Publ. Math. Debrecen
80/3-4 (2012), 343–345
DOI: 10.5486/PMD.2012.5037

A note on two conjectures associated to Goldbach's problem

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Abstract. Chen and Chen recently proposed two conjectures on the structure of multiplicative functions f for which $f(p)+f(q)=f(p+q)$ for all odd primes p and q. In this note, we show that the second conjecture is either true unconditionally or follows from the first conjecture, depending on whether or not there is an odd prime p_0 such that $f(p_0)\neq 0$.

Define $g_1(n):=n$ for $n\geq 1$ and define $g_2(n)$ by

$$g_2(n)=\begin{cases}1 & \text{if } n \text{ is odd}\\ 2 & \text{if } n \text{ is even.}\end{cases}$$

Note that both g_1 and g_2 are multiplicative functions.

Chen and Chen [1] gave the following result.

Theorem 1 (Chen and Chen [1])**.** *Let f be a multiplicative function for which*

$$f(p)+f(q)=f(p+q)$$

for all odd primes p and q. If there is an odd prime p_0 for which $f(p_0)\neq 0$, then either $f=g_1$ or $f=g_2$. Moreover, $f=g_1$ if and only if $f(3)=3$.

As probable extensions of this theorem, Chen and Chen gave the following two conjectures (see [1, Conjectures 1 and 2]).

Mathematics Subject Classification: Primary 11A25.
Key words and phrases: multiplicative function, recurrence sequences.
The research of M. Coons is supported by a Fields–Ontario Fellowship and NSERC.

Conjecture 1 (Chen and Chen [1]). *If f is a multiplicative function such that $f(2) \neq 0$, $f(3) = 0$ and $f(p) + f(q) = f(p+q)$ for all odd primes p and q, then $f(n) = 0$ for all $n \geq 5$.*

Conjecture 2 (Chen and Chen [1]). *If f is a multiplicative function such that $f(2) = 2$ and $f(p) + f(q) = f(p+q)$ for all odd primes p and q, then for $n \geq 3$*

$$f(2n) = \frac{f(3)}{3}\left((n-3)f(4) + 12 - 2n\right),$$

and

$$f(2n-1) = \frac{f(3)}{3}\left((n-2)f(4) + 7 - 2n\right).$$

Concerning Conjecture 1, Chen and Chen remark that *if f satisfies the conditions of Conjecture 1, then $f(p) = 0$ for all primes $p \geq 5$. Thus by induction on n we can prove that the Goldbach conjecture implies Conjecture 1. This implies that if Conjecture* 1 *is false, then the Goldbach conjecture is false.*

It seems that the use of the recurrence relations in Conjecture 2 may be a bit misleading. Indeed, Conjecture 2 can be considered following two cases: if $f(p_0) \neq 0$ for some odd prime p_0, then Conjecture 2 is true by Theorem 1, and if $f(p) = 0$ for all odd primes p, then Conjecture 2 is implied by Conjecture 1, which we will now show.

Proposition 1. *Let f be a function satisfying the assumptions of Conjecture 2 and suppose further that there is an odd prime p_0 such that $f(p_0) \neq 0$. Then the conclusion of Conjecture 2 holds unconditionally.*

Proof. If there is an odd prime p_0 such that $f(p_0) \neq 0$, then by Theorem 1 f is one of g_1 or g_2. If $f = g_1$, then for $n \geq 3$ we have that both

$$\frac{f(3)}{3}\left((n-3)f(4) + 12 - 2n\right) = (n-3)4 + 12 - 2n = 2n = g_1(2n) = f(2n),$$

and

$$\frac{f(3)}{3}\left((n-2)f(4) + 7 - 2n\right) = (n-2)4 + 7 - 2n = 2n-1 = g_1(2n-1) = f(2n-1),$$

so that Conjecture 2 holds. Now if $f = g_2$, then for $n \geq 3$, we have that both

$$\frac{f(3)}{3}\left((n-3)f(4) + 12 - 2n\right) = \frac{(n-3)2 + 12 - 2n}{3} = 2 = g_2(2n) = f(2n),$$

and

$$\frac{f(3)}{3}\left((n-2)f(4) + 7 - 2n\right) = \frac{(n-2)2 + 7 - 2n}{3} = 1 = g_2(2n-1) = f(2n-1),$$

so that in either case since $f(p_0) \neq 0$ for some odd prime p_0, Conjecture 2 holds. $\square$

Proposition 2. *Let f be a function satisfying the assumptions of Conjecture 2 and suppose further that $f(p) = 0$ for all odd primes p. Then the conclusion of Conjecture 2 follows from Conjecture 1.*

Proof. Suppose that Conjecture 1 holds and let f be a multiplicative function such that $f(2) = 2$ and $f(p) + f(q) = f(p + q)$ for all odd primes p and q, and suppose further that $f(p) = 0$ for all odd primes p.

Since 3 is an odd prime $f(3) = 0$ and it is easy to see for $n \geq 3$ that

$$\frac{f(3)}{3}\left((n-3)f(4) + 12 - 2n\right) = 0 = f(2n),$$

and

$$\frac{f(3)}{3}\left((n-2)f(4) + 7 - 2n\right) = 0 = f(2n-1),$$

where in each of these equations the last equals sign is given by Conjecture 1. This finishes the proof. □

References

[1] Kang-Kang Chen and Yong-Gao Chen, On $f(p) + f(q) = f(p + q)$ for all odd primes p and q, *Publ. Math. Debrecen* **76** (2010), 425–430.

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(Received November 5, 2010; revised August 15, 2011)