A characterization of finite supersolvable groups

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Abstract. We give a characterization of supersolvable groups, generalizing those by Huppert and Kramer. At the end, we give an application of this result.

1. Introduction

All groups considered in this paper will be finite. We use conventional notion and notations from HUPPERT [7]. G denotes a finite group; $M \lessdot G$ means that M is a maximal subgroup of G; G_p is a Sylow p-subgroup of G and $\Phi(G)$ is the Frattini subgroup of G. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ *a formation* provided that (i) $G \in \mathcal{F}$ and $H \triangleleft G$ implies $G/H \in \mathcal{F}$, and (ii) G/M and G/N are in $\mathcal{F}$ implies that $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is referred to as *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (ref. [7, p. 713, Satz 8.6]).

For the characterizations of a supersolvable group, the most famous result is Huppert's Theorem: a finite group G is supersolvable if and only if every maximal subgroup of G has prime index in G ([7] or [19]). Kramer generalized Huppert's result in the solvable case by proving that the supersolvability of a solvable group G can be concluded if only those maximal subgroups of G not containing the Fitting subgroup $F(G)$ have prime index in G: a finite solvable

group G is supersolvable if and only if, for any maximal subgroup M of G, $|F(G) :
F(G) \cap M| = 1$ or a prime. The main aim of this paper is to go further to consider
the general case by dropping the hypothesis of solvability of G in Kramer's result.
We give a new characterization of supersovable groups which generalizes both
Huppert's and Kramer's characterizations. As applications of our main result,
some new interesting results under the assumption that all maximal subgroup of
any Sylow subgroups of some normal subgroups of G are well-suited in G are given.
We know that the Fitting subgroup $F(G)$ of G is an important concept in the
study of solvable groups. In [14], the subgroup $\widetilde{F}(G)$ of G was introduced, where
$\widetilde{F}(G)$ satisfies $\widetilde{F}(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$. It is easy to see that $\widetilde{F}(G) = F(G)$
when G is solvable. We now define a sequence of subgroups $\{\widetilde{F}_i(G)\}$ of G by the
rule that

$$\widetilde{F}_1(G) = \widetilde{F}(G), \widetilde{F}_i(G)/\widetilde{F}_{i-1}(G) = \widetilde{F}(G/\widetilde{F}_{i-1}(G)) \quad \text{for } i > 1.$$

Obviously $\widetilde{F}_n(G)$ is a generalization of $F_n(G)$, the Fitting subgroup of degree n
([1]). Since G is finite and $\widetilde{F}_i(G) > \widetilde{F}_{i-1}(G)$, there exists an integer m such that
$\widetilde{F}_m(G) = G$. Our main results are as follows.

Theorem 1.1. *Suppose that G is a finite group. Then G is supersolvable
if and only if, for any maximal subgroup M of G, there holds that $|\widetilde{F}(G) :
\widetilde{F}(G) \cap M| = 1$ or a prime.*

By Theorem 1.1, we give a generalization of [19, Ch 1, Corollary 3.2].

Corollary 1.2. *Suppose that G is a finite group. Then G is supersolvable
if and only if, for each maximal subgroup M of G and each normal subgroup H
of G, there holds that $|H : H \cap M| = 1$ or a prime.*

The following result is a slight generalization of Huppert's theorem.

Theorem 1.3. *Suppose that G is a finite group. Then G is supersolvable if
and only if there exists a normal subgroup H of G such that G/H is supersolvable
and, for any maximal subgroup M of G, there holds that $|H : H \cap M| = 1$ or a
prime.*

Applying Theorem 1.1 and Theorem 1.3, we can obtain

Theorem 1.4. *Suppose that G is a finite group Then G is supersolvable if
and only if there exists a positive integer n such that, for any maximal subgroup
M of G, $|\widetilde{F}_n(G) : \widetilde{F}_n(G) \cap M| = 1$ or a prime.*

Corollary 1.5. *Suppose that G is a solvable group. Then G is supersolvable if and only if there exists a positive integer n such that, for any maximal subgroup M of G, $|F_n(G) : F_n(G) \cap M| = 1$ or a prime.*

Remark 1.6. Both Huppert and Kramer's characterizations can be derived from our Theorem 1.4: if n is sufficient large such that $\widetilde{F}_n(H) = G$, then our Theorem 1.4 is precisely Huppert's theorem; if G is solvable and $n = 1$, then our Theorem 1.4 is precisely Kramer's theorem.

Remark 1.7. We know that the Fitting subgroup $F(G)$ of G was usually generalized to as $F^*(G)$, the unique maximal normal quasinilpotent subgroup of G ([8]), which has played an important role in the proof of the theorem of the classification of finite simple groups ([4]). The definition and important properties of $F^*(G)$ can be found in [8, X, 13]. Then some natural questions arise, what is the relation of $F^*(G)$ and $\widetilde{F}(G)$? Does it hold if we replace $\widetilde{F}(G)$ by $F^*(G)$ in our Theorem 1.1? We shall show that $F^*(G) \leq \widetilde{F}(G)$ (see Lemma 2.1 (2) below). The following example indicates that $F^*(G) \neq \widetilde{F}(G)$; and the answer to the second question is negative in general.

Example 1.8. Suppose that N is a non-trivial 2-Frattini-module of A_5 (ref. [5]). Denote the Frattini 2-elementary A_5-extension via N by G. Then we have $\Phi(G) = N$ and $G/N \cong A_5$. Hence $\widetilde{F}(G) = G$. Since $N \leq F(G)$, $F^*(G) = N$ or G. If $F^*(G) = G$, then every chief factor of G is cyclic or non-abelian ([8, X, 13]). By the Maschke Theorem ([7, Satz I.17.5]), $N = \mathrm{Soc}(G) = N_1 \times \cdots \times N_t$, where N_i is 2-chief factor of G below N. So N_i is of order 2 and then $N_i \leq Z(G)$. This implies N is a trivial A_5-module, contrary to the choice. So $F^*(G) = N$ and $F^*(G) \neq \widetilde{F}(G)$. For any maximal subgroup M of G, we have $|F^*(G) : F^*(G) \cap M| = 1$. But G is not solvable.

The following is a similar example.

Example 1.9. Suppose that G is a non-split extension $(Z_2)^3 L_3(2)$ of an elementary abelian subgroup $(Z_2)^3$ of order 2^3 by $L_3(2)$ (ref. [2, page 61]). Then $\widetilde{F}(G) = G$, but $F^*(G) = F(G) = \Phi(G) = (Z_2)^3$.

Two subgroups H and K of G are said to permute if $HK = KH$. A subgroup of G is said to be *quasinormal* in G if it permutes with every subgroup of G, a subgroup of G is called *S-quasinormal* (or *π-quasinormal* or *s-permutable*) in G if it permutes with every Sylow subgroup of G ([9]). A subgroup H of G is said to be *c-supplemented* in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = \mathrm{Core}_G(H)$ ([1]). Recently, SKIBA in [15] introduced the following

concept, which covers both s-permutability and c-supplementedness: a subgroup H of G is called *weakly s-supplemented* in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \le H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G.

2. The proofs

We first give two lemmas.

Lemma 2.1. (1) *Suppose that* N *is a normal subgroup of* G *contained in* $\Phi(G)$. *Then* $\widetilde{F}(G/N) = \widetilde{F}(G)/N$. (2) $F(G) \le F^*(G) \le \widetilde{F}(G)$. (3) $C_G(\widetilde{F}(G)) \le \widetilde{F}(G)$.

PROOF. (1) Noticing that $\Phi(G/N) = \Phi(G)/N$, we can easily obtain (1).

(2) Obviously, $F(G) \le F^*(G)$. Now we prove that $F^*(G) \le \widetilde{F}(G)$. Let $E(G)$ be the layer of G (ref. [8, X. 13]) and let M be a maximal subgroup of G such that $Z(E(G)) \not\le M$. Then $G = MZ(E(G))$. Therefore $E(G) = Z(E(G))(M \cap E(G))$. Thus $E(G) = (E(G))' = (M \cap E(G))' \le M$, contrary to $Z(E(G)) \not\le M$. This shows that $Z(E(G)) \le \Phi(G)$. If $E(G)$ is solvable, then obviously $F^*(G) = F(G) \le \widetilde{F}(G)$. Hence suppose that $E(G)$ is not solvable. Then $E(G)/Z(E(G))$ is a direct product of non-abelian simple groups (ref. [8, X, 13. 18]). Hence $E(G)/Z(E(G))$ can be written as a direct product of minimal normal subgroups of $G/Z(E(G))$. Therefore $F(G)E(G)/\Phi(G)$ is a direct product of minimal normal subgroups of $G/\Phi(G)$. Hence $F^*(G)/\Phi(G) = F(G)E(G)/\Phi(G) \le \mathrm{Soc}(G/\Phi(G))$. It follows that $F^*(G) = F(G)E(G) \le \widetilde{F}(G)$.

(3) By [8, X, 13. 12], we know that $C_G(F^*(G)) \le F^*(G)$. Applying (2), we have $C_G(\widetilde{F}(G)) \le \widetilde{F}(G)$. $\square$

Lemma 2.2 ([6, Lemma 3.5]). *Suppose that* G *is a group with two maximal subgroups with different prime indices in* G. *Then* G *is not a simple group.*

PROOF OF THEOREM 1.1. By Kramer's theorem, we only need to prove the sufficiency. Suppose that it is false and G is a counter-example with minimal order.

Step 1. $\Phi(G) = 1$, hence $\widetilde{F}(G) = Soc(G)$. Otherwise, consider the factor group $G/\Phi(G)$. It follows from Lemma 2.1(1) that $G/\Phi(G)$ would satisfy the hypothesis of the theorem. The minimality of G implies that $G/\Phi(G)$ is supersolvable. Since $\mathcal{U}$ is saturated, G is supersolvable, a contradiction.

Step 2. $\widetilde{F}(G)$ is solvable, hence $\widetilde{F}(G) = F(G)$. If $\widetilde{F}(G) = \mathrm{Soc}(G)$ is not solvable, then there is a non-solvable minimal normal subgroup N of G contained

in $\widetilde{F}(G)$. Denote $N = N_1 \times \cdots \times N_s$, where $N_1, \ldots, N_s$ are conjugated non-abelian simple groups. Take a maximal subgroup M of G such that $N \nleq M$. Then $G = NM$ and $|NM : M| = |N : N \cap M|$. Since $\widetilde{F}(G) = N(\widetilde{F}(G) \cap M)$, $|\widetilde{F}(G) : \widetilde{F}(G) \cap M| = |N(\widetilde{F}(G) \cap M) : \widetilde{F}(G) \cap M| = |N : N \cap M|$. So $|N : N \cap M|$ is a prime by the hypothesis of the theorem. Therefore there exists N_i such that $N_i \nleq M$. It follows that $|N_i : N_i \cap M| = |N_i(N \cap M) : N \cap M| = |N : N \cap M|$ is a prime. Assume that $|N : N \cap M| = p$, where p is a prime. Then N_i is a non-abelian simple group with a maximal subgroup of index p in N_i. On the other hand, by the Frattini's argument we have $G = N_G(N_p)N$. Take a maximal subgroup K of G such that $N_G(N_p) \leq K$. With the similar argument, we can see that there exists N_j such that $|N_j : N_j \cap K|$ is a prime q. By the choice of K, we have $q \neq p$. Since $N_i \cong N_j$, we have a non-abelian simple group with two maximal subgroups having different indices p and q. By Lemma 2.2, this is impossible. Thus $\widetilde{F}(G)$ is solvable and then $\widetilde{F}(G) = F(G)$ by Lemma 2.1(2).

Step 3. Final contradiction. By Steps 1 and 2, we have $\widetilde{F}(G) = \mathrm{Soc}(G) = R_1 \times \cdots \times R_s$, where all R_i are solvable minimal normal subgroups of G. Take an arbitrary minimal normal subgroup R_i of G contained in $F(G)$. It follows from Step 1 that there is some maximal subgroup M of G such that $G = R_iM$. Obviously, $R_i \cap M = 1$. Hence $|R_i| = |R_i : R_i \cap M| = |\widetilde{F}(G) : \widetilde{F}(G) \cap M|$ is a prime by the hypothesis of the theorem. Therefore $\widetilde{F}(G) = F(G)$ is a direct product of some minimal normal subgroups of G of prime order. Hence $\widetilde{F}(G) \leq Z_{\mathcal{U}}(G)$, the $\mathcal{U}$-hypercenter of G (ref. [3, pp. 389]).

Because $G/C_G(R_i)$ is isomorphic to a subgroup of $Aut(R_i)$, $G/C_G(R_i)$ is cyclic and so is supersolvable. This implies that $G/\cap_{i=1}^{s} C_G(R_i) = G/C_G(\widetilde{F}(G))$ is supersolvable. Since $C_G(\widetilde{F}(G)) \leq \widetilde{F}(G)$ by Lemma 2.1 (3), we have $G/\widetilde{F}(G)$ is supersolvable. We have

$$G/\widetilde{F}(G) = Z_{\mathcal{U}}(G/\widetilde{F}(G)) = Z_{\mathcal{U}}(G)/\widetilde{F}(G).$$

Then $G = Z_{\mathcal{U}}(G)$ and G is supersolvable, a contradiction. These complete the proof of the theorem. $\qquad\square$

PROOF OF COROLLARY 1.2. "Sufficiency" Let H be $\widetilde{F}(G)$. Then G is supersolvable by Theorem 1.1. "Necessity" Let G be supersolvable and M a maximal subgroup of G and H a normal subgroup of G. If M contains H, then $[H : H \cap M] = 1$. If M does not contain H, then $G = HM$ and $[H : H \cap M] = [HM : M] = [G : M]$ is a prime by Huppert's theorem. This completes the proof of the theorem. $\qquad\square$

PROOF OF THEOREM 1.3. Let H be G. Then G is supersolvable by Huppert's theorem. Now we show the sufficiency. Suppose that it is false and G is a counter-example with minimal order.

Step 1. The minimal normal subgroup G, N say, is unique and G/N is supersolvable and $\Phi(G) = 1$. Suppose that N is a minimal normal subgroup G. Consider the factor groups G/N and HN/N. Since $(G/N)/(HN/N) \cong G/HN \cong (G/H)/(HN/H)$, we have $(G/N)/(HN/N)$ is supersolvable. For any maximal subgroup M/N of G/N, since $|HN/N : HN/N \cap M/N| = |HN : HN \cap M| = |HN : (H \cap M)N|$ is a factor of $|H : H \cap M|$, $|HN/N : HN/N \cap M/N| = 1$ or a prime. Hence G/N satisfies the hypotheses. By the minimal choice of G, we have G/N is supersolvable. It is easy to see that N is the unique minimal subgroup of G and $\Phi(G) = 1$

Step 2. G is solvable. If G is not solvable, then N is not solvable. Denote $N = N_1 \times \cdots \times N_s$, where $N_1, \ldots, N_s$ are conjugated non-abelian simple groups. Take a maximal subgroup M of G such that $N \not\leq M$. Then $G = NM = HM$. Then $|N : N \cap M| = |NM : M| = |HM : M| = |H : H \cap M|$ is a prime by the hypothesis of the theorem. Now repeat the same arguments in the proof of Step 2 of Theorem 1.1, we get a contradiction.

Step 3. The final contradiction. By Step 2 we know that N is an elementary abelian group. Since $\Phi(G) = 1$, there exist a maximal subgroup M of G such that $N \not\leq M$. Then $G = NM$ and $N \cap M = 1$. So $|N| = |N : N \cap M| = |NM : M| = |HM : M|$ is a prime. Hence G is supersolvable, the final contradiction. This completes the proof of Theorem 1.3. $\qquad\square$

PROOF OF THEOREM 1.4. If $n = 1$, by Theorem 1.1, we have that G is supersolvable. Thus suppose that $n \geq 2$. Denote $N = \widetilde{F}_{n-1}(G)$ and consider the factor group $\overline{G} = G/N$. Since $\widetilde{F}(\overline{G}) = \widetilde{F}(G/N) = \widetilde{F}_n(G)/N$, applying Theorem 1.1 to $\overline{G}$ we have $\overline{G}$ is supersolvable. So the homomorphic image $G/\widetilde{F}_n(G)$ of $\overline{G} = G/\widetilde{F}_{n-1}(H)$ is supersolvable. Applying Theorem 1.3 to G and $\widetilde{F}_n(G)$ we have that G is supersolvable. This completes the proof of Theorem 1.4. $\qquad\square$

PROOF OF COROLLARY 1.5. obvious. $\qquad\square$

3. Applications

Recently, many people have presented a lot of new characterizations of supersolvable groups by giving conditions to certain subgroups of $F^*(G)$, for examples, [11], [12], [13], [18], etc. Now we consider a similar problem in $\widetilde{F}(G)$ and give

a theorem which is analogous with the results in the papers mentioned above.

We first collect some properties of weakly s-supplemented subgroups.

Lemma 3.1 ([15, Lemma 2.10]). *Let U be a weakly s-supplemented subgroup of G and N a normal subgroup of G. Then*

(1) *If $U \leq H \leq G$, then U is weakly s-supplemented in H.*

(2) *If $N \leq U$, then U/N is weakly s-supplemented in G/N.*

(3) *Let π be a set of primes, U a π-subgroup and N a π'-subgroup. Then UN/N is weakly s-supplemented in G/N.*

Lemma 3.2. *Let G be a group, p the smallest prime dividing $|G|$ and G_p a Sylow p-subgroup of G. If every maximal subgroup of G_p is weakly s-supplemented in G, then $G/O_p(G)$ is p-nilpotent; in particular, G is solvable.*

PROOF. If $O_p(G) = G_p$, then obviously $G/O_p(G)$ is p-nilpotent. So we assume that $O_p(G) < G_p$. For any maximal subgroup $T/O_p(G)$ of $G_p/O_p(G)$, by Lemma 3.1(2) and the hypothesis, we have that $T/O_p(G)$ is weakly s-supplemented in $G/O_p(G)$. Hence there exists a subgroup $K/O_p(G)$ such that

$$G/O_p(G) = T/O_p(G) \cdot K/O_p(G) \quad \text{and} \quad T/O_p(G) \cap K/O_p(G) \leq (T/O_p(G))_{s\overline{G}}.$$

Since $(T/O_p(G))_{s\overline{G}} \leq O_p(G/O_p(G)) = 1$, we know that every maximal subgroup of $G_p/O_p(G)$ is complemented in $G/O_p(G)$. Now from [16, Theorem 3.1 and Corollary 3.2], our lemma follows. $\square$

Theorem 3.3. *Suppose that G is a group. If all maximal subgroups of all Sylow subgroups of $\widetilde{F}(G)$ are weakly s-supplemented in G, then G is supersolvable.*

PROOF. Assume that the theorem is not true and let G be a counterexample of minimal order.

(1) $\Phi(G) = 1$, hence $\widetilde{F}(G) = \mathrm{Soc}(G)$: Assume that $\Phi(G) \neq 1$. Then there exists a prime p such that $p\|\Phi(G)|$. Let $P_0 \in \mathrm{Syl}_p(\Phi(G))$. Then $P_0 \lhd G$. By Lemma 2.1(1), we have $\widetilde{F}(G/P_0) = \widetilde{F}(G)/P_0$. Let P_1/P_0 be a maximal subgroup of the Sylow p-subgroup of $\widetilde{F}(G)/P_0$. Then P_1 is a maximal subgroup of the Sylow p-subgroup of $\widetilde{F}(G)$. Hence P_1 is weakly s-supplemented in G by hypothesis. Thus P_1/P_0 is weakly s-supplemented in G/P_0 by Lemma 3.1(2). Let Q^*/P_0 be a maximal subgroup of the Sylow q-subgroup of $\widetilde{F}(G)/P_0$, where $q \neq p$. It is clear that $Q^* = Q_1 P_0$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $\widetilde{F}(G)$. Then Q_1 is weakly s-supplemented in G by hypothesis, therefore $Q_1 P_0/P_0$ is weakly s-supplemented in G/P_0 by Lemma 3.1(3). Hence, we have proved that

G/P_0 satisfies the hypothesis of the theorem. Therefore G/P_0 is supersolvable by minimal choice of G. Since $P_0 \leq \Phi(G)$ and $\mathcal{U}$ is a saturated formation, we have G is supersolvable, a contradiction.

(2) $\widetilde{F}(G) = F(G) < G$: suppose that p is the smallest prime in $\pi(\widetilde{F}(G))$. We know every maximal subgroup of any Sylow p-subgroup of $\widetilde{F}(G)$ is weakly s-supplemented in G, thus in $\widetilde{F}(G)$ by Lemma 3.1(2). Applying Lemma 3.2 in $\widetilde{F}(G)$, we have $\widetilde{F}(G)$ is solvable. Hence $\widetilde{F}(G) = F(G)$. Obviously, $\widetilde{F}(G) < G$. Hence (2) holds.

(3) $\widetilde{F}(G) = F(G) = R_1 \times \cdots \times R_t$, where every R_i is a minimal normal subgroup of G of prime order. By (1) we may and shall assume that $\widetilde{F}(G) = F(G) = R_1 \times \cdots \times R_t$, where every R_i is a minimal normal subgroup of G. We now prove that all R_i are of prime order. Suppose that there exists an index i such that R_i is not of prime order. Without loss of generality, we may and shall suppose that $i = 1$ and P is the Sylow subgroup of $\widetilde{F}(G)$ containing R_1. We write P for $R_1 \times \cdots \times R_s$, where all the $R_j (j = 1, 2, \ldots, s)$ are minimal normal subgroups of G for some $s \leq t$. Pick a maximal subgroup R_1^* of R_1 such that R_1^* is normal in G_p. Then $P^* = R_1^* R_2 \ldots R_s$ is a maximal subgroup of P. By the hypothesis of the theorem, P^* is weakly s-supplemented in G. Then there exists a subgroup K such that $G = P^* K$ and $P^* \cap K \leq (P^*)_{sG}$. By an easy calculation we have $(P^*)_{sG} = (R_1^*)_{sG} R_2 \ldots R_s$. Denote $K_1 = K R_2 \ldots R_s$. Then $G = R_1^* K_1$ and

$$R_1^* \cap K_1 = R_1^* \cap K_1 \cap P^* = R_1^* \cap R_2 \ldots R_s (K \cap P^*) \leq R_1^* \cap (P^*)_{sG} = (R_1^*)_{sG}.$$

Since $R_1 \cap K_1$ is normal in G, we have $R_1 \cap K_1 = 1$ or R_1 by the minimality of R_1. If $R_1 \cap K_1 = 1$, then $R_1 = R_1 \cap G = R_1^*(R_1 \cap K_1) = R_1^*$, a contradiction. Hence $R_1 \cap K_1 = R_1$, that is, $R_1 \leq K_1$. Then $R_1^* = (R_1^*)_{sG}$ is s-permutable in G. So $N_G(R_1^*) \geq O^p(G)$. By the choice of R_1^*, we have R_1^* is normal in G. By the minimality of R_1 we get that $R_1^* = 1$ and R_1 is of prime order, as desired.

(4) The final step.

For any maximal subgroup M of G, if $\widetilde{F}(G) = F(G) \leq M$, then $|F(G) : F(G) \cap M| = 1$. If $\widetilde{F}(G) = F(G) \nleq M$, then there exists a minimal normal subgroup R_i of G contained in $F(G)$ such that $G = R_i M$. Then $|F(G) : F(G) \cap M| = |F(G)M : M| = |R_i M : M| = |R_i : R_i \cap M| = |R_i| = $ a prime. Hence, by Theorem 1.1, we have that G is supersolvable, the final contradiction. $\square$

Corollary 3.4. *Let G be a group. If all maximal subgroups of every Sylow subgroup of $\widetilde{F}(G)$ are either S-quasinormal or c-supplemented in G, then G is supersolvable.*

4. Final remark

Consider our Theorem 1.1 and the main result of [17, Theorem 3.1], we naturally have the following conjecture.

Conjecture 1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that H is a normal subgroup of G such that $G/H \in \mathcal{F}$. If, for any maximal subgroup M of G, there holds that $|\widetilde{F}(H) : \widetilde{F}(H) \cap M| = 1$ or a prime, then $G \in \mathcal{F}$.*

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