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A refinement of a double inequality for the gamma function

By JIAO-LIAN ZHAO (Weinan), BAI-NI GUO (Jiaozuo) and FENG QI (Jiaozuo)

Abstract. In the paper, we present a monotonicity result of a function involving the gamma and logarithmic functions, refine a double inequality for the gamma function, improve some known results for bounding the gamma function, and pose an open problem and three conjectures.

1. Introduction

In [10], the following double inequality was obtained in a complicated way: For $x \in (0,1)$,

$$\frac{x^2+1}{x+1} < \Gamma(x+1) < \frac{x^2+2}{x+2}, \tag{1.1}$$

where $\Gamma(x)$ stands for the classical Euler's gamma function, which may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}\, t. \tag{1.2}$$

The aim of this paper is to simply and concisely generalize, refine and sharpen the double inequality (1.1).

Our main results are stated in the following theorems.

Theorem 1. *The function*

$$Q(x) = \frac{\ln\Gamma(x+1)}{\ln(x^2+1) - \ln(x+1)} \tag{1.3}$$

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is strictly increasing on $(0,1)$, *with the limits*

$$\lim_{x\to 0^+} Q(x)=\gamma \quad \text{and} \quad \lim_{x\to 1^-} Q(x)=2(1-\gamma), \tag{1.4}$$

where $\gamma=0.57\dots$ *denotes Euler–Mascheroni's constant.*

Theorem 2. *The double inequality*

$$\left(\frac{x^2+1}{x+1}\right)^{\alpha}<\Gamma(x+1)<\left(\frac{x^2+1}{x+1}\right)^{\beta} \tag{1.5}$$

holds on $(0,1)$ *if and only if* $\alpha\ge 2(1-\gamma)$ *and* $\beta\le\gamma$. *Consequently, the double inequality*

$$\left[\frac{(x-\lfloor x\rfloor)^2+1}{x-\lfloor x\rfloor+1}\right]^{\alpha}\prod_{i=0}^{\lfloor x\rfloor-1}(x-i)<\Gamma(x+1)<\left[\frac{(x-\lfloor x\rfloor)^2+1}{x-\lfloor x\rfloor+1}\right]^{\beta}\prod_{i=0}^{\lfloor x\rfloor-1}(x-i) \tag{1.6}$$

holds for $x\in(0,\infty)\setminus\mathbb{N}$ *if and only if* $\alpha\ge 2(1-\gamma)$ *and* $\beta\le\gamma$, *where* $\lfloor x\rfloor$ *represents the largest integer less than or equal to* x.

In Section 2, we cite three lemmas, which are utilized in Section 3 to prove Theorem 1. In Section 4, we compare Theorem 2 with several known results. In Section 5 we pose an open problem and three conjectures.

2. Lemmas

In order to prove Theorem 1, we need the following lemma, which can be found in [3], [19, pp. 9–10, Lemma 2.9], [20, p. 71, Lemma 1] or closely-related references therein.

Lemma 1. *Let* f *and* g *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $g'(x)\ne 0$ *on* (a,b). *If* $\frac{f'(x)}{g'(x)}$ *is increasing (or decreasing) on* (a,b), *then so are the functions* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *and* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *on* (a,b).

We also need the following elementary conclusions.

Lemma 2. *For* $x\in(0,1)$, *we have*

$$h_1(x)=x^4+4x^3-2x^2-4x-3<0, \tag{2.1}$$

$$h_2(x)=(x-1)\bigl(x^2+2x-1\bigr)-(x+1)\bigl(x^2+1\bigr)\ln\frac{x^2+1}{x+1}>0, \tag{2.2}$$

$$h_3(x) = x^6 + 6x^5 - 3x^4 - 16x^3 - 21x^2 - 6x - 1 < 0, \tag{2.3}$$

$$h_4(x) = x^5 + 5x^4 - 2x^3 - 8x^2 - 7x - 1 < 0, \tag{2.4}$$

$$h_5(x) = 5x^7 + 34x^6 + 27x^5 - 62x^4 - 205x^3 - 198x^2 - 83x - 6 < 0. \tag{2.5}$$

Proof. By Descartes' Rule of Signs, the function $h_1(x)$ has just one possible positive root. Since $h_1(1) = -4$ and $h_1(2) = 29$, the function $h_1(x)$ is negative on $[0, 1]$.

A straightforward calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\frac{h_2(x)}{(x+1)(x^2+1)}\right] = \frac{(1-x)h_1(x)}{(x+1)^2(x^2+1)^2},$$

so the function $\frac{h_2(x)}{(x+1)(x^2+1)}$ is strictly increasing on $[0, 1]$. Since $h_2(0) = 1$, it follows that $h_2(x) > 0$ on $(0, 1)$.

Since

$$h_3(1) = -40, \qquad h_3(3) = 1304, \qquad h_4(1) = -12,$$
$$h_4(2) = 49, \qquad h_5(1) = -488, \qquad h_5(2) = 84,$$

using Descartes' Rule of Signs again yields the negativity of the functions $h_i(x)$ for $3 \le i \le 5$ on $(0, 1)$. The proof of Lemma 2 is complete. □

For our own convenience, we also recite the following double inequality for polygamma functions $\psi^{(k)}(x)$ on $(0, \infty)$.

Lemma 3. *The double inequality*

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{2.6}$$

holds for $x > 0$ and $k \in \mathbb{N}$.

For the proof of the inequality (2.6), please refer to [5, p. 131], [6, p. 223, Lemma 2.3], [7, p. 107, Lemma 3], [8, p. 853], [12, p. 55, Theorem 5.11], [13, p. 1625], [16, p. 79], [18, p. 2155, Lemma 3] and closely-related references therein.

3. Proofs of Theorems 1 and 2

Now we are in a position to prove our main results in Theorems 1 and 2.

Proof of Theorem 1. It is easy to see that

$$Q(x)=\frac{\frac{1}{x-1}\ln\Gamma(x+1)}{\frac{1}{x-1}\ln\frac{x^2+1}{x+1}}=\frac{f(x)-f(0)}{g(x)-g(0)}, \tag{3.1}$$

where

$$f(x)=\begin{cases}\dfrac{1}{x-1}\ln\Gamma(x+1), & 0\le x<1,\\ 1-\gamma, & x=1\end{cases}$$

and

$$g(x)=\begin{cases}\dfrac{1}{x-1}\ln\dfrac{x^2+1}{x+1}, & 0\le x<1,\\ \dfrac{1}{2}, & x=1.\end{cases}$$

Easy computation and simplification yield

$$\frac{f'(x)}{g'(x)}=\frac{(x+1)\bigl(x^2+1\bigr)[(x-1)\psi(x+1)-\ln\Gamma(x+1)]}{h_2(x)}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\frac{f'(x)}{g'(x)}\right]=\frac{(1-x)h_1(x)q(x)}{[h_2(x)]^2},$$

where

$$q(x)=\ln\Gamma(x+1)-(x-1)\psi(x+1)-\frac{(x+1)\bigl(x^2+1\bigr)h_2(x)}{h_1(x)}\psi'(x+1).$$

Further computation and simplification give

$$q'(x)=-\frac{h_2(x)}{h_1(x)^2}q_1(x),$$

where

$$q_1(x)=2h_3(x)\psi'(x+1)+(x+1)\bigl(x^2+1\bigr)h_1(x)\psi''(x+1)$$

and satisfies

$$\begin{aligned}q_1'(x)&=12h_4(x)\psi'(x+1)\\&\quad+h_1(x)\bigl[3\bigl(3x^2+2x+1\bigr)\psi''(x+1)+(x+1)\bigl(x^2+1\bigr)\psi'''(x+1)\bigr].\end{aligned}$$

By virtue of Lemmas 2 and 3, we obtain

$$q_1'(x)<h_1(x)\biggl\{(x+1)\bigl(x^2+1\bigr)\left[\frac{2}{(x+1)^3}+\frac{3}{(x+1)^4}\right]$$

$$- 3\left(3x^2 + 2x + 1\right)\left[\frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}\right]\Bigg\}$$
$$+ 12h_4(x)\left[\frac{1}{x+1} + \frac{1}{2(x+1)^2}\right] = \frac{h_5(x)}{(x+1)^3} < 0$$

on $[0,1]$. So the function $q_1(x)$ is strictly decreasing on $[0,1]$. Since

$$q_1(0) = -2\psi'(1) - 3\psi''(1) = -2\sum_{k=0}^{\infty}\frac{1}{(k+1)^2} + 6\sum_{k=0}^{\infty}\frac{1}{(k+1)^3} = 3.9\dots$$

and

$$q_1(1) = 80\left(1 - \frac{\pi^2}{6}\right) - 16\psi''(2) = 80\left(1 - \frac{\pi^2}{6}\right) + 32\sum_{k=0}^{\infty}\frac{1}{(k+2)^3} = -45.1\dots,$$

the function $q_1(x)$ has a unique zero on $(0,1)$, and so does also the function $q'(x)$. Consequently, the function $q(x)$ has a unique minimum on $(0,1)$. Since

$$q(0) = \frac{1}{3}\left(\frac{\pi^2}{6} - 3\gamma\right) = -0.028\dots$$

and $q(1) = 0$, we find that $q(x) < 0$ on $(0,1)$. Combining this with Lemma 2 leads to

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\frac{f'(x)}{g'(x)}\right] > 0$$

on $(0,1)$, which means that the function $\frac{f'(x)}{g'(x)}$ is strictly increasing on $(0,1)$. Furthermore, from Lemma 1 and the equation (3.1), it follows that the function (1.3) is strictly increasing on $(0,1)$.

By l'Hospital's Rule, we have

$$\lim_{x\to 0^+} Q(x) = \lim_{x\to 0^+}\frac{(x+1)\left(x^2+1\right)\psi(x+1)}{x^2+2x-1} = -\psi(1) = \gamma$$

and

$$\lim_{x\to 1^-} Q(x) = \lim_{x\to 1^-}\frac{(x+1)\left(x^2+1\right)\psi(x+1)}{x^2+2x-1} = 2\psi(2) = 2(1-\gamma).$$

The proof of Theorem 1 is completed. □

Proof of Theorem 2. The double inequality (1.5) and its sharpness follow immediately from considering the monotonicity of the function (1.3) together with the limits in (1.4).

The double inequality (1.6) is deduced from the double inequality (1.5) and the recurrent formula $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. The proof of Theorem 2 is complete. □

4. Comparisons

In this section, we compare Theorem 2 with some known results.

4.1. It is clear that the double inequality (1.5) refines the double inequality (1.1). Moreover, the extreme case of the inequality (1.5) may be rewritten as

$$\frac{1}{x}\left(\frac{x^2+1}{x+1}\right)^{2(1-\gamma)} < \Gamma(x) < \frac{1}{x}\left(\frac{x^2+1}{x+1}\right)^{\gamma}, \quad x \in (0,1). \tag{4.1}$$

4.2. In [1, p. 145, Theorem 2], it was shown that if $x \in (0,1)$, then

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma} \tag{4.2}$$

with the best possible constants

$$\alpha = 1-\gamma = 0.42278\dots \quad \text{and} \quad \beta = \frac{1}{2}\left(\frac{\pi^2}{6}-\gamma\right) = 0.53385\dots, \tag{4.3}$$

and that if $x \in (1,\infty)$, then (4.2) holds with the best possible constants

$$\alpha = \frac{1}{2}\left(\frac{\pi^2}{6}-\gamma\right) \quad \text{and} \quad \beta = 1. \tag{4.4}$$

In [2, p. 780, Corollary], the following conclusion was established: Let α and β be nonnegative real numbers. For $x>0$, we have

$$\sqrt{2\pi}\,x^x \exp\left[-x-\frac{1}{2}\psi(x+\alpha)\right] < \Gamma(x) < \sqrt{2\pi}\,x^x \exp\left[-x-\frac{1}{2}\psi(x+\beta)\right] \tag{4.5}$$

with the best possible constants $\alpha = \frac{1}{3}$ and $\beta = 0$.

In [9, p. 3, Theorem 5], among other things, it was demonstrated that for $x \in (0,1]$ we have

$$\frac{x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) \le \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}}. \tag{4.6}$$

By the Mathematica or other mathematical softwares, we can show that

(1) the double inequalities (4.1) and (4.2) are not included each other on $(0,1)$,
(2) when $x>0$ is smaller, the double inequalities (4.1) is better than (4.2),
(3) the double inequality (4.1) improves (4.5) on $(0,1)$,
(4) the left-hand side inequality in (4.1) refines the corresponding one in (4.6),
(5) the right-hand side inequalities in (4.1) and (4.6) are not contained each other,
(6) when $x>0$ is smaller, the right-hand side inequality in (4.1) is better than the corresponding one in (4.6).

4.3. In [4, Corollary 1.2, Theorem 1.4 and Theorem 1.5], the following sharp inequalities for bounding the gamma function were obtained: For $x > 0$, we have

$$\sqrt{2}\left(x+\frac{1}{2}\right)^{x+1/2}e^{-x} \le \Gamma(x+1) \le e^{\gamma/e^{\gamma}}\left(x+\frac{1}{e^{\gamma}}\right)^{x+1/e^{\gamma}}e^{-x}, \tag{4.7}$$

$$\sqrt{2e}\left(\frac{x+1/2}{e}\right)^{x+1/2} \le \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2} \tag{4.8}$$

and

$$\begin{aligned}\sqrt{2x+1}\,x^{x}\exp\biggl\{-\biggl[x+\frac{1}{6(x+3/8)}-\frac{4}{9}\biggr]\biggr\} &< \Gamma(x+1)\\ &< \sqrt{\pi(2x+1)}\,x^{x}\exp\biggl\{-\biggl[x+\frac{1}{6(x+3/8)}\biggr]\biggr\}.\end{aligned} \tag{4.9}$$

By the Mathematica or other mathematical softwares, we can reveal that

(1) the double inequalities (1.5) and (4.7) do not include each other on $(0, 1)$,
(2) the right-hand side inequality in (1.5) is better than the one in (4.8) on $(0, 1)$,
(3) the left-hand side inequalities in (1.5) and (4.8) are not included each other on $(0, 1)$,
(4) the lower bound in (1.5) improves the corresponding one in (4.9), but the right-hand side inequalities in (1.5) and (4.9) do not contain each other on $(0, 1)$.

4.4. It is clear that when $x \in \mathbb{N}$ the inequality (1.6) becomes equality. This shows us that for $x > 1$ the double inequality (1.6) is better than those double inequalities listed in the above Remarks 4.2 and 4.3.

4.5. In [15, Theorem 1], among other things, it was proved that the function

$$F(x) = \frac{\ln\Gamma(x+1)}{x\ln(2x)} \tag{4.10}$$

is both strictly increasing and strictly concave on $\left(\frac{1}{2}, \infty\right)$. By l'Hospital Rule and the double inequality (2.6) for $k = 1$, we obtain

$$\lim_{x\to\infty}F(x) = \lim_{x\to\infty}\frac{\psi(x+1)}{1+\ln(2x)} = \lim_{x\to\infty}\bigl[x\psi'(x+1)\bigr] = 1,$$

so it follows that $\Gamma(x + 1) < (2x)^x$ on $\left(\frac{1}{2}, \infty\right)$, which is not better than the right-hand side inequality in (1.5) on $\left(\frac{1}{2}, 1\right)$.

Remark 1. For more information on the history, backgrounds, origins, and recent developments of bounding the gamma function, please refer to the expository and survey article [12] and plenty of references therein.

5. An open problem and three conjectures

In this section, we pose an open problem and three conjectures.

5.1. An open problem. By similar argument to the proof of Theorem 1, we may prove that the function

$$\frac{\ln\Gamma(x+1)}{\ln(x^2+6)-\ln(x+6)} \tag{5.1}$$

is strictly decreasing on $(0,1)$. Consequently,

$$\left(\frac{x^2+6}{x+6}\right)^{6\gamma} < \Gamma(x+1) < \left(\frac{x^2+6}{x+6}\right)^{7(1-\gamma)}, \quad x\in(0,1). \tag{5.2}$$

Motivating by monotonic properties of the functions (1.3) and (5.1), we pose the following open problem: What is the largest number $\lambda>1$ (or the smallest number $\lambda<6$ respectively) for the function

$$\frac{\ln\Gamma(x+1)}{\ln(x^2+\lambda)-\ln(x+\lambda)} \tag{5.3}$$

to be strictly increasing (or decreasing respectively) on $(0,1)$?

5.2. Three conjectures. Finally, we pose the following conjectures.

(1) The function (1.3) is strictly increasing not only on $(0,1)$ but also on $(0,\infty)$.

(2) For $\tau>0$, the function

$$\begin{cases} \dfrac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}, & x\neq 1\\ -(1+\tau)\gamma, & x=1 \end{cases} \tag{5.4}$$

is strictly increasing with respect to $x\in(0,\infty)$.

(3) Recall from [11, Chapter XIII], [21, Chapter 1] or [22, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$0\le(-1)^n f^{(n)}(x)<\infty \tag{5.5}$$

for $x\in I$ and $n\ge 0$. We conjecture that the function

$$h(x)=\begin{cases} \dfrac{\ln x}{\ln(1+x^2)-\ln(1+x)}, & x\neq 1\\ 2, & x=1 \end{cases} \tag{5.6}$$

is completely monotonic on $(0,\infty)$.

We remark that numerical experiments with Mathematica or other mathematical softwares support these conjectures.

Remark 2. This paper is a revised version of the preprint [14].

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Addendum. The first conjecture on the function (1.3) in Section 5.2 has been solved in [17].

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JIAO-LIAN ZHAO
DEPARTMENT OF MATHEMATICS AND INFORMATICS
WEINAN TEACHERS UNIVERSITY
WEINAN CITY, SHAANXI PROVINCE, 714000
CHINA

E-mail: zhaojl2004@gmail.com, darren2004@126.com

BAI-NI GUO
SCHOOL OF MATHEMATICS AND INFORMATICS
HENAN POLYTECHNIC UNIVERSITY
JIAOZUO CITY, HENAN PROVINCE,454010
CHINA

E-mail: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com

FENG QI
SCHOOL OF MATHEMATICS AND INFORMATICS
HENAN POLYTECHNIC UNIVERSITY
JIAOZUO CITY, HENAN PROVINCE, 454010
CHINA
AND
DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCE
TIANJIN POLYTECHNIC UNIVERSITY
TIANJIN CITY, 300160
CHINA

E-mail: qifeng618@gmail.com, qifeng618@qq.com, qifeng618@hotmail.com
URL: http://qifeng618.wordpress.com

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