Publ. Math. Debrecen
45 / 1-2 (1994), 19–28

On Lagrange and Hermite interpolation based on the Laguerre abscissas

By XIE-HUA SUN (Hangzhou)

Abstract. A two-sided uniformal estimate of the Lebesgue function for Lagrange and Hermite interpolation based on the Laguerre abscissas is established and an improved rate of convergence for Lagrange and Hermite interpolation is given.

1. Introduction

Let

$$L_n^{(\alpha)}(x) = e^x\, x^{-\alpha}(e^{-x}x^{n+\alpha})^{(n)}/n!, \qquad n = 1, 2, \dots.$$

be the Laguerre polynomial of degree n for $\alpha > -1$, with the normalization

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$$

and

$$(0 <)x_{1n}^{(\alpha)} < x_{2n}^{(\alpha)} < \dots < x_{nn}^{(\alpha)} \tag{1.1}$$

its zeros. We shall write x_k instead of $x_{kn}^{(\alpha)}$ if there is no misunderstanding.

Freud [1] and Névai [2–4] studied Lagrange interpolation based on nodes (1.1). Balázs [5] considered the Hermite interpolation of degree $n+\alpha$ with the nodes (1.1) and 0, if α is an integer and f is α-time differentiable on $[0,\infty)$:

$$Q_{n\alpha}(f,x) = \sum_{k=1}^{n} f(x_k) l_k(x)(x/x_k)^{\alpha+1} + \sum_{i=0}^{\alpha} f^{(i)}(0) r_i(x), \tag{1.2}$$

Research supported by the Natural Science Foundation of Zhejiang, China .

where

$$l_k(x) = L_n^{(\alpha)}(x)/(L_n'^{(\alpha)}(x_k)(x - x_k)), \qquad k = 1, \dots, n$$

and

$$r_i(x) = x^i L_n^{(\alpha)}(x) \,/\, \left(i!\binom{n+\alpha}{n}\right), \qquad i = 0, 1, \dots, \alpha$$

with the properties

$$r_i^{(s)}(0) = \begin{cases} 1, & \text{if } s = i \\ 0, & \text{if } 0 \le s < i \end{cases}$$

and

$$r_i(x_k) = 0, \quad \text{for} \quad k = 1, \dots, n; \quad i = 1, \dots, \alpha.$$

If $\alpha = 0$, $Q_{n0}(f, x)$ is the Lagrange interpolation. Balázs established some estimates of degree of approximation for functions and their derivatives by (1.2).

In this paper we shall first establish a two-sided uniformal estimate of the Lebesgue function

$$\Lambda_n^{(\alpha)}(x) := \sum_{k=1}^{n} |l_k(x)|(x/x_k)^{\alpha+1} + \sum_{i=0}^{\alpha} |r_i(x)|$$

for the interpolation (1.2). Second, we will give degrees of approximation of functions and their derivatives by (1.2) that improve results of [5].

Throughout the paper, $0(1)$ or c are always independent of n, x, f and $f^{(i)}$ but may depend on α. The sign "$A_n(x) \sim B_n(x)$" means that there exist two constants $0 < c_1 < c_2$ independent of n, x such that

$$c_1 B_n(x) \le A_n(x) \le c_2 B_n(x).$$

2. Main Results

Now we state our main results.

Theorem 2.1. *The estimate*

$$\text{(2.1)} \quad \left(\Lambda_n^{(\alpha)}(x) - 1\right) \sim \begin{cases} |L_n^{(\alpha)}(x)|(n^{\frac{1}{4}-\alpha/2}\, x^{\frac{1}{4}+\alpha/2}(|\log nx| + 1) + n^{-\alpha}), \\ \qquad\qquad x \ge x_1 \\ |L_n^{(\alpha)}(x)|(nx^{\alpha+1} + n^{-\alpha}x), \\ \qquad\qquad 0 \le x < x_1 \end{cases}$$

holds uniformly for $0 \le x \le \Delta$, where Δ is an arbitrary but fixed positive number.

Corollary 2.2. *The following estimate is valid:*

$$\left\|\Lambda_n^{(\alpha)}(x)\right\|_{[0,\Delta]} = 0(\log n). \tag{2.2}$$

Theorem 2.3. *Let $f \in C^r[0,\infty)(r \geq \alpha,\ r = 0,1,\dots)$. Then we have for $0 \leq x \leq \Delta$*

$$|f(x) - Q_{n\alpha}(f,x)| = 0(1)\omega(f^{(r)}, n^{-\frac{1}{2}})(x^{\frac{1}{2}r} + x^{\frac{1}{2}\alpha+3/4})n^{-\frac{1}{2}r}\log n, \tag{2.3}$$

where $\omega(f^{(r)},\cdot)$ is the modulus of continuity of $f^{(r)}$ on $[0,x_n]$.

Corollary 2.4. *If f satisfies the condition*

$$\omega(f,n^{-\frac{1}{2}})\log n \to 0 \qquad (n \to \infty),$$

on $[0,x_n]$, then $Q_{n\alpha}(f,x)$ coverges uniformly to $f(x)$ on every finite subinterval of $[0,\infty)$.

Theorem 2.5. *Suppose $f^{(r)} \in C[0,\infty)(0 \leq \alpha \leq r, \alpha$ integer). Then we have*

$$|f^{(i)}(x) - Q^{(i)}_{n\alpha}(f,x)| = 0(1)\omega(f^{(r)}, n^{-\frac{1}{2}})n^{i-\frac{1}{2}r}\log n$$

for $1 \leq i \leq [\frac{1}{2}\alpha]$ and $0 \leq x \leq \Delta$.

3. Preliminaries and Lemmas

In order to prove our theorems, we need some known results:

$$x_k \sim k^2/n \quad (0 < x_k \leq 2\Delta) \qquad [6,(8.9.10)] \tag{3.1}$$

$$|L_n'^{(\alpha)}(x_k)| \sim x_k^{-\frac{1}{2}\alpha-3/4} n^{\frac{1}{2}\alpha+\frac{1}{4}} \sim k^{-\alpha-3/2}n^{\alpha+1} \qquad [6,(8.9.11)] \tag{3.2}$$

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\frac{1}{2}\alpha-\frac{1}{4}}0(n^{\frac{1}{2}\alpha-\frac{1}{4}}), & \text{if } cn^{-1} \leq x \leq \Delta \\ 0(n^\alpha), & \text{if } 0 \leq x \leq cn^{-1} \end{cases} \quad [6,(7.6.8)] \tag{3.3–4}$$

Lemma 3.1 [6,(14.7.5)]**.** *The following equation is valid:*

$$\sum_{k=1}^{n} x_k^{m-1}\left(L_n'^{(\alpha)}(x_k)\right)^{-2} = \Gamma(n+1)\Gamma(m+\alpha+1)/\Gamma(n+\alpha+1) \tag{3.5}$$

$$(m \leq 2n+1)$$

Lemma 3.2 [7]. *Let*

$$|x - x_j| = \min_{1 \le k \le n} |x - x_k|. \tag{3.6}$$

Then for $0 \le x, x_k \le 2\Delta$,

$$|x - x_k| \sim |k^2 - j^2|/n \qquad (k \ne j) \tag{3.7}$$

holds.

Lemma 3.3 [7]. *If* $cn^{-1} \le x \le 2\Delta$, *then*

$$x \sim j^2/n, \tag{3.8}$$

where j *is defined by (3.6).*

Lemma 3.4. *The following estimate is valid for* $\alpha > -1$*:*

$$x^{\frac{1}{2}\alpha + \frac{1}{4}} |L_n^{(\alpha)}(x)| = 0(n^{\frac{1}{2}\alpha - \frac{1}{4}}), \qquad 0 \le x \le 2\Delta. \tag{3.9}$$

PROOF. (3.9) follows from (3.3)–(3.4). □

Lemma 3.5. *If* $f \in C^r[0, \infty)$, *then there exists a polynomial* $G_n(f, x)$ *of degree* $n \ge 4r + 5$ *such that*

$$\begin{aligned} &|f^{(i)}(x) - G_n^{(i)}(f, x)| \\ &\qquad = 0(1)\omega(f^{(i)}, \sqrt{x(x_n - x)}/n)(\sqrt{x(x_n - x)}/n)^{r-i} \end{aligned} \tag{3.10}$$

for $0 \le x \le x_n$ *and* $i = 1, \ldots, r$.

The lemma is an obvious consequence of GOPENGAUZ's theorem [8].

4. Proofs of the Theorems

PROOF of Theorem 2.1. Set $x_0 = 0$. If $x = x_k$ $(k = 0, 1, \ldots, n)$, (2.1) holds obviously. Now suppose $x \ne x_k$ $(k = 0, 1, \ldots, n)$. Since

$$Q_{n\alpha}(1, x) = \sum_{k=1}^{n} l_k(x)(x/x_k)^{\alpha+1} + r_0(x) = 1,$$

observing that for $x_m \le x < x_{m+1}$ $(0 \le m \le n - 1)$

$$\operatorname{sign} l_k(x) = \begin{cases} (-1)^{m+k+1}, & k > m \\ (-1)^{m+k}, & k \le m, \end{cases}$$

where $l_0(x)$ means $r_0(x)$, then we have

$$\begin{aligned}
\Lambda_n^{(\alpha)}(x) - 1 &= \sum_{k=1}^{n} |l_k(x)|(x/x_k)^{\alpha+1} + \sum_{i=0}^{\alpha} |r_i(x)| \\
&\quad - \sum_{k=1}^{n} l_k(x)(x/x_k)^{\alpha+1} - r_0(x) \\
&= \sum_{k=1}^{n} (|l_k(x)| - l_k(x))(x/x_k)^{\alpha+1} + \sum_{i=0}^{\alpha} |r_i(x)| - r_0(x) \\
&= \sum_{k=1}^{m-1} (1-(-1)^{m+k})|l_k(x)|(x/x_k)^{\alpha+1} \qquad (4.1)\\
&\quad + \sum_{k=m+2}^{n} (1-(-1)^{m+k+1})|l_k(x)|(x/x_k)^{\alpha+1} \\
&\quad + \left(\sum_{i=0}^{\alpha} |r_i(x)| - r_0(x)\right) := R_1 + R_2 + R_3.
\end{aligned}$$

First suppose $x_1 \le x \le \Delta$. We have $m = 0(\sqrt{n})$ and $j = 0(\sqrt{n})$. Write (4.2)

$$R_2 = \left(\sum_{x_k \le 2\Delta} + \sum_{x_k > 2\Delta}\right)(1-(-1)^{m+k+1})|l_k(x)|(x/x_k)^{\alpha+1} := R_{21} + R_{22}.$$

For R_{21}, using (3.1)–(3.2) and (3.7),we get

$$\begin{aligned}
R_{21} &\sim nx^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{m+2\le k\le c\sqrt{n}} k^{-\alpha-\frac{1}{2}}|k^2-j^2|^{-1} \\
&= nx^{\alpha+1}|L_n^{(\alpha)}(x)| \left(\sum_{m+2\le k\le 2j} + \sum_{2j+1\le k\le c\sqrt{n}}\right) \qquad (4.3)\\
&:= R'_{21} + R''_{21}. \qquad (4.4)
\end{aligned}$$

If $2j < m+2$, R'_{21} vanishes. Since $j \le m+2 \le k \le 2j$, using (3.8), it follows that

$$\begin{aligned}
R'_{21} &\sim nx^{\alpha+1}|L_n^{(\alpha)}(x)|j^{-\alpha-3/2} \sum_{m+2\le k\le 2j} |k-j|^{-1} \\
&\sim nx^{\alpha+1}j^{-\alpha-3/2}|L_n^{(\alpha)}(x)|\log j \\
&\sim n^{-\frac{1}{2}\alpha+\frac{1}{4}}x^{\frac{1}{2}\alpha+\frac{1}{4}}|L_n^{(\alpha)}(x)|\,|\log(nx)| \qquad (4.5)
\end{aligned}$$

and

$$
\begin{aligned}
R''_{21} &\sim nx^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{2j+1\le k\le c\sqrt{n}} k^{-\alpha-5/2} \\
&\sim nx^{\alpha+1}|L_n^{(\alpha)}(x)| j^{-\alpha-3/2} \sim n^{-\frac{1}{2}\alpha+\frac{1}{4}} x^{\frac{1}{2}\alpha+\frac{1}{4}}|L_n^{(\alpha)}(x)|.
\end{aligned} \tag{4.6}
$$

Combining (4.4)–(4.6) yields

$$
R_{21} \sim n^{-\frac{1}{2}\alpha+\frac{1}{4}} x^{\frac{1}{2}\alpha+\frac{1}{4}}|L_n^{(\alpha)}(x)|(|\log(nx)|+1). \tag{4.7}
$$

For R_{22}, using (3.5), we get

$$
\begin{aligned}
R_{22} &\le cx^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{x_k\ge 2\Delta} x_k^{-\alpha-2}|L_n'^{(\alpha)}(x_k)|^{-1} \\
&\le cx^{\alpha+1}|L_n^{(\alpha)}(x)| \left(\sum_{x_k\ge 2\Delta} x_k^{-1}\left(L_n'^{(\alpha)}(x_k)\right)^{-2}\right)^{\frac{1}{2}} \left(\sum_{k\ge c\sqrt{n}} x_k^{-2\alpha-3}\right)^{\frac{1}{2}} \\
&\le cn^{\frac{1}{4}-\frac{1}{2}\alpha} x^{\alpha+1}|L_n^{(\alpha)}(x)| \le cn^{\frac{1}{4}-\frac{1}{2}\alpha} x^{\frac{1}{4}+\frac{1}{2}\alpha}|L_n^{(\alpha)}(x)|.
\end{aligned} \tag{4.8}
$$

Then combining (4.2) and (4.7)–(4.8) yields

$$
R_2 \sim n^{\frac{1}{4}-\frac{1}{2}\alpha} x^{\frac{1}{4}+\frac{1}{2}\alpha}|L_n^{(\alpha)}(x)|(|\log(nx)|+1). \tag{4.9}
$$

For R_1, we have

$$
\begin{aligned}
R_1 &\le 2x^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{k=1}^{m-1} (x_k^{\alpha+1}|L_n'^{(\alpha)}(x_k)|\,|x-x_k|)^{-1} \\
&\le cnx^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{k=1}^{m-1} k^{-\alpha-3/2}|k-j|^{-1} \\
&\le cnx^{\alpha+1}|L_n^{(\alpha)}(x)| \left(\sum_{k=1}^{[\frac{1}{2}j]} + \sum_{[\frac{1}{2}j]+1\le k\le m-1}\right) \\
&\le cnx^{\alpha+1}|L_n^{(\alpha)}(x)| \left(j^{-\alpha-3/2} + j^{-\alpha-3/2}\log j\right) \\
&\le cn^{-\frac{1}{2}\alpha+\frac{1}{4}} x^{\frac{1}{2}\alpha+\frac{1}{4}}|L_n^{(\alpha)}(x)|\,|\log(nx)|.
\end{aligned} \tag{4.10}
$$

It is clear that

$$
R_3 \le cn^{-\alpha}|L_n^{(\alpha)}(x)| \sum_{i=0}^{\alpha} x^i/i! = 0(n^{-\alpha}|L_n^{(\alpha)}(x)|). \tag{4.11}
$$

Combining (4.1) and (4.9)–(4.11) yields (2.1) for $x \geq x_1$.

If $0 < x < x_1$, R_1 vanishes in (4.1) and $j = 1$. From (4.3) and (4.8) we obtain

$$R_{21} \sim nx^{\alpha+1}|L_n^{(\alpha)}(x)| \sum_{2\leq k\leq c\sqrt{n}} (k^2-1)^{-1}\, k^{-\alpha-\frac{1}{2}} \sim nx^{\alpha+1}|L_n^{(\alpha)}(x)| \tag{4.12}$$

and

$$R_{22} \sim n^{\frac{1}{4}-\frac{1}{2}\alpha}\, x^{\alpha+1}|L_n^{(\alpha)}(x)| \leq nx^{\alpha+1}|L_n^{(\alpha)}(x)|. \tag{4.13}$$

Hence

$$R_2 \sim nx^{\alpha+1}|L_n^{(\alpha)}(x)|. \tag{4.14}$$

Observing that $r_0(x)$ vanishes in this case, it is obvious that

$$R_3 \sim n^{-\alpha}\, x|L_n^{(\alpha)}(x)|. \tag{4.15}$$

Finally, collecting (4.1) and (4.14)–(4.15) implies

$$(\Lambda_n^{(\alpha)}(x) - 1) \sim x|L_n^{(\alpha)}(x)|(nx^\alpha + n^{-\alpha}) \quad (0 \leq x \leq x_1). \qquad \square$$

Proof of Theorem 2.3. Applying (3.10) and the fact $x_n \leq cn$, we have

$$|f^{(i)}(x) - G_n^{(i)}(f,x)| = 0(1)\omega(f^{(r)}, \sqrt{x/n})(\sqrt{x/n})^{r-i}.$$

Observing the invariability of Hermite interpolation (1.2) for a polynomial of degree $\leq n+\alpha$, we obtain

$$\begin{aligned}
|f(x) - Q_{n\alpha}(f,x)| &\leq |f(x) - G_{n+\alpha}(f,x)| + |G_{n+\alpha}(f,x) - Q_{n\alpha}(f,x)| \\
&= 0(1)\Bigg\{\omega(f^{(r)}, \sqrt{x/n})(\sqrt{x/n})^r \\
&\qquad + \sum_{k=1}^{n} |f(x_k) - G_{n+\alpha}(f,x_k)|\,|l_k(x)|(x/x_k)^{\alpha+1}\Bigg\} \\
&= 0(1)\Bigg\{\omega(f^{(r)}, n^{-\frac{1}{2}})(x/n)^{\frac{1}{2}r} \\
&\qquad + \sum_{k=1}^{n} \omega(f^{(r)}, \sqrt{x_k/n})(\sqrt{x_k/n})^r |l_k(x)|(x/x_k)^{\alpha+1}\Bigg\}
\end{aligned}$$

$$= 0(1)\omega(f^{(r)}, n^{-\frac{1}{2}})\Bigg\{(x/n)^{\frac{1}{2}r}$$

$$+ \sum_{k=1}^{n}(1 + x_k^{\frac{1}{2}})(x_k/n)^{\frac{1}{2}r}|l_k(x)|(x/x_k)^{\alpha+1}\Bigg\}$$

$$= 0(1)\omega(f^{(r)}, n^{-\frac{1}{2}})\left\{(x/n)^{\frac{1}{2}r} + \sum_{0<x_k\leq 2\Delta} + \sum_{x_k>2\Delta}\right\}$$

$$:= 0(1)\omega(f^{(r)}, n^{-\frac{1}{2}})\left\{(x/n)^{\frac{1}{2}r} + r_1 + r_2\right\}. \tag{4.16}$$

Noting that $x_k \leq 2\Delta$ and using (3.1)–(3.2) and (3.7), it follows

$$r_1 = 0(x^{\alpha+1})n^{-\frac{1}{2}r}|L_n^{(\alpha)}(x)| \sum_{\substack{0<x_k\leq 2\Delta \\ k\neq j}} x_k^{\frac{1}{2}r-\alpha-1}(|L_n'^{(\alpha)}(x_k)(x-x_k)|)^{-1}$$

$$+ n^{-\frac{1}{2}r}x^{\alpha+1}x_j^{\frac{1}{2}r-\alpha-1}|l_j(x)| := r_{11} + r_{12}, \tag{4.17}$$

and

$$r_{11} = 0(x^{\alpha+1})\, n^{1-r}|L_n^{(\alpha)}(x)| \sum_{k=1,\, k\neq j}^{[cn^{\frac{1}{2}}]} k^{r-\alpha-\frac{1}{2}}|k^2 - j^2|^{-1}$$

$$= 0(x^{\alpha+1})n^{1-r}|L_n^{(\alpha)}(x)| \left(\sum_{k=1}^{[\frac{1}{2}j]} + \sum_{k=[\frac{1}{2}j]+1,\, k\neq j}^{2j} + \sum_{k=2j+1}^{[cn^{\frac{1}{2}}]}\right)$$

$$= 0(x^{\alpha+1})n^{1-r}|L_n^{(\alpha)}(x)|(j^{r-\alpha-3/2} + j^{r-\alpha-3/2}\log j +$$

$$+ \max(n^{\frac{1}{2}r-\frac{1}{2}\alpha-3/4}, j^{r-\alpha-3/2}))$$

$$= \begin{cases} 0(x^{\frac{1}{2}r}n^{-\frac{1}{2}r}\log n), & \text{if } r-\alpha-3/2 \leq 0 \\ 0(x^{\frac{1}{2}\alpha+3/4}n^{-\frac{1}{2}r} + x^{\frac{1}{2}r}n^{-\frac{1}{2}r}\log n), & \text{if } r-\alpha-3/2 > 0 \end{cases}$$

$$= 0(n^{-\frac{1}{2}r}\log n)(x^{\frac{1}{2}r} + x^{\frac{1}{2}\alpha+3/4}). \tag{4.18}$$

For r_{12}, if $x \geq c/n$, using Lemma 3.3 and the fact $x_j \sim x$ and $l_j(x) = 0(1)$, we get

$$r_{12} = 0(x^{\frac{1}{2}r}n^{-\frac{1}{2}r}) \qquad (x \geq c/n).$$

If $0 \le x < c/n$, then $j = 1$ and

$$\begin{aligned} r_{12} &= 0(x^{\alpha+1})|L_n^{(\alpha)}(x)|n^{1-r} = 0(x^{\frac{1}{2}\alpha+3/4})(n^{-\frac{1}{2}r}n^{-\frac{1}{2}r+\frac{1}{2}\alpha+3/4}) \\ &= 0(n^{-\frac{1}{2}r})(x^{\frac{1}{2}r}+x^{\frac{1}{2}\alpha+3/4}) \qquad (x < c/n). \end{aligned} \tag{4.19}$$

Combining (4.17)–(4.19) yields

$$r_1 = 0(n^{-\frac{1}{2}r}\log n)(x^{\frac{1}{2}r}+x^{\frac{1}{2}\alpha+3/4}). \tag{4.20}$$

Colleting (2.1), (3.5) and (3.9), we get

$$r_2 = 0(x^{\alpha+1})|L_n^{(\alpha)}(x)|n^{-\frac{1}{2}r}\sum_{x_k>2\Delta} x_k^r(x_k^{\alpha+3/2}|L_n'^{(\alpha)}(x_k)|)^{-1}$$

$$= 0(x^{\alpha+1}n^{-\frac{1}{2}r})|L_n^{(\alpha)}(x)|\left(\sum_{k=1}^n x_k^{2r}(L_n'^{(\alpha)}(x_k))^{-2}\right)^{\frac{1}{2}}\left(\sum_{k>[cn^{\frac{1}{2}}]} x_k^{-2\alpha-3}\right)^{\frac{1}{2}}$$

$$= 0(x^{\frac{1}{2}\alpha+3/4}n^{-\frac{1}{2}r}). \tag{4.21}$$

Finally, collecting (4.16) and (4.20)–(4.21) yields (2.3). □

Proof of Theorem 2.5. Since the proof is similar to [5], we omit the detail. □

References

[1] G. Freud, Convergence of the Lagrange interpolation on the infinite interval, *Mat. Lapok* **18** (1967), 289–292.

[2] P. Névai, On Lagrange interpolation based on Laguerre roots, *Mat. Lapok* **22** (1971), 149–164.

[3] P. Névai, On the convergence of Lagrange interpolation based on Laguerre roots, *Publ. Math.* **20** (1973), 235–239.

[4] P. Névai, Lagrange interpolation at zeros of orthogonal polynomials, in *"Approximation Theory"*, (G. G. Lorentz ed.) *Academic Press*, *New York*, 1976, pp. 163–201.

[5] K. Balázs, Lagrange and Hermite interpolation processes on the positive real line, *J. Approxi. Theory* **50** (1987), 18–24.

[6] G. Szegő, Orthogonal Polynomials, *AMS Colloq. Publ.* **23** (1939).

[7] X. H. Sun, On Hermite-Fejér interpolation based on the Laguerre abscissas, (*submitted*).

[8] I. E. Gopengauz, On Timan's theorem on approximation of functions by polynomials, *Mat. Zametki* **1** (1967), 163–172.

XIE–HUA SUN
DIVISION OF MATHEMATICS,
CHINA INSTITUTE OF METROLOGY,
HANGZHOU, ZHEJIANG 310034,
P. R. CHINA

(Received November 26, 1992)