_1 \wedge b_2) \wedge c \neq 0$. Therefore, $b_1 \wedge b_2 \in v_1$. Finally, for showing that the filter $v_1$ is an ultrafilter, suppose that there exists $b \in B$ such that $b \notin v_1$ and $b^* \notin v_1$. Then there exist

$c, d \in J \setminus J_1$ such that $b \wedge c = 0$ and $b^* \wedge d = 0$. Since $c \wedge d \in J \setminus J_1$, we have that $c \wedge d \neq 0$. On the other hand, $d \leq b$ and hence $c \wedge d \leq c \wedge b = 0$, i.e., $c \wedge d = 0$, a contradiction. Therefore, $v_1$ is a bounded ultrafilter in $(B, J)$ and $v_1 \cap J = J \setminus J_1$. By (b), there exists $a \in I$ such that $a \wedge \varphi(v_1) \neq 0$. Since $a \in I$ and $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$, there exist $b_1, \ldots, b_k \in J_1$ and $a_1, \ldots, a_k \in I$ (where $k \in \mathbb{N}^+$) such that $a = \bigvee\{a_i \mid i = 1, \ldots, k\}$ and $a_i \leq \varphi(b_i)$, $i = 1, \ldots, k$. Set $b = \bigvee\{b_i \mid i = 1, \ldots, k\}$. Then $a \leq \varphi(b)$ and $b \in J_1$. Since $\varphi$ is an injection, we have that $\varphi(v_1 \cap J) = \varphi(J \setminus J_1) = \varphi(J) \setminus \varphi(J_1)$. Thus $\varphi(b) \notin \varphi(v_1 \cap J)$ (because $b \in J_1$). Since $a \leq \varphi(b)$, we get that $\varphi(b) \wedge \varphi(v_1) \neq 0$. The injectivity of $\varphi$ implies that $b \wedge v_1 \neq 0$. Thus $b \in v_1 \cap J_1$, a contradiction. Hence, $J_1 = J$.

(c)⇒(a) Suppose that $f(X) \neq Y$. Then there exists $y \in Y \setminus f(X)$. Set $U = Y \setminus \{y\}$. Thus $f(X) \subseteq U$. Set $J_1 = \{G \in CK(Y) \mid G \subseteq U\}$. Then $J_1$ is a prime ideal of $J(= CK(Y))$. (Indeed, if $G_1, G_2 \in CK(Y)$ and $y \notin G_1 \cap G_2$ then either $y \notin G_1$ or $y \notin G_2$; hence, $G_1 \in J_1$ or $G_2 \in J_1$.) Obviously, $J_1 \neq J$. We will prove that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$, which, by (c), will lead to a contradiction. So, let $F \in CK(X)$. Then $f(F) \subseteq U$. Since $f(F)$ is compact, there exists $G \in CK(Y)$ such that $f(F) \subseteq G \subseteq U$. Then $G \in J_1$ and $F \subseteq f^{-1}(G) = \varphi(G)$. Thus $F \in I_{\varphi(G)}$. Therefore, $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. So, $f(X) = Y$.

(a)⇒(d) Let $f(X) = Y$. Then, by Proposition 4.1, $\varphi$ is an injection. Let $J_1$ be an ideal of $J$ such that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. Suppose that $J_1 \neq J$. Set $U = \bigcup\{G \mid G \in J_1\}$. Then $U \neq Y$. (Indeed, if $U = Y$ then every $H \in CK(Y)$ $(= J)$ will be covered by a finite number of elements of $J_1$; since $J_1$ is an ideal, we will get that $H \in J_1$.) Since $f$ is a surjection, we get that $V = f^{-1}(U) \neq X$. Set $I_V = \{F \in I \mid F \subseteq V\}$. Then, obviously, $I_V$ is a proper ideal of $I$. Let $G \in J_1$ and $F \in I_{\varphi(G)}$. Then $F \subseteq \varphi(G) = f^{-1}(G) \subseteq f^{-1}(U) = V$. Thus $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} \subseteq I_V$. Since $I_V \neq I$, we get a contradiction. Therefore, $J_1 = J$.

(d)⇒(c) It is obvious. □

*Remark 4.2.* In [18, Theorem 7] M. Stone proved a result which is equivalent to our assertion that (a)⇔(d) in the previous theorem. More precisely, M. Stone proved the following (in our notation): the map $f$ is a surjection iff the map $\psi = \varphi_{|J} : J \longrightarrow A$ is a 0-pseudolattice monomorphism and for every ideal $J_1$ of $J$, $[(\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I) \leftrightarrow (J_1 = J)]$. The Stone's condition "$(J_1 = J) \rightarrow (\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I)$", i.e., "$\bigvee\{I_{\varphi(b)} \mid b \in J\} = I$", is equivalent (as it is easy to see) to our condition (LBA) (see Definition 2.1) which is automatically satisfied by the morphisms of the category $\mathsf{ZLBA}$ and thus it appears in our Theorem 4.5 in another form. Further, when $\varphi$ is an injection then, obviously, $\psi = \varphi_{|J}$ is an injection; in the converse direction we have the following: the map $\psi$ can be extended to a homomorphism $\varphi : B \longrightarrow A$ (by the result proved below) and then $\varphi$ is obliged to be an injection (indeed, if $b \in B \setminus \{0\}$ and $\varphi(b) = 0$ then the density of $J$ in $B$ implies that there exists $c \in J \setminus \{0\}$ such that $c \leq b$; then $\psi(c) = \varphi(c) = 0$, a contradiction). So, our condition (d) is equivalent to the cited above Stone condition from [18, Theorem 7].

**Proposition 4.6.** *Let $(A, I)$ be a ZLBA, $(B, J)$ be an LBA and $\psi : J \longrightarrow A$ be a 0-pseudolattice homomorphism satisfying condition (LBA) (i.e., $\forall a \in I$ $\exists b \in J$ such that $a \leq \psi(b)$). Then $\psi$ can be extended to a homomorphic map $\varphi : B \longrightarrow A$.*

Proof. For every $a \in A$ and every $b \in B$, set $I_a = \{c \in I \mid c \leq a\}$ and $J_b = \{c \in J \mid c \leq b\}$. It is easy to see that $I_a$ and $J_b$ are simple ideals of $I$ and $J$ respectively. Note also that $\neg I_a = I_{a^*}$ and analogously for $J_b$.

Let $b \in B$. Since $J$ is dense in $B$, we have that $b = \bigvee J_b$. We will show that $I(b) = \bigvee\{I_{\psi(c)} \mid c \in J_b\}$ is a simple ideal of $I$. It is easy to see that $I(b) = \bigcup\{I_{\psi(c)} \mid c \in J_b\}$. Let now $a \in I$. Then, by condition (LBA), there exists $c \in J$ such that $a \leq \psi(c)$. We have that $c = (c \wedge b) \vee (c \wedge b^*)$, $c_1 = c \wedge b \in J_b$, $c_2 = c \wedge b^* \in \neg J_b$ and $c = c_1 \vee c_2$. Thus $a \leq \psi(c) = \psi(c_1) \vee \psi(c_2)$. We obtain that $a = a_1 \vee a_2$, where $a_1 = a \wedge \psi(c_1)$ and $a_2 = a \wedge \psi(c_2)$. Obviously, $a_1 \in I(b)$. We will show that $a_2 \in \neg I(b)$. Indeed, let $a' \in I(b)$; then there exists $d \in J_b$ such that $a' \leq \psi(d)$. Since $c_2 \in \neg J_b$, we get that $d \wedge c_2 = 0$. Thus $\psi(d) \wedge \psi(c_2) = 0$. Hence $a' \wedge a_2 \leq \psi(d) \wedge a \wedge \psi(c_2) = 0$. Therefore, for every $a' \in I(b)$ we have that $a_2 \wedge a' = 0$. This means that $a_2 \in \neg I(b)$. Therefore, $I(b) \vee \neg I(b) = I$, i.e., $I(b)$ is a simple ideal. Since $(A, I)$ is a ZLBA, we get that $\bigvee I(b)$ exists in $A$. We set now $\varphi(b) = \bigvee I(b)$. Obviously, $\varphi(0) = 0$. Further, $\varphi(1) = \bigvee I(1)$. We have that $I(1) = \bigcup\{I_{\psi(c)} \mid c \in J\}$. Applying condition (LBA), we get that $I(1) = I$. Now, using the density of $I$ in $A$, we obtain that $\varphi(1) = 1$. Finally, the fact that $\varphi$ preserves finite meets and finite joins can be easily proved. Hence $\varphi : B \longrightarrow A$ is a Boolean homomorphism and the definition of $\varphi$ together with the density of $I$ in $A$ imply that $\varphi$ extends $\psi$. □

*Remark 4.3.* Note that Remark 4.2 and Proposition 4.6 imply that in Theorem 4.5 we can obtain new conditions equivalent to the condition (a) by replacing in (b), (c) and (d) the phrase "$\varphi$ is an injection" by the phrase "$\varphi_{|J}$ is an injection".

**Theorem 4.7.** *Let $f \in \mathsf{OpenBoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then $f$ is an injection $\iff$ $\varphi(J) \supseteq I \iff \varphi(B) \supseteq I$.*

Proof. Note that, by Remark 4.1, conditions “$\varphi(J) \supseteq I$” and “$\varphi(B) \supseteq I$” are equivalent.

Let $f$ be an injection and $F \in CK(X)$. Then $f(F) \in CK(Y)$ and

$$f^{-1}(f(F)) = F.$$

Hence, $\varphi(J) \supseteq I$. Conversely, let $\varphi(J) \supseteq I$. Then, by Proposition 4.2, we get that $f$ is an injection. $\square$

**Theorem 4.8.** *Let $f \in \mathsf{PerfBoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then $f$ is a surjection $\iff$ $\varphi$ is an injection $\iff$ $\varphi_{|J}$ is an injection.*

Proof. By Proposition 4.1, if $f$ is a surjection then $\varphi$ is an injection. Hence $\varphi_{|J}$ is an injection.

Let now $\varphi_{|J}$ be an injection. Then, by Proposition 4.1, $\mathrm{cl}(f(X)) = Y$. Since $f$ is a closed map, we get that $f$ is a surjection. $\square$

**Theorem 4.9.** *Let $f \in \mathsf{PerfBoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then $f$ is an injection iff $\varphi(J) = I$.*

Proof. Let $f$ be an injection. Then $f_{\restriction X} : X \longrightarrow f(X)$ is a homeomorphism. Let $F' \in CK(X)$. Then $F = f(F')$ is compact. Since $F$ is open in $f(X)$, there exists an open set $U$ in $Y$ such that $U \cap f(X) = F$. Then there exists $G \in CK(Y)$ such that $F \subseteq G \subseteq U$. Then, clearly, $f^{-1}(G) = f^{-1}(F) = F'$. Hence $\varphi(G) = F'$. Therefore, $\varphi(J) \supseteq I$. Since $f$ is perfect, we have that $\varphi(J) \subseteq I$. Thus $\varphi(J) = I$. Conversely, let $\varphi(J) = I$. Then Proposition 4.2 implies that $f$ is an injection. $\square$

Obviously, the last two theorems imply the well-known Stone's results that a $\mathsf{Stone}$-morphism $f$ is an injection (resp., a surjection) iff $\varphi = S^t(f)$ is a surjection (resp., an injection).

Now we will be occupied with the homeomorphic embeddings. We will call them shortly *embeddings*.

**Theorem 4.10.** *Let $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then $f$ is a dense embedding iff $\varphi$ is an injection and $\varphi(J) \supseteq I$.*

Proof. Let $f$ be a dense embedding. Then $f(X)$ is open in $Y$ and thus $f$ is an open injection. Now, Theorem 4.7 implies that $\varphi(J) \supseteq I$. Since $\mathrm{cl}(f(X)) = Y$, we get, by Proposition 4.1, that $\varphi$ is an injection.

Conversely, let $\varphi$ be an injection and $\varphi(J) \supseteq I$. Then, by Proposition 4.1, $\mathrm{cl}(f(X)) = Y$. We will show that $\varphi$ has a lower pre-adjoint. Indeed, for every

$a \in I$ there exists a unique $b_a \in J$ such that $\varphi(b_a) = a$. Let $\psi : I \longrightarrow J$ be defined by $\psi(a) = b_a$ for every $a \in I$. Then, obviously, $\varphi(\psi(a)) = a$, for every $a \in I$. Thus condition (OZL2) (see Theorem 3.5) is satisfied. Further, let $a \in I$, $b \in J$ and $a \wedge \varphi(b) = 0$. Since $a = \varphi(\psi(a))$, we get that $\varphi(\psi(a) \wedge b) = 0$. This implies, by the injectivity of $\varphi$, that $\psi(a) \wedge b = 0$. So, condition (OZL1) (see Theorem 3.5) is also satisfied. Therefore, $\psi$ is a lower pre-adjoint of $\varphi$. Hence, by Theorem 3.5, $f$ is an open map. Now, using the condition $\varphi(J) \supseteq I$, we get, by Theorem 4.7, that $f$ is an injection. Hence, $f$ is a dense embedding. $\square$

**Theorem 4.11** (M. Stone [18]). *Let $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then $f$ is a closed embedding iff $\varphi(J) = I$.*

Proof. Let $f$ be a closed embedding. Then $f$ is a perfect injection. Hence, by Theorem 4.9, $\varphi(J) = I$.

Conversely, let $\varphi(J) = I$. Then, by Theorem 2.10, $f$ is a perfect map. Using Proposition 4.2, we get that $f$ is an injection. Hence, $f$ is a closed embedding. $\square$

**Proposition 4.12.** *Let $f \in \mathsf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $\Theta^t(Y) = (B, J)$. Then $f$ is an embedding iff there exists a ZLBA $(A_1, I_1)$ and two $\mathsf{ZLBA}$-morphisms $\varphi_1 : (A_1, I_1) \longrightarrow (A, I)$ and $\varphi_2 : (B, J) \longrightarrow (A_1, I_1)$ such that $\varphi = \varphi_1 \circ \varphi_2$, $\varphi_1$ is an injection, $\varphi_1(I_1) \supseteq I$ and $\varphi_2(J) = I_1$.*

Proof. Obviously, $f$ is an embedding iff $f = i \circ f_1$ where $f_1$ is a dense embedding and $i$ is a closed embedding. (Indeed, when $f$ is an embedding then let $f_1 : X \longrightarrow \mathrm{cl}_Y(f(X))$ be the restriction of $f$ and $i : \mathrm{cl}_Y(f(X)) \longrightarrow Y$ be the inclusion map; the converse is also clear.) Setting $\varphi_1 = \Theta^t_d(f_1)$ and $\varphi_2 = \Theta^t_d(i)$, we get, by Theorem 2.7, that $\varphi = \varphi_1 \circ \varphi_2$. Now our assertion follows from Theorems 4.10 and 4.11. $\square$

## 5. The construction of the dual objects of the closed, regular closed and open subsets

The next theorem is the well-known result of M. Stone [18] (written in our terms and notation) that open sets correspond to the ideals .

**Theorem 5.1** (Stone [18]). *Let $I$ be a GBA and $(X, \mathcal{O}) = \Theta_s^a(I)$. Then there exists a frame isomorphism $\iota_s : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq)$, $J \mapsto \bigcup\{\gamma_I(a) \mid a \in J\}$. If $U \in \mathcal{O}$ then $J = \iota_s^{-1}(U)) = \{b \in I \mid \gamma_I(b) \subseteq U\}$, $J$ is isomorphic to the ideal $J_U = \{F \in CK(X) \mid F \subseteq U\}$ of $CK(X)$ $(= \Theta_g^t(X))$ and $J_U = CK(U)$, i.e., $J_U = \Theta_g^t(U)$.*

**Corollary 5.2.** *Let $(A, I)$ be a ZLBA and $(X, \mathcal{O}) = \Theta^a(A, I)(= \Theta_g^a(I))$. Then there exists a frame isomorphism*

$$\iota : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq), \quad J \mapsto \bigcup\{\lambda_{(A,I)}(a) \mid a \in J\}.$$

*If $U \in \mathcal{O}$ then $J = \iota^{-1}(U) = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq U\}$, $J$ is isomorphic to the ideal $J_U = \{F \in CK(X) \mid F \subseteq U\}$ of $CK(X)$ $(= \Theta_g^t(X))$ and $J_U = CK(U)$, i.e., $J_U = \Theta_g^t(U)$.*

**Corollary 5.3** (M. Stone [18, Theorem 5]). *Let $I$ be a GBA, $(X, \mathcal{O}) = \Theta_s^a(I)$, $J$ be an ideal of $I$ and $U = \iota_s(J)$. Then:*

(a) *$U$ is a clopen set $\Longleftrightarrow$ $J$ is a simple ideal of $I$;*

(b) *$U$ is a regular open set iff $J$ is a normal ideal of $I$;*

(c) *$U$ is a compact open set iff $J$ is a principal ideal of $I$.*

If $(A, I)$ is an LBA and $a \in A$ then the ideal $I_a = \{b \in I \mid b \leq a\}$ of $I$ will be called an *$A$-principal ideal of $I$*.

**Corollary 5.4.** *Let $(A, I)$ be a ZLBA, $(X, \mathcal{O}) = \Theta^a(A, I)$ $(= \Theta_g^a(I))$, $J$ be an ideal of $I$ and $U = \iota(J)$. Then:*

(a) *$U$ is a clopen set $\Longleftrightarrow$ $J$ is a simple ideal of $I$ $\Longleftrightarrow$ $J$ is an A-principal ideal;*

(b) *$U$ is a regular open set iff $J$ is a normal ideal of $I$;*

(c) *$U$ is a compact open set iff $J$ is a principal ideal of $I$.*

Proof. We need only to prove the second assertion in (a). By Proposition 2.6, we have that $\lambda_{(A,I)}(A) = CO(X)$. Let $U$ be a clopen set. There exists $a \in A$ such that $U = \lambda_{(A,I)}(a)$. Then $J = \iota^{-1}(U) = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq U\} = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq \lambda_{(A,I)}(a)\} = \{b \in I \mid b \leq a\}$, i.e., $J$ is an A-principal ideal. Conversely, let $J$ be an A-principal ideal. Then there exists $a \in A$ such that $J = \{b \in I \mid b \leq a\}$. Since $I$ is dense in $A$, we get that $a = \bigvee J$. Using again Proposition 2.6, we get that $\lambda_{(A,I)}(a) = \bigvee_{CO(X)}\{\lambda_{(A,I)}(b) \mid b \in J\} = \bigvee_{RC(X)}\{\lambda_{(A,I)}(b) \mid b \in J\} = \mathrm{cl}_X(\bigcup\{\lambda_{(A,I)}(b) \mid b \in J\}) = \mathrm{cl}_X(U)$. If there exists $x \in \lambda_{(A,I)}(a) \setminus U$ then there exists $b \in I$ such that $x \in \lambda_{(A,I)}(b) \subseteq \lambda_{(A,I)}(a)$ (since $\lambda_{(A,I)}(a)$ is open). Thus $b \leq a$, i.e., $b \in J$, a contradiction. Therefore, $U = \lambda_{(A,I)}(a)$, i.e., $U$ is a clopen set. $\square$

The above results show that if $X \in |\mathsf{BoolSp}|$ and $U$ is an open subset of $X$ then $\iota^{-1}(U)$ (or, equivalently, $\iota_s^{-1}(U)$) is $\mathsf{GenBoolAlg}$-isomorphic to $\Theta^t_g(U)$. Then the dual object $\Theta^t_d(U)$ of $U$ can be obtained with the help of the following fact which was proved in Section 2: if $I \in |\mathsf{GenBoolAlg}|$ is the $\Theta^t_g$-dual object of some $Y \in |\mathsf{BoolSp}|$ then $(Si(I), e_I(I))$ is its $\Theta^t_d$-dual object in the category $\mathsf{ZLBA}$.

Now, for every $X \in |\mathsf{BoolSp}|$, we will find the connections between the dual objects $\Theta^t_g(F)$ of the closed or regular closed subsets $F$ of $X$ and the dual object $\Theta^t_g(X)$ of $X$. The obtained result for regular closed subsets of $X$ seems to be new even in the compact case.

**Theorem 5.5.** *Let $I, J \in |\mathsf{GenBoolAlg}|$, $X = \Theta^a_g(I)$ and $F = \Theta^a_g(J)$. Then:*

(a)(M. Stone [18, Theorem 4(4)]) *$F$ is homeomorphic to a closed subset of $X$ iff there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ (i.e., iff $J$ is a quotient of $I$);*

(b) *$F$ is homeomorphic to a regular closed subset of $X$ if and only if there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ which preserves all meets that happen to exist in $I$.*

Proof. (a) Let $F$ be homeomorphic to a closed subset of $X$, i.e. there exists a closed embedding $f : F \longrightarrow X$. Then, by Theorem 4.11, $\varphi' = \Theta^t_g(f) : \Theta^t_g(X) \longrightarrow \Theta^t_g(F)$ is a surjective 0-pseudolattice homomorphism. Thus, by the duality, there exists a a surjective 0-pseudolattice homomorphism $\varphi : I \longrightarrow J$.

Conversely, if $\varphi : I \longrightarrow J$ is a surjective 0-pseudolattice homomorphism then, by Theorem 4.11, $F$ is homeomorphic to a closed subset of $X$.

(b) Having in mind the assertion (a) above and Theorem 3.4, it is enough to show that if $f : F \longrightarrow X$ is a closed injection then $f(F) \in RC(X)$ iff $f$ is a quasi-open map. This can be easily done, so that the proof of assertion (b) is complete. $\square$

We will finish with mentioning some assertions about isolated points. All these statements have easy proofs which will be omitted.

**Proposition 5.6.** *Let $(A,I)$ be a ZLBA, $X = \Theta^a(A,I)$ and $a \in A$. Then $a$ is an atom of $A$ iff $\lambda_{(A,I)}(a)$ is an isolated point of the space $X$. Also, for every isolated point $x$ of $X$ there exists an $a \in I$ such that $a$ is an atom of $I$ (equivalently, of $A$) and $\{x\} = \lambda_{(A,I)}(a)$.*

**Proposition 5.7.** *Let $(A,I)$ be a ZLBA and $X = \Theta^a(A,I)(= \Theta^a_g(I))$. Then $X$ is a discrete space $\iff$ the elements of $I$ are either atoms of $I$ or finite sums of atoms of $I$.*

**Proposition 5.8** (M. Stone [18])**.** *Let $(A,I)$ be a ZLBA and $X = \Theta^a(A,I)$ $(= \Theta^a_g(I))$. Then $X$ is an extremally disconnected space iff $A$ is a complete Boolean algebra.*

**Proposition 5.9.** *Let $(A,I)$ be a ZLBA and $X = \Theta^a(A,I)(= \Theta^a_g(I))$. Then the set of all isolated points of $X$ is dense in $X$ iff $A$ is an atomic Boolean algebra iff $I$ is an atomic 0-pseudolattice.*

Acknowledgements. The author is very grateful to the referee for his/her helpful suggestions and remarks.

## References


[1] J. Adámek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, *Wiley Interscience*, *New York*, 1990..

[2] B. Banaschewski, Über nulldimensionale Räume, *Math. Nachr.* **13** (1955), 129–140.

[3] G. Dimov, A Generalization of De Vries' Duality Theorem, *Applied Categorical Structures* **17** (2009), 501–516.

[4] G. Dimov, Some generalizations of the Fedorchuk Duality Theorem - I, *Topology Appl.* **156** (2009), 728–746.

[5] G. Dimov, A de Vries-type duality theorem for the category of locally compact spaces and continuous maps – I, *Acta Math. Hungar.* **129** (2010), 314–349.

[6] G. Dimov, A de Vries-type duality theorem for the category of locally compact spaces and continuous maps – II, *Acta Math. Hungar.* **130** (2011), 50–77.

[7] H. Doctor, The categories of Boolean lattices, Boolean rings and Boolean spaces, *Canad. Math. Bulletin* **7** (1964), 245–252.

[8] Ph. Dwinger, Introduction to Boolean Algebras, *Physica Verlag*, *Würzburg*, 1961.

[9] R. Engelking, General Topology, *PWN*, *Warszawa*, 1977.

[10] V. V. Fedorchuk, Boolean $\delta$-algebras and quasi-open mappings, *Sibirsk. Mat. Ž.* **14** (1973), 1088–1099.

[11] D. Hofmann, On a generalization of the Stone-Weierstrass theorem, *Appl. Categ. Structures* **10** (2002), 569–592.

[12] P. T. Johnstone, Stone Spaces, *Cambridge Univ. Press*, *Cambridge*, 1982.

[13] S. Mardešic and P. Papic, Continuous images of ordered compacta, the Suslin property and dyadic compacta, *Glasnik Mat.-Fiz. Astronom.* **17** (1962), 3–25.

[14] J. D. Monk, Completions of Boolean algebras with operators, *Math. Nachr.* **46** (1970), 47-55.

[15] J. Mioduszewski and L. Rudolf, H-closed and extremally disconnected Hausdorff spaces, *Dissert. Math. (Rozpr. Mat.)* **66** (1969), 1–52.

[16] R. Sikorski, Boolean Algebras, *Springer-Verlag*, *Berlin*, 1964.

[17] M. H. Stone, Theory of representation for Boolean algebras, *Trans. Amer. Math. Soc.* **40** (1936), 37–111.

[18] M. H. Stone, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* **41** (1937), 375–481.

GEORGI D. DIMOV
DEPARTMENT OF MATHEMATICS
AND INFORMATICS
UNIVERSITY OF SOFIA
BLVD. J. BOURCHIER 5
1164 SOFIA
BULGARIA

*E-mail:* gdimov@fmi.uni-sofia.bg

*(Received January 11, 2010; revised June 23, 2011)*