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FP-injectivity relative to a semidualizing bimodule

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Abstract. In this paper, we extend the notion of C-injective module to that of C-FP-injective module, where ${}_SC_R$ is a semidualizing bimodule over associative rings. We first obtain a result on the (pre)covering and (pre)enveloping properties of C-FP-injective modules. Then we study right C-FP-injective dimension of modules via right derived functors of Hom. As applications, a necessary and sufficient condition of semidualizing modules having finite FP-injective dimension and a new characterization of left noetherian rings are given.

1. Introduction and preliminaries

Since Foxby, Vasconcelos and Golod independently initiated the study of semidualizing modules, there have been numerous publications concerning these objects (e.g. [1], [4], [8], [21]). Holm and White [9] extended the notion of a semidualizing module to associative rings and furthered the study of Foxby equivalence theory. On the other hand, semidualizing modules, under another name by Wakamatsu [22] (Generalized tilting modules or Wakamatsu tilting modules), proved to be powerful tools in tilting theory. Huang and Tang [12] proved that if S is left coherent, R is right coherent and ${}_SW_R$ is a faithfully balanced self-orthogonal bimodule (semidualizing bimodule), then $FP-\mathrm{id}({}_SW)\leqslant n$ and $FP-\mathrm{id}(W_R)\leqslant n$ if and only if every finitely presented left S-module and every finitely presented right R-module have finite generalized Gorenstein dimension at most n. This paper is motivated by this result and aims at describing the FP-injective dimension of a semidualizing module from another point of view.

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Throughout this paper, we assume that all rings have non-zero unities, and that all modules are unitary. For a ring R, we denote by $\mathrm{Mod}-R$ (resp., $R-\mathrm{Mod}$) the category of right (resp., left) R-modules. A *degreewise finite projective resolution* of an R-module M is a projective resolution $\mathbf{P}$ of M such that each P_i is finitely generated (projective). Let S be a ring. A left S-module M is called *FP-injective* (or *absolutely pure*) if $\mathrm{Ext}_S^1(N, M) = 0$ for all finitely presented left S-modules N. We write $\mathcal{FI}(S)$ for the class of all FP-injective left S-modules. The FP-injective dimension of M, denoted by $FP-\mathrm{id}(M)$, is defined to be the smallest non-negative integer n such that $\mathrm{Ext}_S^{n+1}(N, M) = 0$ for every finitely presented left S-module N (if no such n exists, set $FP-\mathrm{id}(M) = \infty$).

Let R be a ring and $\mathcal{F}$ be a class of R-modules, by an *$\mathcal{F}$-preenvelope* of an R-module M we mean a homomorphism $\varphi : M \to F$ with $F \in \mathcal{F}$ such that for any homomorphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a homomorphism $g : F \to F'$ such that $g \circ \varphi = f$. If furthermore, when $F = F'$ and $f = \varphi$ the only such g are automorphisms of F, then $\varphi : M \to F$ is called an *$\mathcal{F}$-envelope* of M. So if envelopes exist, they are unique up to isomorphism. We say that $\mathcal{F}$ is *(pre)enveloping* if every R-module has an $\mathcal{F}$-(pre)envelope. Dually we have the definitions of an *$\mathcal{F}$-precover* and an *$\mathcal{F}$-cover*.

By a right *$\mathcal{F}$-resolution* of M, we will mean a $\mathrm{Hom}(-, \mathcal{F})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each $F^i \in \mathcal{F}$. Let $L^0 = M, L^1 = \mathrm{coker}(M \to F^0)$, $L^i = \mathrm{coker}(F^{i-2} \to F^{i-1})$ for $i \geq 2$. The nth cokernel L^n ($n \geq 0$) is called the *nth $\mathcal{F}$-cosyzygy of M*. M is said to have *right $\mathcal{F}$-dimension*$\leq n$, denoted right $\mathcal{F}-\dim M \leq n$, if there is a right $\mathcal{F}$-resolution of the form $0 \to M \to F^0 \to \cdots \to F^{n-1} \to F^n \to 0$ of M. If n is the least, then we set right $\mathcal{F}-\dim M = n$ and if there is no such n, we set right $\mathcal{F}-\dim M = \infty$. In a similar manner, we can define the *left $\mathcal{F}$-dimension* of M, denoted by left $\mathcal{F}-\dim M$.

In Section 2 of this paper, we introduce the concept of C-FP-injective modules. It is shown that the class $\mathcal{FI}_C(R)$ of all C-FP-injective left R-modules is both enveloping and covering under some conditions.

Section 3 is devoted to investigating right $\mathcal{FI}_C(R)$-dimension in terms of right derived functors $\mathrm{Ext}^n_{\mathcal{FI}_C}(-,-)$. If S is left coherent and ${}_SC_R$ is a faithfully semidualizing bimodule, for a left R-module M, it is shown that right $\mathcal{FI}_C(R)-\dim M \leq n$ if and only if $\mathrm{Ext}^{n+1}_{\mathcal{FI}_C}(-, M) = 0$ if and only if there exists a right $\mathcal{FI}_C(R)$-resolution of M such that the nth $\mathcal{FI}_C(R)$-cosyzygy is C-FP-injective.

In the final section, we focus on the applications of results obtained in Section 2 and 3. The main results of this section are Corollary 4.4 and Proposition 4.5.

Now, let us recall the concept of a semidualizing bimodule over arbitrary rings, which is taken from [9, Definition 2.1].

Definition 1.1. An $(S-R)$-bimodule $C = {}_SC_R$ is *semidualizing* if

(a1) ${}_SC$ admits a degreewise finite S-projective resolution.

(a2) C_R admits a degreewise finite R-projective resolution.

(b1) The homothety map ${}_SS_S \overset{S\gamma}{\to} \mathrm{Hom}_R(C,C)$ is an isomorphism.

(b2) The homothety map ${}_RR_R \overset{\gamma_R}{\to} \mathrm{Hom}_S(C,C)$ is an isomorphism.

(c1) $\mathrm{Ext}_S^{\geqslant 1}(C,C) = 0$.

(c2) $\mathrm{Ext}_R^{\geqslant 1}(C,C) = 0$.

Definition 1.2 ([9]). A semidualizing bimodule $C = {}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R.

(a) If $\mathrm{Hom}_S(C,N) = 0$, then $N = 0$.

(b) If $\mathrm{Hom}_R(C,M) = 0$, then $M = 0$.

${}_RR_R$ is a typical faithfully semidualizing bimodule, and more examples can be found in [9] and [15]. It is recommended to consult [9, Section 3] for more properties of faithfully semidualizing bimodules.

Fact 1.3. Related to a bimodule ${}_SC_R$ with $R = \mathrm{End}_S\, C$ we have the adjoint pair of functors

$$C \otimes_R - : R - \mathrm{Mod} \to S - \mathrm{Mod}, \qquad \mathrm{Hom}_S(C,-) : S - \mathrm{Mod} \to R - \mathrm{Mod},$$

and for any $M \in R - \mathrm{Mod}$ and $N \in S - \mathrm{Mod}$, the canonical homomorphisms

$$\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M), \qquad m \mapsto [c \mapsto c \otimes m],$$

$$\nu_N : C \otimes_R \mathrm{Hom}_S(C,N) \to N, \qquad c \otimes f \mapsto (c)f.$$

We recall from [23] that $M \in R - \mathrm{Mod}$ (resp., $N \in S - \mathrm{Mod}$) is called *C-adstatic* (resp., *C-static*) if μ_M (resp., ν_N) is an isomorphism. The class of all C-adstatic (resp., static) left R-modules (resp., S-modules) is denoted by $\mathrm{Adst}(C)$ (resp., $\mathrm{Stat}(C)$). The functor $C \otimes_R - : \mathrm{Adst}(C) \to \mathrm{Stat}(C)$ defines an equivalence with inverse $\mathrm{Hom}_S(C,-)$ (see [23, 2.4]). It is straightforward to check the following:

$$\nu_{C \otimes_R M} \circ (C \otimes \mu_M) = \mathrm{id}_{C \otimes_R M} \text{ and } \mathrm{Hom}_S(C, \nu_N) \circ \mu_{\mathrm{Hom}_S(C,N)} = \mathrm{id}_{\mathrm{Hom}_S(C,N)}.$$

In what follows, let us consider two classes of modules related to a semidualizing bimodule.

Definition 1.4 ([9]). The *Auslander class* $\mathcal{A}_C(R)$ with respect to a semidualizing bimodule C consists of all left R-modules M satisfying

(A1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0$,

(A2) $\mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M) = 0$, and

(A3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of R-modules).

Definition 1.5 ([9]). The *Bass class* $\mathcal{B}_C(S)$ with respect to a semidualizing bimodule C consists of all left S-modules N satisfying

(B1) $\mathrm{Ext}^{\geqslant 1}_S(C, N) = 0$,

(B2) $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C, N)) = 0$, and

(B3) The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C, N) \to N$ is an isomorphism (of S-modules).

Lemma 1.6 ([9, Proposition 4.3]). *Let ${}_SC_R$ be a semidualizing bimodule. There are equivalences of categories*

$$\mathcal{A}_C(R) \underset{G=\mathrm{Hom}_S(C,-)}{\overset{F=C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S).$$

By an argument similar to the proof of [21, Theorem 2.8], we have the following result in non-commutative setting.

Lemma 1.7. *Let ${}_SC_R$ be a faithfully semidualizing R-module, $M \in R-\mathrm{Mod}$ and $N \in S-\mathrm{Mod}$. Then the following hold.*

(a) *$M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(S)$.*

(b) *$N \in \mathcal{B}_C(S)$ if and only if $\mathrm{Hom}_S(C, N) \in \mathcal{A}_C(R)$.*

Using [11, Lemma 3], we obtain the following result.

Lemma 1.8. *Let R and S be rings. If S is left coherent, in the situation $(A_R,\ {}_SB_R,\ {}_SI)$, for $n \geqslant 0$, $\mathrm{Tor}^R_n(A, \mathrm{Hom}_S(B, I)) \cong \mathrm{Hom}_S(\mathrm{Ext}^n_R(A, B), I)$, where A_R admits a degreewise finite R-projective resolution, ${}_SB$ is finitely presented, and ${}_SI$ is FP-injective.*

The next three classes of modules have been studied extensively in, for example, [4], [9], [21].

Definition 1.9 ([9]). Let ${}_SC_R$ be a semidualizing bimodule, a left S-module is *C-flat* (resp., *C-projective*) if it has the form $C \otimes {}_RF$ for some flat (resp., projective) module ${}_RF$. A left R-module is *C-injective* if it has the form $\mathrm{Hom}_S(C, E)$

for some injective module ${}_SE$. Dually, the above notions can be defined for right modules. Set the notation

$$\begin{array}{ll}
\mathcal{F}_C(S) = \{C \otimes_R F \mid {}_RF \text{ is flat}\} & \mathcal{F}_C(R) = \{F \otimes_S C \mid F_S \text{ is flat}\} \\
\mathcal{P}_C(S) = \{C \otimes_R P \mid {}_RP \text{ is projective}\} & \mathcal{P}_C(R) = \{P \otimes_S C \mid P_S \text{ is projective}\} \\
\mathcal{I}_C(R) = \{\mathrm{Hom}_S(C, E) \mid {}_SE \text{ is injective}\} & \mathcal{I}_C(S) = \{\mathrm{Hom}_R(C, E) \mid E_R \text{ is injective}\}.
\end{array}$$

Analogously, we say that a left R-module is *C-FP-injective* if it has the form $\mathrm{Hom}_S(C, I)$ for some FP-injective module ${}_SI$. By $\mathcal{FI}_C(R) = \{\mathrm{Hom}_S(C, I) \mid_S I$ is FP-injective$\}$ we mean the class of all C-FP-injective left R-modules. Obviously, $\mathcal{P}_C(S) \subseteq \mathcal{F}_C(S)$, and $\mathcal{I}_C(R) \subseteq \mathcal{FI}_C(R)$. Note that $\mathrm{Hom}_S(C, C \otimes_R F) \cong F$ for any flat left R-module F, and so R is left perfect if and only if $\mathcal{P}_C(S) = \mathcal{F}_C(S)$. It will be shown below that S is left noetherian if and only if $\mathcal{I}_C(R) = \mathcal{FI}_C(R)$.

2. *C-FP*-injective modules

In this section we study C-FP-injective modules and their basic properties.

Lemma 2.1. *Let ${}_SC_R$ be a semidualizing bimodule.*

(a) $\mathcal{FI}(S) \subseteq \mathrm{Stat}(C)$ *and* $\mathcal{FI}_C(R) \subseteq \mathrm{Adst}(C)$.

(b) *If S is left coherent, then* $\mathcal{FI}(S) \subseteq \mathcal{B}_C(S)$ *and* $\mathcal{FI}_C(R) \subseteq \mathcal{A}_C(R)$.

(c) $\mathrm{Adst}(C)$ *is closed under products, coproducts and summands.*

Proof. (a). Since C_R is finitely presented, we have an exact sequence $F_1 \to F_0 \to C \to 0$ with F_0, F_1 finitely generated and free. Applying $\mathrm{Hom}_S(-, C)$ to the exact sequence gives an exact sequence $0 \to \mathrm{Hom}_R(C, C) \to \mathrm{Hom}_R(F_0, C) \to \mathrm{Hom}_R(F_1, C)$. Note that all modules in the above exact sequence are finitely presented. For any FP-injective S-module I, we get another exact sequence $\mathrm{Hom}_S(\mathrm{Hom}_R(F_1, C), I) \to \mathrm{Hom}_S(\mathrm{Hom}_R(F_0, C), I) \to \mathrm{Hom}_S(\mathrm{Hom}_R(C, C), I) \to 0$. Then we have the following commutative diagram

$$\begin{tikzcd}
F_1 \otimes_R \mathrm{Hom}(C, I) \arrow[r] \arrow[d, no head] & F_0 \otimes_R \mathrm{Hom}(C, I) \arrow[r] \arrow[d, no head] & C \otimes_R \mathrm{Hom}(C, I) \arrow[r] \arrow[d, no head, "\nu_I"] & 0 \\
T_1 \arrow[r] & T_0 \arrow[r] & \mathrm{Hom}_S(\mathrm{Hom}_R(C, C), I) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where $T_i = \mathrm{Hom}_S(\mathrm{Hom}_R(F_i, C), I)$ for $i = 0, 1$. But the first two vertical maps are isomorphisms. So ν_I is an isomorphism and the first inclusion follows. The other inclusion is a consequence of Fact 1.3.

(b) is immediate from Lemmas 1.6 and 1.8.

(c). It is straightforward. □

Proposition 2.2. *Let ${}_SC_R$ be a semidualizing bimodule, then S is left noetherian if and only if $\mathcal{I}_C(R) = \mathcal{FI}_C(R)$.*

Proof. "Only if" part is trivial.

"If" part. Let $\{E_i\}_{i\in A}$ be a family of injective left S-modules, then $\coprod_A E_i$ is *FP*-injective by [20, Corollary 2.4]. Thus $\mathrm{Hom}_S(C, \coprod_A E_i) = \mathrm{Hom}_S(C, E)$ for some injective module E. $\coprod_A E_i \cong C \otimes_R \mathrm{Hom}_S(C, \coprod_A E_i) = C \otimes_R \mathrm{Hom}_S(C, E) \cong E$ by Lemma 2.1(a). Hence S is left noetherian. □

Lemma 2.3. *Let ${}_SC_R$ be a semidualizing bimodule. For $U \in R-\mathrm{Mod}$, the following hold.*

(a) *If $C \otimes_R U$ is an FP-injective left S-module and $U \in \mathrm{Adst}(C)$, then $U \in \mathcal{FI}_C(R)$.*

(b) *If S is left coherent, then $U \in \mathcal{FI}_C(R)$ if and only if $C \otimes_R U$ is an FP-injective left S-module and $U \in \mathcal{A}_C(R)$.*

Proof. (a) is trivial.

(b) follows from Lemma 2.1 and part (a). □

In what follows, we turn to study the existence of $\mathcal{FI}_C(R)$-envelopes and $\mathcal{FI}_C(R)$-covers.

Proposition 2.4. *Let ${}_SC_R$ be a semidualizing bimodule.*

(a) *$\mathcal{FI}_C(R)$ is closed under products, coproducts and summands.*

(b) *S is left coherent if and only if $\mathcal{FI}_C(R)$ is closed under direct limits.*

Proof. (a). Since $\mathrm{Adst}(C)$ and $\mathcal{FI}(S)$ are closed under products, coproducts and summands, the statement holds by Lemmas 2.1 and 2.3.

(b). "Only if" part. Let $(U_i)_I$ be a direct system of C-*FP*-injective left R-modules. Then $C \otimes_R \varinjlim U_i \cong \varinjlim (C \otimes_R U_i)$ is an *FP*-injective left S-module (for S is left coherent). On the other hand, $\varinjlim U_i \in \mathcal{A}_C(R)$ by [9, Proposition 4.5(a)]. Hence $\varinjlim U_i \in \mathcal{FI}_C(R)$ by Lemma 2.3.

"If" part. By [20, Theorem 3.2], it is enough to check that $\mathcal{FI}(S)$ is closed under direct limits. Let $\{I_i\}$ be a direct system of *FP*-injective left S-modules, then $\varinjlim \mathrm{Hom}_S(C, I_i) = \mathrm{Hom}_S(C, I)$ for some *FP*-injective left S-module I. So

$$\varinjlim I_i \cong \varinjlim (C \otimes_R \mathrm{Hom}_S(C, I_i)) \cong C \otimes_R \varinjlim \mathrm{Hom}_S(C, I_i) = C \otimes_R \mathrm{Hom}_S(C, I) \cong I.$$
$\square$

Proposition 2.5. *Let ${}_SC_R$ be a semidualizing bimodule. Consider the exact sequence $0 \to U' \to U \to U'' \to 0$ of left R-modules with $U' \in \mathcal{FI}_C(R)$.*

(a) *If $U'' \in \mathcal{FI}_C(R)$, then $U \in \mathcal{FI}_C(R)$.*

(b) *If S is left coherent and C is faithfully semidualizing, then $U \in \mathcal{FI}_C(R)$ implies $U'' \in \mathcal{FI}_C(R)$.*

Proof. (b). If U' and U are both in $\mathcal{FI}_C(R)$, then they are in $\mathcal{A}_C(R)$ by Lemma 2.1(b). It follows from [9, Theorem 6.5] that U'' is in $\mathcal{A}_C(R)$. Hence we get a short exact sequence $0 \to C \otimes_R U' \to C \otimes_R U \to C \otimes_R U'' \to 0$. Note that S is left coherent, $C \otimes_R U''$ is *FP*-injective by [16, Proposition 4.2]. So we are done.

(a) can be proved similarly because $\mathcal{A}_C(R)$ is closed under extensions by [9, Theorem 6.3], so we omit its proof. $\square$

Lemma 2.6. *Let S be a left coherent ring and ${}_SC_R$ be a faithfully semidualizing bimodule, then the class $\mathcal{FI}_C(R)$ is closed under pure submodules and pure quotients.*

Proof. Consider a pure exact sequence $\mathbf{Y} = 0 \to U' \to U \to U'' \to 0$ with $U \in \mathcal{FI}_C(R)$. Clearly, $C \otimes_R \mathbf{Y}$ is also pure exact. In the pure exact sequence $C \otimes_R \mathbf{Y}$, $C \otimes_R U$ is *FP*-injective by Lemma 2.3 as U is in $\mathcal{FI}_C(R)$. Because the class $\mathcal{FI}(S)$ over a coherent ring S is closed under pure submodules and pure quotients, $C \otimes_R U'$ and $C \otimes_R U''$ are both *FP*-injective. Therefore, the result holds by Lemmas 1.7, 2.1 and 2.3. $\square$

Proposition 2.7. *Let ${}_SC_R$ be a semidualizing bimodule. Then the following hold.*

(a) *The class $\mathcal{FI}_C(R)$ is preenveloping on $R-\mathrm{Mod}$. In particular, every C-adstatic R-module M has a monic $\mathcal{FI}_C(R)$-preenvelope.*

(b) *If every S-module has an FP-injective envelope, then $\mathcal{FI}_C(R)$ is enveloping on $R-\mathrm{Mod}$.*

(c) *If S is left coherent and C is faithfully semidualizing, then the class $\mathcal{FI}_C(R)$ is covering on $R-\mathrm{Mod}$.*

Proof. (a) By [5, Proposition 6.2.4], the class of all *FP*-injective left S-modules is preenveloping. Thus, for any R-module M, the S-module $C \otimes_R M$

has an FP-injective preenvelope $\alpha : C \otimes_R M \to I$. Define β to be the composite homomorphism

$$M \xrightarrow{\mu_M} \mathrm{Hom}_S(C, C \otimes_R M) \xrightarrow{\mathrm{Hom}_S(C,\alpha)} \mathrm{Hom}_S(C, I).$$

Hence β is an $\mathcal{FI}_C(R)$-preenvelope of M in virtue of [2, Proposition 2.6]. From the construction of $\mathcal{FI}_C(R)$-preenvelope, the other statement is quite easy.

(b) This proof is analogous to that of [9, Proposition 5.10(c)] (and dual to that of [9, Proposition 5.10(a)]). $\square$

Remark 2.8. Rada and Saorin [17] asked whether every module over an arbitrary ring S has an FP-injective envelope. The answer is negative (see [6, Corollary 6.3.19]), but from [14, Theorem 5] (i.e., S is von Neumann regular if and only if every S-module is FP-injecitve) we easily deduce that this statement holds over a von Neumann regular ring.

3. Characterizing right $\mathcal{FI}_C(R)$-dimension via right derived functors

Consider an additive functor $T : \mathcal{C} \to \mathcal{E}$ between module categories. Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{C}$ be two full subcategories and $\mathbf{F}_\bullet$ be a deleted complex corresponding to a left $\mathcal{F}$-resolution of an object of $\mathcal{C}$. If T is covariant, then the nth homology groups of $T(\mathbf{F}_\bullet)$ give left derived functors L_nT of T. Similarly, the right derived functors R^nT are the nth cohomology groups of $T(\mathbf{G}_\bullet)$, where $\mathbf{G}_\bullet$ corresponds to a deleted right $\mathcal{G}$-resolution. The situation where T is contravariant is handled similarly. We refer to [5, Section 8.2] for a more detailed discussion on this matter. In this section, $(-, I)$ stands for the functor $\mathrm{Hom}_R(-, I)$ or $\mathrm{Hom}_S(-, I)$. Let $\mathrm{Ext}^n_{\mathcal{FI}}(M, -)$, $\mathrm{Ext}^n_{\mathcal{FI}_C}(M, -)$ and $\mathrm{Ext}^n_{\mathcal{I}_C}(M, -)$ denote the nth right derived functors of $\mathrm{Hom}(M, -)$ respectively, since $\mathcal{FI}(S)$, $\mathcal{FI}_C(R)$ and $\mathcal{I}_C(R)$ (see Definition 1.9) are preenveloping by Proposition 2.7 and [9, Proposition 5.10(c)].

Lemma 3.1. *Let ${}_SC_R$ be a semidualizing bimodule, $M \in R-\mathrm{Mod}$ and $N \in S-\mathrm{Mod}$.*

(a) *If M has a right $\mathcal{FI}_C(R)$-resolution $\mathbf{X}$, then $C \otimes_R \mathbf{X}$ is an exact right $\mathcal{FI}(S)$-resolution of $C \otimes_R M$.*
(b) *If N is C-static and has a right $\mathcal{FI}(S)$-resolution $\mathbf{Y}$, then $\mathrm{Hom}_S(C, \mathbf{Y})$ is a right $\mathcal{FI}_C(R)$-resolution of $\mathrm{Hom}_S(C, N)$.*

Proof. (a). Suppose $\mathbf{X} = 0 \to M \to \mathrm{Hom}_S(C, I^0) \to \mathrm{Hom}_S(C, I^1) \to \cdots$, where $\mathrm{Hom}_S(C, I^n) \in \mathcal{FI}_C(R)$ for $n \geq 0$. Tensoring by C yields a complex

$C\otimes_R \mathbf{X}=0\to C\otimes_R M\to C\otimes_R \mathrm{Hom}_S(C,I^0)\to C\otimes_R \mathrm{Hom}_S(C,I^1)\to\cdots$. For any $I\in\mathcal{FI}(S)$, applying $(-,I)$ to $C\otimes_R \mathbf{X}$, we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (C\otimes_R (C,I^1),I) \arrow[r] \arrow[d, dash, "\cong"] & (C\otimes_R (C,I^0),I) \arrow[r] \arrow[d, dash, "\cong"] & (C\otimes_R M,I) \arrow[r] \arrow[d, dash, "\cong"] & 0 \\
\cdots \arrow[r] & ((C,I^1),(C,I)) \arrow[r] & ((C,I^0),(C,I)) \arrow[r] & (M,(C,I)) \arrow[r] & 0.
\end{tikzcd}$$

The exactness of the bottom row implies that of the top row. Because injective S-modules belong to $\mathcal{FI}(S)$, the assertion is true.

(b). Suppose $\mathbf{Y}=0\to N\to I^0\to I^1\to\cdots$, then $\mathrm{Hom}_S(C,\mathbf{Y})=0\to \mathrm{Hom}_S(C,N)\to \mathrm{Hom}_S(C,I^0)\to \mathrm{Hom}_S(C,I^1)\to\cdots$. For any $I\in\mathcal{FI}(S)$, applying $(-,(C,I))$ to the complex $\mathrm{Hom}_S(C,\mathbf{Y})$, we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (I^1,I) \arrow[r] \arrow[d, dash, "\cong"] & (I^0,I) \arrow[r] \arrow[d, dash, "\cong"] & (N,I) \arrow[r] \arrow[d, dash, "(\nu_N,\ I)"] & 0 \\
\cdots \arrow[r] & (C\otimes_R (C,I^1),I) \arrow[r] \arrow[d, dash, "\cong"] & (C\otimes_R (C,I^0),I) \arrow[r] \arrow[d, dash, "\cong"] & (C\otimes_R (C,N),I) \arrow[r] \arrow[d, dash, "\cong"] & 0 \\
\cdots \arrow[r] & ((C,I^1),(C,I)) \arrow[r] & ((C,I^0),(C,I)) \arrow[r] & ((C,N),(C,I)) \arrow[r] & 0.
\end{tikzcd}$$

Since N is C-static, $\mathrm{Hom}_S(\nu_N,I)$ is an isomorphism. It follows that the bottom row is exact, hence we are done. □

We now investigate how right $\mathcal{FI}_C(R)$-dimension, right $\mathcal{FI}(S)$-dimension and *FP*-injective dimension relate.

Theorem 3.2. *Let ${}_SC_R$ be a semidualizing bimodule, $M\in R-\mathrm{Mod}$ and $N\in S-\mathrm{Mod}$. The following equalities hold.*

(a) *right $\mathcal{FI}_C(R)-\dim M=$ right $\mathcal{FI}(S)-\dim(C\otimes_R M)$. Moreover, if S is left coherent, right $\mathcal{FI}_C(R)-\dim M=FP-\mathrm{id}(C\otimes_R M)$.*

(b) *If N is C-static, then right $\mathcal{FI}(S)-\dim N=$ right $\mathcal{FI}_C(R)-\dim(\mathrm{Hom}_S(C,N))$.*

Proof. (a). Assume that right $\mathcal{FI}(S)-\dim(C\otimes_R M)=n$, there is an exact right $\mathcal{FI}(S)$-resolution $\mathbf{Y}$ of $C\otimes_R M$, that is, $\mathbf{Y}=0\to C\otimes_R M\xrightarrow{f} I^0\xrightarrow{d^0} I^1\to\cdots\to I^n\to 0$, then we claim that the complex $0\to M\xrightarrow{\beta}(C,I^0)\xrightarrow{\delta^0}(C,I^1)\to\cdots\to(C,I^n)\to 0$ is a right $\mathcal{FI}_C(R)$-resolution of M, where $\beta=\mathrm{Hom}_S(C,f)\circ\mu_M$, $\delta^i=\mathrm{Hom}_S(C,d^i)$ for $0\le i\le n$. For any *FP*-injective left S-module I, we get a commutative diagram after applying $(-,(C,I))$ to the complex above

$$\begin{tikzcd}
\cdots \arrow[r] & (C\otimes_R(C,I^1),I)) \arrow[r, "(C\otimes_R\delta^0)^*"] \arrow[d, dash, "\cong"] & (C\otimes_R(C,I^0),I)) \arrow[r, "(C\otimes_R\beta)^*"] \arrow[d, dash, "\cong"] & (C\otimes_R M,I) \arrow[d, dash, "\cong"] \\
\cdots \arrow[r] & ((C,I^1),(C,I)) \arrow[r] & ((C,I^0),(C,I)) \arrow[r] & (M,(C,I)).
\end{tikzcd}$$

Because the assignment ν is natural, it is routine to check that $(C\otimes_R\beta)=(\nu_{I^0}^{-1}\circ f)$ and $C\otimes_R\delta^i=\nu_{I^{i+1}}^{-1}\circ d^i\circ\nu_{I^i}$ for $0\le i\le n-1$. Then we have $(C\otimes_R\beta)^*=(\nu_{I^0}^{-1}\circ f)^*=f^*\circ\nu_{I^0}^{-1\,*}$, and $(C\otimes_R\delta^i)^*=(\nu_{I^{i+1}}^{-1}\circ d^i\circ\nu_{I^i})^*=\nu_{I^i}{}^*\circ d^{i*}\circ\nu_{I^{i+1}}^{-1}{}^*$. Hence the bottom row is exact, as $\mathbf{Y}$ is a $\mathrm{Hom}_S(-,I)$ exact complex. This means that right $\mathcal{FI}_C(R)-\dim M\le n$. Conversely, it is straightforward to get that right $\mathcal{FI}(S)-\dim(C\otimes_R M)\le$ right $\mathcal{FI}_C(R)-\dim M$ by Lemma 3.1 (a). Hence we have the first equality in (a). Furthermore, if S is left coherent, it follows from [13, Lemma 3.4] that right $\mathcal{FI}(S)-\dim(C\otimes_R M)=FP-\mathrm{id}(C\otimes_R M)$, and so we get the other equality.

(b) Since N is C-static, $N\cong C\otimes_R\mathrm{Hom}_S(C,N)$. Hence, by (a), we have

$$\begin{aligned}\text{right}\,\mathcal{FI}_C(R)-\dim(\mathrm{Hom}_S(C,N)) &= right\mathcal{FI}(S)-\dim(C\otimes_R\mathrm{Hom}_S(C,N))\\ &= \text{right}\,\mathcal{FI}(S)-\dim N\end{aligned}$$ □

Motivated by [21], we then study how vanishing of the relative cohomology functor $\mathrm{Ext}^i_{\mathcal{FI}_C}(-,M)$ characterizes the finiteness of $\mathcal{FI}_C(R)-\dim M$.

Theorem 3.3. *Let S be a left coherent ring and ${}_SC_R$ be a faithfully semidualizing bimodule. The following are equivalent for a left R-module M.*

(a) $\mathrm{Ext}^1_{\mathcal{FI}_C}(-,M)=0$.

(b) $\mathrm{Ext}^{\geqslant 1}_{\mathcal{FI}_C}(-,M)=0$.

(c) *M is C-FP-injective.*

Proof. (a)⇒(c). Suppose $\mathbf{X}=0\to M\xrightarrow{f}\mathrm{Hom}_S(C,I^0)\xrightarrow{d^0}\mathrm{Hom}_S(C,I^1)\xrightarrow{d^1}\cdots$ is a right $\mathcal{FI}_C(R)$-resolution of M. Let L be the cokernel of f and $g:\mathrm{Hom}_S(C,I^0)\to L$ the natural epimorphism. There is a homomorphism $l:L\to\mathrm{Hom}_S(C,I^1)$ such that $d^0=l\circ g$. Noting that $d^1\circ l\circ g=d^1\circ d^0=0$, the surjectivity of g implies that $d^1\circ l=0$. Since $\mathrm{Ext}^1_{\mathcal{FI}_C}(L,M)=0$, the induced sequence $\mathrm{Hom}_R(L,\mathrm{Hom}_S(C,I^0))\to\mathrm{Hom}_R(L,\mathrm{Hom}_S(C,I^1))\to\mathrm{Hom}_R(L,\mathrm{Hom}_S(C,I^2))$ is exact. Hence, there exists $h\in\mathrm{Hom}_R(L,\mathrm{Hom}_S(C,I^0))$ such that $l=d^0\circ h=l\circ g\circ h$. There is an equality $C\otimes_R l=(C\otimes_R l)\circ(C\otimes_R g)\circ(C\otimes_R h)$ and so $(C\otimes_R g)\circ(C\otimes_R h)=\mathrm{id}_{C\otimes_R L}$ (for $C\otimes_R l$ is monic by Lemma 3.1). Therefore, the exact sequence $0 \to C \otimes_R M \to C \otimes_R (C, I^0) \to C \otimes_R L \to 0$ splits. Thus we have $C \otimes_R M$ is *FP*-injective, and so M is C-*FP*-injective by Lemmas 1.7, 2.1 and 2.3.

(b)⇒(a) and (c)⇒(b) are trivial. □

It should be noted that if one replaces $\mathcal{FI}_C(R)$ with an arbitrary preenveloping class, then one does not always have (a) ⇒ (c) as in Theorem 3.3 (see [7, Lemma 3.3 and Remark 5.6]). Using dimension shifting, we can easily obtain the following results.

Theorem 3.4. *Let S be a left coherent ring, ${}_SC_R$ a faithfully semidualizing bimodule and n a non-negative integer. The following are equivalent for a left R-module M.*

(a) $\mathrm{Ext}^{n+1}_{\mathcal{FI}_C}(-, M) = 0$.

(b) $\mathrm{Ext}^{\geqslant n+1}_{\mathcal{FI}_C}(-, M) = 0$.

(c) *right* $\mathcal{FI}_C(R) - \dim M \leq n$.

(d) *There exists a right $\mathcal{FI}_C(R)$-resolution of M with the nth $\mathcal{FI}_C(R)$-cosyzygy C-FP-injective.*

(e) *Every right $\mathcal{FI}_C(R)$-resolution of M has a C-FP-injective nth $\mathcal{FI}_C(R)$-cosyzygy.*

Proposition 3.5. *Let ${}_SC_R$ be a semidualizing bimodule, $M, N \in R - \mathrm{Mod}$. The following hold.*

(a) $\mathrm{Ext}^i_{\mathcal{FI}_C}(M, N) \cong \mathrm{Ext}^i_{\mathcal{FI}}(C \otimes_R M, C \otimes_R N)$.

(b) *If S is left coherent and M is finitely presented, then* $\mathrm{Ext}^i_{\mathcal{FI}_C}(M, N) \cong \mathrm{Ext}^i_{\mathcal{FI}}(C \otimes_R M, C \otimes_R N) \cong \mathrm{Ext}^i_S(C \otimes_R M, C \otimes_R N)$.

Proof. (a). Let $\mathbf{X}$ be a right $\mathcal{FI}_C(R)$-resolution of N. By Lemma 3.1(a), $C \otimes_R \mathbf{X}$ is a right $\mathcal{FI}(S)$-resolution of $C \otimes_R N$. Thus, by definition, we have

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{FI}}(C \otimes_R M, C \otimes_R N) &= H^i(\mathrm{Hom}_S(C \otimes_R M, C \otimes_R X^\bullet)) \\ &\cong H^i(\mathrm{Hom}_R(M, \mathrm{Hom}_S(C, C \otimes_R X^\bullet)) \\ &\cong H^i(\mathrm{Hom}_R(M, X^\bullet)) = \mathrm{Ext}^i_{\mathcal{FI}_C}(M, N)\end{aligned}$$

where $X^\bullet$ denotes the deleted complex of $\mathbf{X}$.

(b). By hypothesis, $C \otimes_R M$ is also finitely presented. Hence we get $\mathrm{Ext}^i_{\mathcal{FI}}(C \otimes_R M, C \otimes_R N) \cong \mathrm{Ext}^i_S(C \otimes_R M, C \otimes_R N)$ by [19, Theorem C]. □

4. Applications

Lemma 4.1. *Let S be a left coherent ring and ${}_SC_R$ be a faithfully semidualizing bimodule. Then M is in $\mathcal{FI}_C(R)$ if and only if the Pontryagin dual M^+ is in $\mathcal{F}_C(R)$.*

Proof. Assume that M is C-*FP*-injective, so there exists an *FP*-injective left S-module I such that $M = \mathrm{Hom}_S(C, I)$. Thus $M^+ \cong \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) \otimes_S C$. Since I is *FP*-injective and S is left coherent, $\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat. Hence, M^+ is in $\mathcal{F}_C(R)$. Conversely, if $M^+ = F \otimes_S C$ with F a flat right S-module, then $M^{++} \cong \mathrm{Hom}_S(C, F^+) \in \mathcal{FI}_C(R)$. But $C \otimes_R M$ is a pure submodule of $C \otimes_R M^{++} \cong F^+$. So $C \otimes_R M$ is *FP*-injective. Hence $M \in \mathcal{FI}_C(R)$. $\square$

Following [5, Definition 8.2.13], let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be abelian categories and let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathbf{C}$ and $\mathbf{D}$ respectively. Let $T : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ be an additive functor contravariant in the first variable and covariant in the second. Then T is said to be *right balanced* by $\mathcal{F} \times \mathcal{G}$ if for each object M of $\mathbf{C}$, there is a $T(-, \mathcal{G})$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathcal{F}$, and if for every object N of $\mathbf{D}$, there is a $T(\mathcal{F}, -)$ exact complex $0 \to N \to G^0 \to G^1 \to \cdots$ with each $G^i \in \mathcal{G}$. Similarly, the definition above is easily modified to give the definitions of a left or right balanced functor relative to $\mathcal{F} \times \mathcal{G}$ with other choices of variances and complexes.

Proposition 4.2. *Let S be a left coherent ring and ${}_SC_R$ be a faithfully semidualizing bimodule. Then $- \otimes -$ is right balanced on* $\mathrm{Mod}\,{-R} \times R - \mathrm{Mod}$ *by $\mathcal{F}_C(R) \times \mathcal{FI}_C(R)$.*

Proof. We need to show that if $0 \to M \to X^0 \to X^1 \to \cdots$ is a right $\mathcal{F}_C(R)$-resolution of a right R-module M, which exists by [9, Proposition 5.10(d)], and G is a C-*FP*-injective module, then $0 \to M \otimes_R G \to X^0 \otimes_R G \to X^1 \otimes_R G \to \cdots$ is exact. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and using adjoint isomorphism, we get the sequence $\cdots \to \mathrm{Hom}_R(X^0, G^+) \to \mathrm{Hom}_R(M, G^+) \to 0$. But G^+ is in $\mathcal{F}_C(R)$ by Lemma 4.1 and so this sequence is exact. This means the desired sequence is exact. On the other hand, given a right $\mathcal{FI}_C(R)$-resolution $\mathbf{X} = 0 \to N \to U^0 \to U^1 \to \cdots$ of a left R-module N. Applying $C \otimes -$ to this sequence, we obtain an exact sequence by Lemma 3.1(a). Note that $F \otimes_S C \cong \varinjlim C^{(n_i)}$ for any flat right S-module F, hence $F \otimes_S C \otimes_R \mathbf{X}$ is exact. Therefore the result follows. $\square$

Let $\mathrm{Tor}^n_R(-,-)$ denote the nth right derived functor of $- \otimes -$ with respect to the pair $\mathcal{F}_C(R) \times \mathcal{FI}_C(R)$. Based in part on an idea of Enochs and Jenda in [5, Theorem 8.4.31], we now come to the first main result of this section.

Theorem 4.3. *Let S be a left coherent ring and ${}_SC_R$ be a faithfully semidualizing bimodule and $n \geq 0$. Then the following are equivalent.*

(a) *For every flat left R-module F, right $\mathcal{FI}_C(R) - \dim F \leq n$.*

(b) *If $0 \to M \to X^0 \to X^1 \to \cdots$ is a right $\mathcal{F}_C(R)$-resolution of M_R, then the sequence is exact at X^k for $k \geq n-1$ where $X^{-1} = M$.*

(c) *If $0 \to M \to W^0 \to W^1 \to \cdots$ is a right $\mathcal{P}_C^f(R)$-resolution of a finitely presented right R-module M, then the sequence is exact at W^k for $k \geq n-1$ where $W^{-1} = M$.*

(d) *right $\mathcal{FI}_C(R) - \dim_R R \leq n$.*

Proof. (a)$\Rightarrow$(d) is immediate.

(d)$\Rightarrow$(b). Suppose $0 \to R \to U^0 \to \cdots \to U^n \to 0$ is an exact right $\mathcal{FI}_C(R)$-resolution of R by [9, Theorem 6.5], Proposition 2.7 and Theorem 3.4. If $n \geq 2$, we get $\mathrm{Tor}_R^k(M, R) = 0$ for $k \geq n-1$. Computing using $0 \to M \to X^0 \to X^1 \to \cdots$ as in (b), we see that $\mathrm{Tor}_R^k(M, R)$ is just the kth homology group of this complex, giving the desired result.

For $n = 1$, $R \to U^0 \to U^1 \to 0$ exact gives $\mathrm{Tor}_R^{\geqslant 1}(M, R) = 0$ so that, as above, $X^0 \to X^1 \to X^2 \to \cdots$ is exact at X^k for $k \geq 1$ and $M \otimes_R R \to \mathrm{Tor}_R^0(M, R)$ is onto. Computing the latter homomorphism using $0 \to M \to X^0 \to X^1$ shows that $0 \to M \to X^0 \to X^1$ is exact at X^0.

If $n = 0$ then (d) means R is a C-*FP*-injective module. But the balance of $- \otimes -$ then gives $0 \to M \otimes_R R \to X^0 \otimes_R R \to X^1 \otimes_R R \to \cdots$ is exact. That is, $0 \to M \to X^0 \to X^1 \to \cdots$ is exact.

(b)$\Rightarrow$(c) is trivial.

(c)$\Rightarrow$(a). Assume (c) with $n \geq 2$. By [9, Theorem 6.5], Proposition 2.7 and the fact that flat modules are in $\mathcal{A}_C(R)$, we suppose that $0 \to F \to U^0 \to U^1 \to \cdots$ is an exact right $\mathcal{FI}_C(R)$-resolution of a flat left R-module F. Then by (c), we get $\mathrm{Tor}_R^k(M, F) = 0$ for $k \geq n-1$ and all finitely presented right R-modules M, as F is flat. Computing using $0 \to U^0 \to U^1 \to U^2 \to \cdots$ and using [5, Lemma 8.4.23], we get $K = \ker(U^n \to U^{n+1})$ is pure in U^n and so K is also C-*FP*-injective. Hence $0 \to F \to U^0 \to \cdots U^{n-1} \to K \to 0$ gives the desired exact sequence.

Now let $n = 1$. Then (c) says $M \to W^0 \to W^1 \to \cdots$ is exact. So $\mathrm{Tor}_R^k(M, F) = 0$ for $k \geq 1$ and $M \otimes_R F \to \mathrm{Tor}_R^0(M, F)$ is onto. Hence $M \otimes_R F \to M \otimes_R U^0 \to M \otimes_R U^1 \to M \otimes_R U^2$ is exact for all finitely presented right

R-modules M. By [5, Lemma 8.4.23], we again get the desired exact sequence $0 \to F \to U^0 \to K \to 0$ with $K = \ker(U^1 \to U^2)$.

If $n = 0$ then $0 \to M \to W^0 \to W^1 \to \cdots$ exact means $\mathrm{Tor}^k(M, F) = 0$ for $k \geq 1$ and $M \otimes_R F \to \mathrm{Tor}^0_R(M, F)$ is an isomorphism. This gives that $0 \to M \otimes_R F \to M \otimes_R U^0 \to M \otimes_R U^1$ is exact for all M. Thus F is a pure submodule of U^0 and so it is C-FP-injective. $\square$

Following the definition of a noetherian pair of rings in [3, Definition 1.1], we call an ordered pair $\langle S, R\rangle$ a coherent pair of rings provided that S is left coherent and R is right coherent.

Corollary 4.4. *Let ${}_SC_R$ be a faithfully semidualizing bimodule over a coherent pair $\langle S, R\rangle$. Then the following are equivalent.*

(a) $\inf\{n \mid \mathrm{Tor}^{\geqslant n}_R(-, F) = 0$ *for every flat left R-module F*$\} =$ $\inf\{n \mid \mathrm{Tor}^{\geqslant n}_S(F', -) = 0$ *for every flat right S-module F'*$\} < \infty$.

(b) FP-$\mathrm{id}({}_SC) < \infty$, FP-$\mathrm{id}(C_R) < \infty$.

Proof. (a)⇔(b) follows from Theorems 3.2, 4.3 and [25, Theorem 2.6] (i.e., FP-$\mathrm{id}({}_SC) = FP$-$\mathrm{id}(C_R)$ if both of them are finite). $\square$

We conclude this section with a new characterization of a left noetherian ring. In fact, the result of [16, Theorem 5.4] is just a particular case of our conclusion, when we take $C = {}_SS_S$ in Proposition 4.5 below. Suppose that we have a right $\mathcal{FI}_C(R)$-resolution $0 \to N \to \mathrm{Hom}_S(C, I^0) \to \mathrm{Hom}_S(C, I^1) \to \cdots$ of N. Suppose further that we have a right $\mathcal{I}_C(R)$-resolution $0 \to N \to \mathrm{Hom}_S(C, E^0) \to \mathrm{Hom}_S(C, E^1) \to \cdots$ of N. Because $\mathcal{I}_C(R) \subseteq \mathcal{FI}_C(R)$, we are able to complete the following diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, dash, "\mathrm{id}_N"] & (C, I^0) \arrow[r] \arrow[d, dash, "\phi_0"] & (C, I^1) \arrow[r] \arrow[d, dash, "\phi_1"] & \cdots \\
0 \arrow[r] & N \arrow[r] & (C, E^0) \arrow[r] & (C, E^1) \arrow[r] & \cdots .
\end{tikzcd}$$

Now applying $\mathrm{Hom}_S(M, -)$ to the diagram gives natural maps $\mathrm{Ext}^n_{\mathcal{FI}_C}(M, N) \to \mathrm{Ext}^n_{\mathcal{I}_C}(M, N)$ for all $n \geq 0$.

Proposition 4.5. *Let ${}_SC_R$ be a faithfully semidualizing bimodule, then the following conditions are equivalent.*

(a) *S is left noetherian.*

(b) $\mathrm{Ext}^1_{\mathcal{FI}_C}(M, N) \to \mathrm{Ext}^1_{\mathcal{I}_C}(M, N)$ *is an isomorphism for all M and N.*

(c) $\mathrm{Ext}^n_{\mathcal{FI}_C}(M,N) \to \mathrm{Ext}^n_{\mathcal{I}_C}(M,N)$ *is an isomorphism for all non-negative integer n, M, and N.*

Proof. (a)$\Rightarrow$(c) and (c)$\Rightarrow$(b) are trivial.

(b)$\Rightarrow$(a). Let $N \in \mathcal{FI}_C(R)$, then $\mathrm{Ext}^1_{\mathcal{FI}_C}(M,N) = 0$ for all M. So by assumption $\mathrm{Ext}^1_{\mathcal{I}_C}(M,N) = 0$ for all M. Observe that C is faithful, N is C-injective by a similar argument in the proof of [21, Theorem 3.1(b)]. Hence the class $\mathcal{FI}_C(R)$ is equal to the class $\mathcal{I}_C(R)$, and S is left noetherian by Proposition 2.2. □

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References

[1] T. Araya, R. Takahashi and Y. Yoshino, Homological invariants associated to semi-dualizing bimodules, *J. Math. Kyoto Univ.* **45** (2005), 287–306.

[2] M. J. Asensio, J. A. López-Ramos and B. Torrecillas, Covers and Envelopes over *gr*-Gorenstein Rings, *J. Algebra* **215** (1999), 437–459.

[3] L. W. Christensen, A. Frankild and H. Holm, On Gorenstein projective, injective and flat dimensions-A functorial description with applications, *J. Algebra* **302** (2006), 231–279.

[4] E. E. Enochs and H. Holm, Cotorsion pairs associated with Auslander categories, *Israel J. Math.* **174** (2009), 253–268.

[5] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, *Walter de Gruyter, Berlin – New York*, 2000.

[6] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, *Walter de Gruyter, Berlin – New York*, 2006.

[7] H. Holm, Relative Ext groups, resolutions, and Schanuel classes, *Osaka J. Math.* **45** (2008), 719–735.

[8] H. Holm and P. Jørgensen, Semi-dualizing modules and related Gorenstein homological dimensions, *J. Pure Appl. Algebra* **205** (2006), 423–445.

[9] H. Holm and D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.* **47** (2007), 781–808.

[10] H. Holm and P. Jørgensen, Covers preenvelopes and purity, *Illinois J. Math.* **52** (2008), 691–703.

[11] Z. Y. Huang, On homological dimensions of simple modules over non-commutative rings, *Comm. Algebra* **24** (1996), 3259–3264.

[12] Z. Y. Huang and G. H. Tang, Self-orthogonal modules over coherent rings, *J. Pure Appl. Algebra* **161** (2001), 167–176.

[13] L. X. Mao and N. Q. Ding, FI-injective and FI-flat modules, *J. Algebra* **309** (2007), 367–385.

[14] C. Megibben, Absolutely pure modules, *Proc. Amer. Math. Soc.* **26** (1970), 561–566.

[15] J. Miyachi, Duality for derived categories and cotilting bimodules, *J. Algebra* **185** (1996), 583–603.

[16] K. R. Pinzon, Absolutely pure modules, Thesis, *University of Kentucky*, 2005.

[17] J. Rada and M. Saorin, Rings characterized by (pre)envelopes and (pre)covers of their modules, *Comm. Algebra* **26** (1998), 899–912.

[18] J. J. Rotman, An Introduction to Homological Algebra, *Academic Press, New York*, 1979.

[19] S. Sather-Wagstaff, T. Sharif and D. White, Comparison of relative cohomology theories with respect to semidualizing modules, *Math. Z.* **264** (2010), 571–600.

[20] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* **2** (1970), 323–329.

[21] R. Takahashi and D. White, Homological aspects of semidualizing modules, *Math. Scand.* **106** (2010), 5–22.

[22] T. Wakamatsu, Tilting modules and Auslander's Gorenstein property, *J. Algebra* **275** (2004), 3–39.

[23] R. Wisbauer, Static modules and equivalences, *Lecture Notes in Pure and Appl. Math.* **210** (2000), 423–449.

[24] J. Xu, Flat Covers of Modules, *Springer-Verlag, Berlin*, 1996.

[25] H. Y. Zhu and N. Q. Ding, Generalized tilting modules with finite FP-injective dimension, *Forum Math.* **21** (2009), 101–116.

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