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Fractional part integral representation for derivatives of a function related to $\ln\Gamma(x+1)$

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Abstract. For $0 \neq x > -1$ let

$$\Delta(x) = \frac{\ln\Gamma(x+1)}{x}.$$

Recently Adell and Alzer proved the complete monotonicity of Δ' on $(-1,\infty)$ by giving an integral representation of $(-1)^n\Delta^{(n+1)}(x)$ in terms of the Hurwitz zeta function $\zeta(s,a)$. We reprove this integral representation in different ways, and then re-express it in terms of fractional part integrals. Special cases then have explicit evaluations. Other relations for $\Delta^{(n+1)}(x)$ are presented, including its leading asymptotic form as $x \to \infty$.

Introduction and statement of results

For $0 \neq x > -1$ let

$$\Delta(x) = \frac{\ln\Gamma(x+1)}{x}, \quad \Delta(0) = -\gamma, \tag{1.1}$$

where Γ is the Gamma function, $\gamma = -\psi(1)$ is the Euler constant, and $\psi(x) = \Gamma'/\Gamma$ is the digamma function. The study of the convexity and monotonicity of the functions Γ and Δ and of their derivatives is of interest [8], [13], [14], [17]. For instance, the paper [8] gave an analog of the well known Bohr–Mollerup theorem for the function $\Delta(x)$. Monotonicity and convexity are very useful properties for developing a variety of inequalities. Completely monotonic functions have

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applications in several branches, including complex analysis, potential theory, number theory, and probability (e.g., [5]). In [4], $-\Delta(x)$ was shown to be a Pick function, with integral representation

$$-\Delta(x) = -\frac{\pi}{4} + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \frac{dt}{-t}.$$

I.e., this function is holomorphic in the upper half plane with nonnegative imaginary part.

Recently Adell and Alzer [2] proved the complete monotonicity of Δ' on $(-1, \infty)$ by demonstrating the following integral representation.

Proposition 1 (Adell and Alzer)**.** *For $x > -1$ and $n \geq 0$ an integer one has*

$$(-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1) du, \tag{1.2}$$

where $\zeta(s, a)$ is the Hurwitz zeta function (e.g., [10]). The complete monotonicity of Δ', the statement $(-1)^n \Delta^{(n+1)}(x) \geq 0$, then follows from $\zeta(n+2, xu+1) \geq 0$ for $x > -1$.

We reprove the result (1.2) in two other ways, and in so doing illustrate properties of the ζ function.

Corollary 1. *We have the following recurrence:*

$$\frac{(-1)^n}{(n+1)!} \Delta^{(n+1)}(x) = \frac{1}{x} \frac{(-1)^{n-1}}{n!} \Delta^{(n)}(x) - \frac{\zeta(n+1, x+1)}{(n+1)x}. \tag{1.3}$$

We then relate cases of (1.2) to fractional part integrals, including the following, wherein we let $\{x\} = x - [x]$ denote the fractional part of x.

Proposition 2. *Let $k \geq 1$ be an integer. Then we have*

$$\int_0^1 u^{n+1} \zeta(n+2, ku+1) du = \frac{1}{k^{n+2}} \left[\int_1^\infty \frac{\{w\}^{n+1}}{w^{n+2}} dw + \sum_{j=1}^{k-1} \int_0^\infty \frac{(\{x\}+j)^{n+1}}{(x+j+1)^{n+2}} dx \right]. \tag{1.4}$$

As a further special case we have

Corollary 2. *We have*

$$(-1)^n \Delta^{(n+1)}(1) = (n+1)! \int_0^1 y^{n+1} \zeta(n+2, y+1) dy = (n+1)! \int_0^\infty \frac{\{x\}^{n+1}}{(x+1)^{n+2}} dx$$
$$= (n+1)! \left[1 - \gamma - \sum_{j=2}^{n+1} \frac{1}{j} [\zeta(j) - 1] \right], \tag{1.5}$$

where $\zeta(s) = \zeta(s, 1)$ *is the Riemann zeta function* [7], [10], [15], [16].

More generally, we have the following, wherein we put $P_1(x) = \{x\} - 1/2$. Let ${}_2F_1$ be the Gauss hypergeometric function [3], [9].

Proposition 3. *We have for integers* $n \geq 0$

$$\int_0^1 u^{n+1} \zeta(n+2, xu+1) du = \frac{1}{2(n+2)} \frac{1}{(x+1)^{n+2}} {}_2F_1 \left(1, n+2; n+3; \frac{x}{x+1} \right)$$
$$+ \frac{1}{(n+1)(n+2)} \frac{1}{(x+1)^{n+1}} {}_2F_1 \left(1, n+1; n+3; \frac{x}{x+1} \right)$$
$$- \int_0^\infty \frac{1}{(t+1)} \frac{P_1(t)}{(t+x+1)^{n+2}} dt. \tag{1.6}$$

From this Proposition we may then determine the following asymptotic form:

Corollary 3. *We have*

$$\Delta^{(n+1)}(x) \sim (-1)^n \frac{n!}{(x+1)^{n+1}}, \quad x \to \infty, \tag{1.7}$$

in agreement with Corollary 1.2 of [2].

In fact, the proof shows how higher order terms may be systematically found.

Many expressions may be found for the ${}_2F_1$ functions in (1.6) and (2.20) below, and we present a sample of these in an Appendix.

A simple property of Δ is given in the following.

Proposition 4. *We have* (a)

$$\int_0^1 \Delta(x) dx = -\gamma + \sum_{k=2}^\infty \frac{(-1)^k}{k^2} \zeta(k), \tag{1.8a}$$

and (b)

$$\int_0^1 \Delta^2(x) dx = \gamma^2 - 2\gamma \sum_{k=2}^\infty \frac{(-1)^k}{k^2} \zeta(k) + \sum_{m=4}^\infty \frac{(-1)^m}{(m-1)} \sum_{\ell=2}^{m-2} \frac{\zeta(m-\ell)\zeta(\ell)}{(m-\ell)\ell}. \tag{1.8b}$$

Throughout we let $\psi^{(j)}$ denote the polygamma functions (e.g., [1]), and we note the relation for integers $n > 0$

$$\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x). \tag{1.9}$$

Therefore, as to be expected, (1.2) could equally well be written as an integral over $\psi^{(n+1)}(xu+1)$. The polygamma functions possess the functional equation

$$\psi^{(j)}(x+1) = \psi^{(j)}(x) + (-1)^j \frac{j!}{x^{j+1}}. \tag{1.10}$$

For a very recent development of single- and double-integral and series representations for the Gamma, digamma, and polygamma functions, [6] may be consulted.

Proof of Propositions

Proposition 1. *We provide two alternative proofs of this result. The result holds for $n = 0$, and for the first proof we proceed by induction. For the inductive step we have*

$$\Delta^{(n+2)}(x) = \frac{d}{dx}\Delta^{(n+1)}(x) = (-1)^n (n+1)! \int_0^1 u^{n+1} \frac{d}{dx}\zeta(n+2, xu+1)du$$
$$= (-1)^{n+1}(n+2)! \int_0^1 u^{n+2} \zeta(n+3, xu+1)du. \tag{2.1}$$

In the last step, we used $\partial_a \zeta(s,a) = -s\zeta(s+1,a)$.

We remark that this first method shows that (1.2) may be evaluated by repeated integration by parts, for we have

$$(n+1)! \int_0^1 u^{n+1}\zeta(n+2, xu+1)du = \frac{(-1)^n}{x^n} \int_0^1 u^{n+1} \left(\frac{\partial}{\partial u}\right)^n \zeta(2, xu+1)du. \tag{2.2}$$

Second method. By the product rule we have

$$\Delta^{(n+1)}(x) = \sum_{j=0}^{n+1} \binom{n}{j} [\ln\Gamma(x+1)]^{(n-j)} \frac{(-1)^j j!}{x^{j+1}}$$
$$= \sum_{j=0}^{n+1} \binom{n+1}{j} \psi^{(n-j)}(x+1) \frac{(-1)^j j!}{x^{j+1}}. \tag{2.3}$$

Here, it is understood that $\psi^{(-1)}(x) = \ln\Gamma(x)$. We now apply (1.9) and the integral representation

$$(n-j)!\zeta(n-j+1, x+1) = \int_0^\infty \frac{t^{n-j}e^{-xt}}{e^t - 1}dt, \tag{2.4}$$

so that

$$\begin{aligned}\Delta^{(n+1)}(x) &= (-1)^{n+1}\sum_{j=0}^{n+1}\frac{j!}{x^{j+1}}\int_0^\infty \frac{t^{n-j}e^{-xt}}{e^t-1}dt \\ &= \frac{(-1)^{n+1}}{x^{n+2}}\int_0^\infty \frac{e^{-xt}}{(e^t-1)}[e^{xt}\Gamma(n+2, xt) - (n+1)!]\frac{dt}{t},\end{aligned} \tag{2.5}$$

where the incomplete Gamma function $\Gamma(x, y)$ has the property [9] (p. 941)

$$\Gamma(n+1, x) = n!e^{-x}\sum_{m=0}^{n}\frac{x^m}{m!}. \tag{2.6}$$

Now we use a Laplace transform,

$$\int_0^1 u^{n+1}e^{-xut}du = \frac{1}{(xt)^{n+2}}[(n+1)! - \Gamma(n+2, xt)], \tag{2.7}$$

to write

$$\begin{aligned}\Delta^{(n+1)}(x) &= (-1)^n\int_0^\infty \frac{e^{-xt}}{(e^t-1)}t^{n+1}\int_0^1 u^{n+1}e^{-xtu}dudt \\ &= (-1)^n\int_0^1 u^{n+1}\int_0^\infty \frac{t^{n+1}}{e^t-1}e^{-xut}dtdu \\ &= -(-1)^n\int_0^1 u^{n+1}\zeta(n+2, xu+1)du.\end{aligned} \tag{2.8}$$

By absolute convergence and the Tonelli–Hobson theorem, the interchange of integrations is justified. In the last step, we applied the representation (2.4).

Corollary 1. *This is proved by integrating by parts in* (1.2).

Remark. It is possible to find explicit expressions for the values $\Delta^{(n+1)}$ $(j+1/2)$ with half-integer argument. This is due to the functional equation (1.10) along with the values $\psi^{(-1)}(1/2) = \ln\sqrt{\pi}$, $\psi(1/2) = -\gamma - 2\ln 2$, and [1] (p. 260)

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1}n!(2^{n+1}-1)\zeta(n+1), \quad n \geq 1. \tag{2.9}$$

We then obtain, for instance, by using (2.3)

$$\frac{\Delta^{(n+1)}\left(-\frac{1}{2}\right)}{(n+1)!} = \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j+1)}(2^{n+2}-2^{j+1})\zeta(n-j+1) + 2^{n+1}(\gamma+2\ln 2) - 2^{n+2}\ln\sqrt{\pi}. \tag{2.10}$$

Similarly, it is possible to find explicit expressions for the values $\Delta^{(n+1)}(j+1/4)$ and $\Delta^{(n+1)}(j+3/4)$ by using the corresponding values of $\psi^{(k)}$ [11].

Proposition 2. *We use two Lemmas.*

Lemma 1. *When the integrals involved are convergent, we have for integrable functions f and g*

$$\int_1^\infty f(\{x\})g(x)dx = \int_0^1 f(y)\sum_{\ell=1}^\infty g(y+\ell)dy. \tag{2.11}$$

Lemma 2. *For $b>0$, $\lambda>1$, and $c \geq 0$ we have for integrable functions f*

$$\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right)\frac{dx}{(x+c)^\lambda} = \frac{1}{b^{\lambda-1}}\int_0^1 f(y)\zeta(\lambda, y+c/b)dy. \tag{2.12}$$

This holds when the integrals are convergent.

Proof. For Lemma 1 we have

$$\int_1^\infty f(\{x\})g(x)dx = \sum_{\ell=1}^\infty \int_\ell^{\ell+1} f(\{x\})g(x)dx$$
$$= \sum_{\ell=1}^\infty \int_\ell^{\ell+1} f(x-\ell)g(x)dx = \sum_{\ell=1}^\infty \int_0^1 f(y)g(y+\ell)dy. \tag{2.13}$$

For Lemma 2 we first have

$$\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right)g(x)dx = b\int_0^\infty f(\{v\})g(bv)dv$$
$$= b\sum_{\ell=0}^\infty \int_\ell^{\ell+1} f(v-\ell)g(bv)dv = b\sum_{\ell=0}^\infty \int_0^1 f(y)g[b(y+\ell)]dy. \tag{2.14}$$

We now put $g(x) = 1/(x+c)^\lambda$, so that

$$\sum_{\ell=0}^\infty g[b(y+\ell)] = \frac{1}{b^\lambda}\sum_{\ell=0}^\infty \frac{1}{(y+\ell+c/b)^\lambda} = \frac{1}{b^\lambda}\zeta(\lambda, y+c/b). \tag{2.15}$$

$\square$

Proof of Proposition 2. We have for integers $k \geq 1$

$$\int_0^1 u^{n+1}\zeta(n+2,ku+1)du = \frac{1}{k^{n+2}}\int_0^k v^{n+1}\zeta(n+2,v+1)dv$$

$$= \frac{1}{k^{n+2}}\sum_{\ell=0}^{k-1}\int_\ell^{\ell+1} v^{n+1}\zeta(n+2,v+1)dv$$

$$= \frac{1}{k^{n+2}}\sum_{\ell=0}^{k-1}\int_0^1 (w+\ell)^{n+1}\zeta(n+2,w+\ell+1)dv. \tag{2.16}$$

We now apply Lemma 2 with $b=1$, $c=\ell+1$, and $f(w)=(w+\ell)^{n+1}$, giving

$$\int_0^1 u^{n+1}\zeta(n+2,ku+1)du = \frac{1}{k^{n+2}}\sum_{\ell=0}^{k-1}\int_0^\infty \frac{(\{x\}+\ell)^{n+1}}{(x+\ell+1)^{n+2}}dx$$

$$= \frac{1}{k^{n+2}}\left[\int_1^\infty \frac{\{w\}^{n+1}}{w^{n+2}}dw + \sum_{\ell=1}^{k-1}\int_0^\infty \frac{(\{x\}+\ell)^{n+1}}{(x+\ell+1)^{n+2}}dx\right]. \tag{2.17}$$

In the last step we used the periodicity $\{w-1\}=\{w\}$ for $w \geq 1$. □

For Corollary 2, we apply Lemma 2 of [6].

Proposition 3. *We start from the integral representation*

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s\int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}}dx, \quad Re\ s > -1. \tag{2.18}$$

Then

$$\int_0^1 u^{n+1}\zeta(n+2,xu+1)du = \int_0^1 u^{n+1}\left[\frac{1}{2(xu+1)^{n+2}} + \frac{1}{(n+1)(xu+1)^{n+1}}\right.$$
$$\left. -(n+2)\int_0^\infty \frac{P_1(t)dt}{(t+xu+1)^{n+3}}\right] du. \tag{2.19}$$

By using a standard integral representation of ${}_2F_1$ (e.g., [9], p. 1040 or [3] p. 65) we have

$$\int_0^1 u^{n+1}\zeta(n+2,xu+1)du = \frac{1}{2(n+2)}{}_2F_1(n+2,n+2;n+3;-x)$$
$$+\frac{1}{(n+1)(n+2)}{}_2F_1(n+1,n+2;n+3;-x) - \int_0^\infty \frac{1}{(t+1)}\frac{P_1(t)}{(t+x+1)^{n+2}}dt. \tag{2.20}$$

By applying a standard transformation rule [9] (p. 1043) to the ${}_2F_1$ functions, we obtain the Proposition.

Corollary 2. *We give the detailed asymptotic forms as $x \to \infty$ of the hypergeometric functions in* (1.6)*. We easily have that ${}_2F_1(1, n{+}1; n{+}3; 1) = n{+}2$ and these forms will then show that the corresponding term in* (1.6) *gives the leading term as $x \to \infty$. We let $(a)_j = \Gamma(a + j)/\Gamma(a)$ be the Pochhammer symbol. The following expansions are valid for $|z - 1| < 1$ and $|\arg(1 - z)| < \pi$:*

$$ {}_2F_1(1, y; 1{+}y; z) = y \sum_{k=0}^{\infty} \frac{(y)_k}{k!} [\psi(k + 1) - \psi(k + y) - \ln(1{-}z)](1{-}z)^k, \quad (2.21a) $$

and

$$ {}_2F_1(1, y; 2 + y; z) = y + 1 - y(y + 1) \sum_{k=0}^{\infty} \frac{(y + 1)_k}{k!} [\psi(k + 1) - \psi(k + y + 1) - \ln(1 - z)](1 - z)^{k+1}, \quad (2.21b) $$

where $(y)_0 = 1$. These expansions are the $n = 0$ and $n = 1$ cases of (9.7.5) *in* [12] *(p. 257), respectively. We put $y = n + 1$, $z = x/(x + 1)$, $\ln(1 - z) = -\ln(x + 1)$ and then find*

$$ {}_2F_1\left(1, n + 1; n + 3; \frac{x}{x + 1}\right) $$
$$ = n + 2 + (n + 1)(n + 2)[\gamma - \ln x + \psi(n + 2)]\frac{1}{x} + O\left(\frac{\ln x}{x^2}\right), \quad (2.22a) $$

and

$$ {}_2F_1\left(1, n{+}2; n{+}3; \frac{x}{x{+}1}\right) = -(n{+}2)[\gamma - \ln x + \psi(n + 2)] + O\left(\frac{\ln x}{x}\right). \quad (2.22b) $$

The integral term in (1.6) *is at most $O[(x{+}1)^{-(n+2)}]$, and is actually much smaller due to cancellation within the integrand, and the Corollary then follows.*

Remarks. Of course we have from (1.2) $\Delta^{(n+1)}(0) = (-1)^n(n + 1)!\zeta(n + 2)/(n + 2)$, in agreement with the expansion (2.27). This special case is recovered from Proposition 3 in the following way. We have ${}_2F_1(a, b; c; 0) = 1$ and the representation [16] (p. 14)

$$ \zeta(s) = \frac{1}{s - 1} + \frac{1}{2} - s \int_1^{\infty} \frac{P_1(x)}{x^{s+1}} dx, \quad (2.23) $$

the $a = 1$ case of (2.18), giving the identity $\int_0^1 u^{n+1}\zeta(n{+}2, 1)du = \zeta(n{+}2)/(n{+}2)$.

In connection with Propositions 2 and 3, another representation that might be employed is [7]

$$\ln \Gamma(x+1) = \left(x + \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi - \int_0^\infty \frac{P_1(t)}{t+x} dt. \tag{2.24}$$

We may note that representations (2.19) or (2.20), for instance, provide another basis for proving integral representations for $(-1)^n \Delta^{(n+1)}(x)$ by induction. When using (2.20), we use the derivative property

$$\frac{d}{dx} {}_2F_1(a, b; c; -x) = -\frac{ab}{c} {}_2F_1(a+1, b+1; c+1; -x). \tag{2.25}$$

The ${}_2F_1$ function in (1.6) can be written in other ways, including using a transformation formula [9] (p. 1043), so that

$${}_2F_1\left(1, n+2; n+3; \frac{x}{x+1}\right) = (x+1) {}_2F_1(1, 2; n+3; -x). \tag{2.26}$$

Proposition 4. *The result uses the expansion* [9] *(p. 939)*

$$\ln \Gamma(x+1) = -\gamma x + \sum_{k=2}^\infty \frac{(-1)^k}{k} \zeta(k) x^k. \tag{2.27}$$

Remark. Let $\mathrm{Ei}(x)$ be the exponential integral function (e.g., [9], p. 925). Given the relations [9] (pp. 927, 942)

$$\Gamma(0, x) = -\mathrm{Ei}(-x) = -\left(\gamma + \ln x + \sum_{k=1}^\infty \frac{(-x)^k}{k k!}\right), \tag{2.28}$$

it is possible to write

$$\int_0^1 \Delta(x) dx = -\gamma - \int_0^\infty \frac{[\gamma - t + \Gamma(0, t) + \ln t]}{t(e^t - 1)} dt. \tag{2.29}$$

This follows by inserting a standard integral representation for the values $\zeta(k)$ into the right side of (1.8a).

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Appendix

Here we present illustrative relations for the sort of hypergeometric functions appearing in (1.6) and (2.20).

The contiguous relations [9] (pp. 1044–45) may be readily applied. As well, we have for instance [9] (p. 1043)

$$ {}_2F_1(n+2,n+2;n+3;-x) = (1+x)^{-(n+1)} {}_2F_1(1,1;n+3;-x). \tag{A.1}$$

The next result provides a type of recurrence relation in the first parameter of the ${}_2F_1$ function.

Proposition A1. *For integers $n \geq -1$ we have*

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^{n+2}} du = \frac{1}{(n+2)} {}_2F_1(n+2,n+2;n+3;-x)$$
$$= \frac{1}{(n+1)}\left[\frac{1}{(x+1)^{n+1}} - \frac{1}{(n+2)} {}_2F_1(n+1,n+2;n+3;-x)\right]. \tag{A.2}$$

Proof. With $v = xu$ in (A.2), we have

$$\frac{1}{x^{n+2}}\int_0^x \frac{v^{n+1}}{(v+1)^{n+2}} dv = \frac{1}{x^{n+2}}\int_0^x [(v+1)-v]\frac{v^{n+1}}{(v+1)^{n+2}} dv$$
$$= \frac{1}{x^{n+2}}\left[\int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv - \int_0^x \frac{v^{n+2}}{(v+1)^{n+2}} dv\right]$$
$$= \frac{1}{x^{n+2}}\left[\int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv - \frac{(n+2)}{(n+1)}\int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv + \frac{1}{(n+1)}\frac{x^{n+2}}{(x+1)^{n+1}}\right], \tag{A.3}$$

where we integrated by parts. Using a standard integral representation for ${}_2F_1$ [9] (p. 1040) leads to the Proposition. □

Proposition A1 may be iterated in the first parameter of the ${}_2F_1$ function. Then the following relation may be applied:

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = \frac{1}{1+x} - \left(\frac{n+1}{n+2}\right) {}_2F_1(1,n+2;n+3;-x). \tag{A.4}$$

The ${}_2F_1$ functions of concern here may be written with one or more terms containing $\ln(x+1)$. One way to see this is the following. We have for the function of (A.4), first integrating by parts,

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = -\frac{1}{x}\left[(n+1)\int_0^1 \frac{u^n}{xu+1} du + \frac{1}{x+1}\right]$$

$$= -\frac{1}{x}\left[\frac{(n+1)}{x^{n+1}}\int_0^x \frac{v^n}{v+1}dv + \frac{1}{x+1}\right]$$
$$= -\frac{1}{x}\left[\frac{(n+1)}{x^{n+1}}\int_0^x \frac{[1-(1-v^n)]}{v+1}dv + \frac{1}{x+1}\right]$$
$$= -\frac{1}{x}\left\{\frac{(n+1)}{x^{n+1}}\left[\ln(x+1) - \int_0^x \frac{(1-v^n)}{v+1}dv\right] + \frac{1}{x+1}\right\}. \quad \text{(A.5)}$$

For $0 \le x \le 1$ we may note the following simple inequality for the integral of (A.5):

$$\int_0^x \frac{(1-v^n)}{v+1}dv \le \int_0^x (1-v^n)dv \le x. \quad \text{(A.6)}$$

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