Publ. Math. Debrecen
80/3-4 (2012), 369–384
DOI: 10.5486/PMD.2012.5052

Kähler and para-Kähler curvature Weyl manifolds

By PETER GILKEY (Eugene) and STANA NIKČEVIĆ (Belgrade)

Abstract. We show that the Weyl structure of an almost pseudo-Hermitian Weyl manifold of dimension $n \geq 6$ is an exact Weyl structure if the associated curvature operator satisfies the Kähler identity. Similarly if the curvature of an almost para-Hermitian Weyl manifold of dimension $n \geq 6$ satisfies the para-Kähler identity, then the Weyl structure is an exact Weyl structure as well.

1. Introduction

1.1. Pseudo-Riemannian Weyl geometry. Let N be a smooth manifold of dimension $n \geq 3$. Let ∇ be a torsion free connection on the tangent bundle TN of N and let g be a pseudo-Riemannian metric on N of signature (p, q). Motivated by the seminal paper of WEYL [26], the triple $\mathcal{W} := (N, g, \nabla)$ is said to be a *Weyl manifold* if there exists a smooth 1-form $\phi_{\nabla,g} \in C^\infty(T^*N)$ so that:

$$\nabla g = -2\phi_{\nabla,g} \otimes g. \tag{1.a}$$

WEYL [26] used these geometries in an attempt to unify gravity with electromagnetism – although this approach failed for physical reasons, the resulting geometries are still an active area of investigation today. We refer, for example, to [6] which studies Weyl geometry in the context of contact manifolds, to [16] where Einstein–Weyl structures are examined in Lorentzian signature, to [17] where projectively flat Weyl manifolds are investigated, and to [25] where the associated mass of an asymptotically flat Weyl structure is defined.

Mathematics Subject Classification: 53B05, 15A72, 53A15, 53B10, 53C07, 53C25.
Key words and phrases: Kaehler–Weyl curvature manifolds, Weyl geometry, Hermitian–Weyl geometry, para-Hermitian–Weyl geometry, Singer–Thorpe theorem, van Hecke–Tricerri theorem, Higa splitting, geometric realizability, curvature decomposition.

Let $[g]$ be the associated conformal class; $g_1 \in [g]$ if and only if there exists a smooth function f so $g_1 = e^{2f} g$. Weyl geometry is linked with conformal geometry as equation (1.a) means that $[g]$ is preserved by covariant differentiation. If $g_1 \in [g]$ and if $\mathcal{W} = (N, g, \nabla)$ is a Weyl manifold, then the triple $\mathcal{W}_1 := (N, g_1, \nabla)$ is again a Weyl manifold where the associated 1-form is given by taking $\phi_{\nabla, g_1} := \phi_{\nabla, g} - df$. We say the Weyl structure is *an exact Weyl structure* or is *trivial* if there exists $g_1 \in [g]$ so that $\nabla = \nabla^{g_1}$ is the Levi–Civita connection of the metric g_1; additional equivalent conditions are given below in Theorem 1.1.

Let $\mathcal{R}$ be the *curvature operator* and let R be the associated *curvature tensor* of the connection ∇ of a Weyl manifold $\mathcal{W} = (N, g, \nabla)$:

$$\begin{aligned} \mathcal{R}(x, y) &:= \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}, \\ R(x, y, z, w) &:= g(\mathcal{R}(x, y)z, w). \end{aligned}$$

We always have the symmetry

$$R(x, y, z, w) = -R(y, x, z, w) \tag{1.b}$$

Since ∇ is torsion free, we also have the symmetries:

$$0 = R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w). \tag{1.c}$$

The Ricci tensor is defined by setting:

$$\mathrm{Ric}(x, y) := \mathrm{Tr}\{z \to \mathcal{R}(z, x)y\}. \tag{1.d}$$

There is an additional well known curvature symmetry which pertains in Weyl geometry (see, for example, the discussion in [8]):

$$R(x, y, z, w) + R(x, y, w, z) = \frac{2}{n}\{\mathrm{Ric}(y, x) - \mathrm{Ric}(x, y)\} g(z, w). \tag{1.e}$$

If the Weyl structure is an exact Weyl structure, then $\nabla = \nabla^{g_1}$ for some $g_1 \in [g]$ and we have the additional curvature symmetry for the curvature R^{g_1} of the Levi–Civita connection:

$$R^{g_1}(x, y, z, w) + R^{g_1}(x, y, w, z) = 0. \tag{1.f}$$

We say that the curvature of $\mathcal{W}$ is *Riemannian* if in addition to the symmetries of equation (1.b) and of equation (1.c), the symmetry of equation (1.f) is satisfied – note that these 3 symmetries are conformal invariants and that equation (1.f) implies equation (1.e). We have the following curvature condition which ensures that the Weyl structure is an exact Weyl structure (see, for example, [8], [14]); we give the proof for the sake of completeness in Section 2.1.

Theorem 1.1. *Let $\mathcal{W} = (N, g, \nabla)$ be a Weyl manifold with $H^1(N;\mathbb{R}) = 0$. The following assertions are equivalent and if any is satisfied, then the Weyl structure is an exact Weyl structure.*

(1) $d\phi_{\nabla,g} = 0$.

(2) $\nabla = \nabla^{g_1}$ *for some* $g_1 \in [g]$.

(3) $\nabla = \nabla^{g_1}$ *for some pseudo-Riemannian metric* g_1.

(4) *The curvature of* ∇ *is Riemannian.*

1.2. Almost para/pseudo-Hermitian Weyl geometry. Let $n = 2\bar{n} \geq 4$. We say that (N, g, ∇, J_-) is an *almost pseudo-Hermitian Weyl manifold* if (N, g, ∇) is a Weyl manifold, if J_- is an *almost complex structure* on TN (i.e. J_- is an endomorphism of TN with $J_-^2 = -\operatorname{id}$), and if $J_-^* g = g$; necessarily g has signature $(2\bar{p}, 2\bar{q})$ in this instance. Similarly, we say that (N, g, ∇, J_+) is an *almost para-Hermitian Weyl manifold* if (N, g, ∇) is a Weyl manifold, if J_+ is a *para-complex structure* on N (i.e. an endomorphism of TN with $J_+^2 = \operatorname{id}$ and $\operatorname{Tr}(J_+) = 0$), and if $J_+^* g = -g$; necessarily g has neutral signature $(\bar{n}, \bar{n})$.

The $\pm$ formalism permits us to discuss para-complex $(+)$ and complex $(-)$ geometry in parallel. For example, *(para)-Nijenhuis tensor* of an almost (para)-complex manifold $(M, J_\pm)$ is given by

$$N_\pm(x, y) := [x, y] \mp J_\pm[J_\pm x, y] \mp J_\pm[x, J_\pm y] \pm [J_\pm x, J_\pm y]. \tag{1.g}$$

It vanishes if and only if $J_\pm$ is an integrable almost (para)-complex structure, i.e. given any point $P \in N$, there exist local coordinates $(x^1, \dots, x^n)$ centered at P so

$$J_\pm \partial_{x_{2i-1}} = \partial_{x_{2i}} \quad \text{and} \quad J_\pm \partial_{x_{2i}} = \pm \partial_{x_{2i-1}} \quad \text{for } 1 \leq i \leq \bar{n}. \tag{1.h}$$

1.3. (Para)-Kähler Weyl geometry. Let $(N, g, \nabla, J_\pm)$ be an almost para/pseudo-Hermitian Weyl manifold. If $\nabla(J_\pm) = 0$, then one says that this is a *(para)-Kähler Weyl manifold.* Note that necessarily $J_\pm$ is integrable in this setting. The study of such manifolds is very much an active research endeavor. See, for example, [13] where the Siu–Beauville theorem is extended to a certain class of compact Kähler–Weyl manifolds.

Pedersen, Poon, and Swann [18] used work of Vaisman [23], [24] to establish the following result in the Hermitian (i.e. positive definite) setting; the extension to the higher signature setting and to the para-Kähler setting is immediate. We shall present their proof in Section 2.3 for the sake of completeness.

Theorem 1.2. *If $(N, g, \nabla, J_\pm)$ is a (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(N;\mathbb{R}) = 0$, then the underlying Weyl structure on N is an exact Weyl structure.*

We remark that Theorem 1.2 fails if $n = 4$; see, for example, [3], [19].

1.4. Curvature (para)-Kähler Weyl manifolds. Suppose $(N, g, \nabla, J_\pm)$ is an almost para/pseudo-Hermitian Weyl manifold. If $\nabla(J_\pm) = 0$, then one has an additional curvature symmetry called the *Kähler identity*:

$$\begin{aligned} &\mathcal{R}(x,y)J_\pm = J_\pm\mathcal{R}(x,y) && \forall \quad x,y, \quad \text{or equivalently} \\ &R(x,y,z,w) = \mp R(x,y,J_\pm z,J_\pm w) && \forall \quad x,y,z,w. \end{aligned} \tag{1.i}$$

We say that $(N, g, \nabla, J_\pm)$ is a *(para)-Kähler curvature Weyl manifold* if equation (1.i) is satisfied. We will show in Section 2.2 that there exist (para)-Kähler curvature Weyl manifolds where $J_\pm$ is not integrable; thus, in particular, these are not (para)-Kähler Weyl manifolds.

The main result of this paper is the extension of Theorem 1.2 to this context. The following result gives a curvature condition in these settings which ensures that the Weyl structure is an exact Weyl structure; again it fails if $n = 4$:

Theorem 1.3. *If $(N, g, \nabla, J_\pm)$ is a curvature (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(N;\mathbb{R}) = 0$, then the underlying Weyl structure on N is an exact Weyl structure.*

1.5. Geometric realization results. It is convenient to work in a purely algebraic context. Let V be a finite dimensional vector space which is equipped with a non-degenerate symmetric bilinear form h that we use to raise and lower indices; the pair (V, h) is said to be an *inner product space*. We say that $A \in \otimes^4 V^*$ is a *affine curvature tensor* if A has the symmetries given in equations (1.b) and (1.c); let $\mathfrak{A}$ be the set of all such tensors. The corresponding *affine curvature operator* $\mathcal{A}$ is defined by raising an index; A and $\mathcal{A}$ are related by the identity:

$$A(x,y,z,w) = h(\mathcal{A}(x,y)z,w) \quad \forall \quad x,y,z,w \in V.$$

Let $\mathfrak{W}$ be the subspace of $\mathfrak{A}$ of all elements which in addition satisfy the symmetry of equation (1.e) and let $\mathfrak{R}$ be the subspace of $\mathfrak{A}$ of elements which in addition satisfy the symmetry of equation (1.f); an element $A \in \mathfrak{R}$ is said to be a *Riemannian curvature tensor* and the associated endomorphism $\mathcal{A}$ to be a *Riemannian curvature operator*. We have proper inclusions:

$$\mathfrak{R} \subset \mathfrak{W} \subset \mathfrak{A}.$$

The relations of equations (1.b) and (1.c) generate the universal symmetries satisfied by the curvature of a torsion free connection, the relations of equations (1.b), (1.c), and (1.e) generate the universal symmetries satisfied by the curvature in Weyl geometry, and the relations of equations (1.b), (1.c), and equation (1.f) generate the universal symmetries satisfied by the curvature in pseudo-Riemannian geometry. We refer to [1], [7], [8], [9] for the proof of the following result (see also [2] for a more complete overview of the field):

Theorem 1.4. *Let (V, h) be an inner product space.*

(1) *If $A \in \mathfrak{A}$, then there exists a manifold N, there exists a point P of N, there exists a torsion free connection ∇ on TN, and there exists an isomorphism $\Phi : T_P N \to V$ so that $\Phi^* A = R_P$.*
(2) *If $A \in \mathfrak{W}$, then there exists a Weyl manifold (N, g, ∇), there exists a point P of N, and there exists an isomorphism $\Phi : T_P N \to V$ so that $\Phi^* h = g_P$ and so that $\Phi^* A = R_P$.*
(3) *If $A \in \mathfrak{R}$, then there exists a pseudo-Riemannian manifold (N, g), there exists a point P of N, and there exists an isomorphism $\Phi : T_N \to V$ so that $\Phi^* h = g_P$ and so that $\Phi^* A = R_P^g$.*

1.6. Para/pseudo-Hermitian curvature models. Let (V, h) be an inner product space. We say that the triple $(V, h, J_\pm)$ is a *para/pseudo-Hermitian vector space* if $J_\pm$ is a (para)-complex structure on V with $J_\pm^* h = \mp h$. Theorem 1.3 will follow from Theorem 1.1 and from the following purely algebraic result:

Theorem 1.5. *Let $n \geq 6$. Let $(V, h, J_\pm)$ be a para/pseudo-Hermitian vector space and let $A \in \mathfrak{W}$. If A satisfies the (para)-Kähler identity of equation* (1.i), *then*

$$A \in \mathfrak{R}.$$

Theorem 1.5 fails if $n = 4$; there are non-trivial elements of $\mathfrak{W} - \mathfrak{R}$ which satisfy the Kähler identity when $n = 4$. We shall investigate this and related questions further in a subsequent paper.

1.7. Outline of the paper. In Section 2, we prove Theorem 1.1, we prove Theorem 1.2, and we exhibit a curvature (para)-Kähler manifold $(N, g, J_\pm)$ with (necessarily if $n \geq 6$) an exact Weyl structure where $J_\pm$ is not integrable. In Section 3, we review the basic group representation theory that we shall need; these results are well known and we refer to the discussion in [2] Chapter 2 for example. We define the orthogonal group $\mathcal{O}$, the (para)-unitary groups $\mathcal{U}_\pm$, and $\mathbb{Z}_2$ extensions $\mathcal{U}_\pm^*$ that play an important role in our discussion. Suppose that $G \in \{\mathcal{O}, \mathcal{U}, \mathcal{U}_\pm^*\}$. Results concerning the theory of submodules of $\otimes^k V$ for the group G are outlined in Section 3.2 and an introduction to the theory of scalar invariants for $\otimes^k V$ is given in Section 3.3. The para unitary group $\mathcal{U}_+$ is exceptional and these results not apply to that group.

In Section 4, we review results of Singer and Thorpe [21] decomposing $\mathfrak{R}$, results of Higa [11], [12] decomposing $\mathfrak{W}$ as orthogonal modules, and an extension of results of Tricerri and Vanhecke [22] decomposing $\mathfrak{R}$ as a $\mathcal{U}_\pm^\star$ module. These results are then used to decompose $\mathfrak{W}$ as a $\mathcal{U}_\pm^\star$ module. Theorem 1.5 is then established Section 5. We refer to [5], [8], [10], [19], [20] for further details concerning Weyl geometry.

2. Geometric considerations

2.1. The proof of Theorem 1.1. Suppose that $d\phi_{\nabla,g}=0$. Since $H^1(N;\mathbb{R})=0$, we can express $\phi_{\nabla,g} = df$ for some function f. Let $g_1 := e^{2f}g \in [g]$. Then $\phi_{\nabla,g_1} = 0$ so $\nabla = \nabla^{g_1}$. Thus Assertion (1) implies Assertion (2); by definition the Weyl structure is an exact Weyl structure (i.e. is trivial) if and only if Assertion (2) holds. Clearly Assertion (2) implies Assertion (3). Since the curvature tensor of the Levi–Civita connection is Riemannian, Assertion (3) implies Assertion (4). Suppose that Assertion (4) holds. We have $d\phi_{\nabla,g} = -\frac{1}{n}\Lambda\,\mathrm{Ric}$ where $\Lambda\,\mathrm{Ric}$ is the alternating part of the Ricci tensor. Since the curvature tensor is Riemannian, the Ricci tensor is symmetric and consequently $\Lambda\,\mathrm{Ric} = 0$. Thus Assertion (4) implies Assertion (1). $\square$

2.2. A curvature (para)-Kähler Weyl manifold which is not integrable. Although relatively elementary, the following example is instructive. Consider the usual coordinates $(x^1,\dots,x^n)$ on $N := \mathbb{R}^n$. Let J_- be the standard complex structure given in equation (1.h). We work first in the positive definite setting. Let

$$g := dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n. \tag{2.a}$$

Let $\Theta : \mathbb{R}^n \to \mathcal{O}$ satisfy $\Theta(0) = \mathrm{id}$. We consider a twisted almost complex structure:

$$J_-^\Theta := \Theta^{-1} J_- \Theta.$$

Let $\theta = \theta(x_1)$. Define $\Theta : \mathbb{R}^n \to \mathcal{O}$ by setting:

$$\Theta\partial_{x_i} := \left\{ \begin{array}{lll} \cos\theta(x_1)\partial_{x_1} + \sin\theta(x_1)\partial_{x_3} & \text{if} & i=1 \\ \cos\theta(x_1)\partial_{x_3} - \sin\theta(x_1)\partial_{x_1} & \text{if} & i=3 \\ \partial_{x_i} & \text{if} & i\neq 1,3 \end{array} \right\}.$$

We compute the Nijenhuis tensor N_- of equation (1.g) for J_-^Θ. We have that $\{N_-(\partial_{x_1}, \partial_{x_3})\}(0)$ consists of 4 parts:

(1) $[\partial_{x_1}, \partial_{x_3}](0) = 0$.

(2) $J_-^\Theta[J_-^\Theta\partial_{x_1}, \partial_{x_3}](0) = -J_-(\partial_{x_3}J_-^\Theta)\partial_{x_1} = 0$.

(3) $$\begin{aligned} J_-^\Theta[\partial_{x_1}, J_-^\Theta\partial_{x_3}](0) &= \left\{J_-(\partial_{x_1}J_-^\Theta)\partial_{x_3}\right\}\big|_{x=0} = \left\{J_-\partial_{x_1}(\Theta^{-1}J_-\Theta)\partial_{x_3}\right\}\big|_{x=0} \\ &= \{(-J_-\partial_{x_1}(\Theta)J_- + J_-J_-\partial_{x_1}(\Theta))\partial_{x_3}\}|_{x=0} \\ &= \{-J_-(\partial_{x_1}\Theta)|_{x=0}\partial_{x_4} - \partial_{x_1}(\Theta)|_{x=0}\partial_{x_3}\} = \partial_{x_1}|_{x=0} \neq 0. \end{aligned}$$

(4) $$\begin{aligned} -[J_-^\Theta\partial_{x_1}, J_-^\Theta\partial_{x_3}](0) &= -\left\{(J_-\partial_{x_1})(J_-^\theta))\partial_{x_3} - (J_-\partial_{x_3})(J_-^\Theta)\partial_{x_1}\right\}\big|_{x=0} \\ &= \left\{(\partial_{x_2}(J_-^\theta)\partial_{x_3} - \partial_{x_4}(J_-^\theta)\partial_{x_1}\right\}\big|_{x=0} = 0. \end{aligned}$$

Thus the Nijenhuis tensor is non-trivial and J_-^Θ is not integrable. Since the curvature vanishes identically, (N, g, J_-^Θ) is necessarily curvature Kähler. If there existed a torsion free connection such that $\nabla J_-^\Theta = 0$, then ∇J_-^Θ would be integrable. Consequently, this structure can not be Kähler since J_-^θ is not in fact integrable. By considering product manifolds, one can create examples which are not flat. Furthermore, by replacing cos and sin by cosh and sinh and modifying the signs appropriately, one can also construct examples in higher signature.

The construction of a curvature para-Kähler manifold which is not para-Kähler is similar. One replaces the complex structure J_- by the para-complex structure J_+ in equation (1.h), one replaces the metric g of equation (2.a) by the metric

$$g := dx^1 \otimes dx^1 - dx^2 \otimes dx^2 + dx^3 \otimes dx^3 - dx^4 \otimes dx^4 \dots,$$

and one replaces the N_- by N_+. The remainder of the construction is unchanged and is therefore omitted.

2.3. The proof of Theorem 1.2. Let $(N, g, \nabla, J_\pm)$ be a (para)-Kähler Weyl manifold. Since $\nabla(J_\pm) = 0$, $J_\pm$ is integrable. Let

$$\Omega_\pm(x, y) := g(x, J_\pm y)$$

be the associated *Kähler form.* We compute:

$$\begin{aligned} (\nabla_z\Omega_\pm)(x, y) &= zg(x, J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, J_\pm\nabla_z y) \\ &= zg(x, J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, \nabla_z J_\pm y) \\ &= (\nabla_z g)(x, J_\pm y) = -2\phi_{\nabla,g}(z)\Omega_\pm(x, y). \end{aligned}$$

Let $\{e_i\}$ be a local frame for TN and let $\{e^i\}$ be the dual frame for the cotangent bundle T^*N. We adopt the *Einstein* convention and sum over repeated indices. Since ∇ is torsion free, $d\Omega_\pm = e^i \wedge \nabla_{e_i}\Omega_\pm$. Consequently

$$d\Omega_\pm = -2\phi_{\nabla,g}(e_i)e^i \wedge \Omega_\pm = -2\phi_{\nabla,g} \wedge \Omega_\pm, \qquad 0 = d^2\Omega_\pm = -2d\phi_{\nabla,g} \wedge \Omega_\pm.$$

Multiplication by $\Omega_\pm^{\frac{1}{2}n-2}$ is an isomorphism between Λ^2 and Λ^{n-2}; this fact is usually cited only in the positive definite setting for J_- but extends to the more general situation. Thus as $n \geq 6$, $d\phi_{\nabla,g} \wedge \Omega_\pm = 0$ implies $d\phi_{\nabla,g} = 0$. $\square$

This argument fails if $n = 4$; we can only conclude from this that $d\phi_{\nabla,g} \perp \Omega_\pm$.

3. Representation theory

In this section, we present the basic results from group representation theory that we shall need; these results are well known and we refer, for example, to [2] Chapter 2 for further details. The structure groups are defined in Section 3.1. The theory of submodules of $\otimes^k V$ is outlined in Section 3.2. Results relating to the theory of scalar invariants are presented in Section 3.3.

3.1. Structure groups. Let h be a non-degenerate symmetric bilinear form on a real vector space V of dimension n. Let $\mathcal{O} = \mathcal{O}(V,h)$ be the associated orthogonal group:

$$\mathcal{O} := \{T \in \mathrm{GL}(V) : T^*h = h\}.$$

If $(V, h, J_\pm)$ is a para/pseudo-Hermitian vector space, there are two associated Lie groups of interest. We define the (para)-unitary group $\mathcal{U}_\pm$ and associated $\mathbb{Z}_2$ extension $\mathcal{U}_\pm^*$ by setting:

$$\begin{aligned}\mathcal{U}_\pm &:= \{T \in \mathcal{O} : TJ_\pm = J_\pm T\},\\ \mathcal{U}_\pm^\star &:= \{T \in \mathcal{O} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}.\end{aligned}$$

3.2. Submodules of $\otimes^k V$. We extend h to $\otimes^k V$ so that

$$h((v_1 \otimes \cdots \otimes v_k), (w_1 \otimes \cdots \otimes w_k)) := \prod_{i=1}^k h(v_i, w_i). \tag{3.a}$$

equation (3.a) defines a non-degenerate symmetric bilinear form on $\otimes^k V$. We use h to identify V with V^* and $\otimes^k V$ with $\otimes^k V^*$. If $\Theta \in \otimes^k V^*$ and if T is a linear map of V, the *pull-back* $T^*\Theta$ is characterized by the identity

$$T^*\Theta(v_1, \ldots, v_k) = \Theta(Tv_1, \ldots, Tv_k).$$

Let G be one of the groups defined in Section 3.1. Then G acts naturally on $\otimes^k V^*$ by pull-back and preserves the canonical inner product defined in equation (3.a). Let ξ be a G-invariant subspace of $\otimes^k V^*$; the natural action of G on $\otimes^k V^*$ makes ξ into a G-submodule of $\otimes^k V$. The following is well known – see, for example, the discussion in [2] Chapter 2:

Lemma 3.1. *Let $G \in \{\mathcal{O}, \mathcal{U}_-, \mathcal{U}_\pm^\star\}$. Let ξ be a non-trivial G-submodule of $\otimes^k V^*$.*

(1) *ξ is not totally isotropic.*

(2) *There is an orthogonal direct sum decomposition $\xi = \eta_1 \oplus \cdots \oplus \eta_k$ where the η_i are irreducible G-modules.*

(3) *If ξ_1 and ξ_2 are inequivalent irreducible submodules of ξ, then $\xi_1 \perp \xi_2$.*

(4) *The multiplicity with which an irreducible representation appears in ξ is independent of the decomposition in* (2).

(5) *If ξ_1 appears with multiplicity* 1 *in ξ and if η is any G-submodule of ξ, then either $\xi_1 \subset \eta$ or else $\xi_1 \perp \eta$.*

Remark 3.2. Much of what we will say subsequently extends to $\mathcal{U}_-$ with minor modifications. As the analysis of $\mathcal{U}_-$ is not needed to establish the results of this paper, we shall not pursue this topic. We note, however, that Lemma 3.1 fails for the group $\mathcal{U}_+$. Let (V, h, J_+) be a para-Hermitian vector space. Decompose $V = V_+ \oplus V_-$ into the ± 1 eigenspaces of J_+. Then $V_\pm$ are totally isotropic subspaces of V which are invariant under $\mathcal{U}_+$.

3.3. Scalar invariants. Let ξ be a G-module. We say that $\Xi : \xi \to \mathbb{R}$ is a *scalar invariant* if $\Xi(g \cdot v) = \Xi(v)$ for every $v \in \xi$ and for every $g \in G$; let $\mathcal{I}^G(\xi)$ be the vector space of all such invariants. Let $\xi \subset \otimes^k V^*$. H. Weyl [27] (see p. 53 and 66) gives a spanning set if $G = \mathcal{O}$ is the orthogonal group; the corresponding result for the unitary group $\mathcal{U}_-$ for in the Hermitian (i.e. positive definite) setting follows from [4], [15] and the extension to the groups $\mathcal{U}_\pm^\star$ in general is straightforward – see [2] for example.

We discuss this spanning set. All invariants arise by using either the metric or the Kähler form to contract indices; invariants of $\mathcal{U}_\pm^\star$ arise when the Kähler form appears an even number of times. It is worth being a bit more formal about this. Let $(V, h, J_\pm)$ be a para/pseudo-Hermitian vector space. Let $\Omega_{\pm,ij}$ be the components of the (para)-Kähler form. If $\{e_i\}$ is any basis for V, let $h_{ij} := h(e_i, e_j)$. The inverse matrix $h^{ij} = h(e^i, e^j)$ gives the components of the dual inner product on V^*. If $\Theta \in \otimes^{2k} V^*$, expand $\Theta = \Theta_{i_1 \ldots i_{2k}} e^{i_1} \otimes \cdots \otimes e^{i_{2k}}$. Let

$\pi \in \mathrm{Perm}(2k)$ be a permutation of $\{1, \dots, 2k\}$. Let $\kappa_0 := h$, let $\kappa_1 := \Omega_\pm$, and let $\vec{a}$ be a sequence of 0's and 1's. Define:

$$\psi_{\pi,\vec{a}}(\Theta) := \kappa_{a_1}^{i_{\pi(1)} i_{\pi(2)}} \dots \kappa_{a_k}^{i_{\pi(2k-1)} i_{\pi(2k)}} \Theta_{i_1 \dots i_{2k}}.$$

Let $n(\vec{a})$ be the number of times $a_i = 1$. One then has:

Lemma 3.3. *If $(V, h, J_\pm)$ is a para/pseudo-Hermitian vector space and if ξ is a $\mathcal{U}_\pm^\star$ submodule of $\otimes^{2k} V^*$, then $\mathcal{I}^{\mathcal{U}_\pm^\star}(\xi) = \mathrm{Span}_{n(\alpha)\ even}\{\psi_{\pi,\alpha}\}$.*

4. Curvature decompositions

In this section, we review the fundamental curvature decompositions that will play an important role our discussion. Section 4.1 treats the Singer–Thorpe [21] decomposition of $\mathfrak{R}$ as an $\mathcal{O}$ module. Section 4.2 presents the Higa decomposition [11], [12] of $\mathfrak{W}$ as an $\mathcal{O}$ module. Section 4.3 discusses a decomposition of $\mathfrak{R}$ as a $\mathcal{U}_\pm^\star$ module which generalizes the original Tricerri–Vanhecke [22] decomposition of $\mathfrak{R}$ as a $\mathcal{U}_-$ module in the positive definite setting. Section 4.4 gives the decomposition of $\mathfrak{W}$ as a $\mathcal{U}_\pm^\star$ module.

4.1. The Singer-Thorpe $\mathcal{O}$ module decomposition of $\mathfrak{R}$. Let $\mathbb{R} \cdot h \subset \otimes^2 V^*$ be the trivial 1-dimensional $\mathcal{O}$ module, let $S_0^2 \subset \otimes^2 V^*$ be the $\mathcal{O}$ module of trace free symmetric 2-tensors, and let $\Lambda^2 \subset \otimes^2 V^*$ be the $\mathcal{O}$ module of alternating 2-tensors. Let $P := \ker\{\mathrm{Ric}\} \cap \mathfrak{R}$ be the $\mathcal{O}$ module of *Weyl conformal curvature tensors.* It follows from [21] that:

Theorem 4.1. *Let $n \geq 4$.*

(1) *We may decompose $\otimes^2 V^* = \mathbb{R} \cdot h \oplus S_0^2 \oplus \Lambda^2$ as the orthogonal direct sum of 3 irreducible and inequivalent $\mathcal{O}$ modules.*
(2) *There is an $\mathcal{O}$ isomorphism $\mathfrak{R} \approx \mathbb{R} \oplus S_0^2 \oplus P$ decomposing $\mathfrak{R}$ as the orthogonal direct sum of 3 irreducible and inequivalent $\mathcal{O}$ modules.*

4.2. The Higa $\mathcal{O}$ module decomposition of $\mathfrak{W}$. If $\psi \in \Lambda^2$, define:

$$\begin{aligned}\sigma(\psi)(x, y, z, w) := {}& 2\psi(x, y)h(z, w) + \psi(x, z)h(y, w) - \psi(y, z)h(x, w)\\ & - \psi(x, w)h(y, z) + \psi(y, w)h(x, z).\end{aligned} \tag{4.a}$$

The map σ is an $\mathcal{O}$ module isomorphism from Λ^2 to $\mathfrak{P} := \sigma(\Lambda^2)$. We have [11], [12]:

Theorem 4.2. *Let $n \geq 4$. We may decompose $\mathfrak{W} = \mathfrak{R} \oplus \mathfrak{P}$ as the orthogonal direct sum of $\mathcal{O}$ modules. This gives a $\mathcal{O}$ module isomorphism $\mathfrak{W} \approx \mathbb{R} \oplus S_0^2 \oplus P \oplus \Lambda^2$ as the orthogonal direct sum of 4 irreducible and inequivalent $\mathcal{O}$ modules.*

4.3. The Tricerri–Vanhecke $\mathcal{U}_\pm^\star$ module decomposition. The results of this section are the natural extension of results of TRICERRI and VANHECKE [22] to the setting at hand and are discussed in [2] in more detail. Let $(V,h,J_\pm)$ be a para/pseudo-Hermitian vector space. Define:

$$S^{2,\mathcal{U}_\pm}_{0,\mp} := \{\theta\in S^2 : J_\pm^*\theta=\mp\theta \text{ and } \theta\perp h\},\qquad S^{2,\mathcal{U}_\pm}_{\pm} := \{\theta\in S^2 : J_\pm^*\theta=\pm\theta\},$$
$$\Lambda^{2,\mathcal{U}_\pm}_{0,\mp} := \{\theta\in \Lambda^2 : J_\pm^*\theta=\mp\theta \text{ and } \theta\perp \Omega_\pm\},\quad \Lambda^{2,\mathcal{U}_\pm}_{\pm} := \{\theta\in \Lambda^2 : J_\pm^*\theta=\pm\theta\}.$$

Lemma 4.3. *Let $n\geq 4$. We may decompose*

$$S^2=\mathbb{R}\cdot h\oplus S^{2,\mathcal{U}_\pm}_{0,\mp}\oplus S^{2,\mathcal{U}_\pm^\star}_{\pm}\quad\textit{and}\quad \Lambda^2=\mathbb{R}\cdot\Omega_\pm\oplus\Lambda^{2,\mathcal{U}_\pm^\star}_{0,\mp}\oplus\Lambda^{2,\mathcal{U}_\pm^\star}_{\pm}$$

as the orthogonal direct sum of 6 irreducible and inequivalent $\mathcal{U}_\pm^\star$ modules.

Remark 4.4. The decomposition given above is also a decomposition of S^2 and Λ^2 into irreducible $\mathcal{U}_-$ modules. However $\mathbb{R}\cdot h\approx\mathbb{R}\cdot\Omega_+$ and $S^{2,\mathcal{U}_-}_{0,+}\approx\Lambda^{2,\mathcal{U}_-}_{0,+}$ as $\mathcal{U}_-$ modules. In the para-Hermitian setting, we note that $S^{2,\mathcal{U}_+}_{+}$ and $\Lambda^{2,\mathcal{U}_+}_{+}$ are not irreducible $\mathcal{U}_+$ modules.

Let $n\geq 8$. One follows [22] to define $\mathcal{U}_-$ modules $W_{-,i}$; these are also $\mathcal{U}_-^*$ modules and there are analogous modules $\mathcal{U}_+^\star$ modules $W_{+,i}$ in the para-Hermitian setting. Set

$$\mathfrak{K}_{\pm,\mathfrak{R}}:=\{A\in\mathfrak{R}: A(x,y,z,w)=\mp A(x,y,J_\pm z,J_\pm w)\};$$

these are the Riemannian curvature tensors which also satisfy the (para)-Kähler identity of equation (1.i).

Theorem 4.5. *Let $(V,h,J_\pm)$ be a para/pseudo Hermitian vector space of dimension $n\geq 8$. We may decompose*

$$\mathfrak{R}=W_{\pm,1}\oplus\cdots\oplus W_{\pm,10}\quad\textit{and}\quad\mathfrak{K}_{\pm,\mathfrak{R}}=W_{\pm,1}\oplus W_{\pm,2}\oplus W_{\pm,3}$$

as the orthogonal direct sum of irreducible $\mathcal{U}_\pm^\star$ modules. We have
(1) $W_{\pm,1}\approx W_{\pm,4}\approx\mathbb{R}$ *and* $W_{\pm,2}\approx W_{\pm,5}\approx S^{2,\mathcal{U}_\pm}_{0,\mp}$.
(2) $W_{\pm,8}\approx S^{2,\mathcal{U}_\pm}_{\pm}$, *and* $W_{\pm,9}\approx\Lambda^{2,\mathcal{U}_\pm}_{\pm}$.
With exception of the isomorphisms in (1), *these are inequivalent $\mathcal{U}_\pm$ modules.*

Remark 4.6.

(1) The original discussion of [22] dealt with the unitary group $\mathcal{U}_-$ in the positive definite setting; we refer to [2] for a discussion of the indefinite Hermitian setting and in the para-Hermitian setting. If $n = 6$, we set $W_{\pm,6} = \{0\}$; if $n = 4$, we set $W_{\pm,5} = W_{\pm,6} = W_{\pm,10} = \{0\}$ to achieve the corresponding decomposition. This does not affect our subsequent analysis.

(2) Let Ψ be the isomorphism from $\Lambda^2_\pm$ to $W_{\pm,9}$ given in (2) above; it is described quite explicitly in [22] (page 372) in the Hermitian setting and extends to our context to become:

$$\begin{aligned}\Psi(\psi)(x,y,z,w) := {} & 2h(x,J_\pm y)\psi(z,J_\pm w) + 2h(z,J_\pm w)\psi(x,J_\pm y)\\ & + h(x,J_\pm z)\psi(y,J_\pm w) + h(y,J_\pm w)\psi(x,J_\pm z)\\ & - h(x,J_\pm w)\psi(y,J_\pm z) - h(y,J_\pm z)\psi(x,J_\pm w). \qquad (4.b)\end{aligned}$$

4.4. The decomposition of $\mathfrak{W}$ as a $\mathcal{U}^\star_\pm$ module. Let σ be as in equation (4.a). We apply Lemma 4.3 to decompose Λ^2 and define:

$$W_{\pm,11} := \sigma(\mathbb{R}\cdot\Omega_\pm), \quad W_{\pm,12} := \sigma(\Lambda^{2,\mathcal{U}_\pm}_{0,\mp}), \quad W_{\pm,13} := \sigma(\Lambda^{2,\mathcal{U}_\pm}_\pm).$$

We combine Lemma 4.3 and Theorem 4.5 to establish:

Theorem 4.7. *Let $(V,h,J_\pm)$ be a para/pseudo Hermitian vector space of dimension $n \geq 8$. We may decompose*

$$\mathfrak{W} = W_{\pm,1} \oplus \cdots \oplus W_{\pm,13}$$

as the orthogonal direct sum of irreducible $\mathcal{U}^\star_\pm$ modules. With the exception of the isomorphisms noted in Theorem 4.5, these are inequivalent $\mathcal{U}^\star_\pm$ modules.

Remark 4.8. As before, we shall set $W_{\pm,6} = \{0\}$ if $n = 6$ and we shall set $W_{\pm,5} = W_{\pm,6} = W_{\pm,10} = \{0\}$ if $n = 4$. The modules $\{\mathbb{R}, S^{2,\mathcal{U}_\pm}_{0,\mp}, \Lambda^{2,\mathcal{U}_\pm}_\pm\}$ appear with multiplicity 2 in the decomposition of $\mathfrak{W}$ as a $\mathcal{U}^\star_\pm$ module; the remaining modules appear with multiplicity 1.

4.5. The modules $\Lambda^{2,\mathcal{U}^\star_\pm}_\pm$. We shall need the following technical result:

Lemma 4.9. *If ξ is a non-trivial proper $\mathcal{U}^\star_\pm$ submodule of $\Lambda^{2,\mathcal{U}_\pm}_\pm \oplus \Lambda^{2,\mathcal{U}_\pm}_\pm$, then there exists $(a,b) \neq (0,0)$ so*

$$\xi = \xi(a,b) := \{(a\theta, b\theta)\}_{\theta\in\Lambda^{2,\mathcal{U}_\pm}_\pm} \subset \Lambda^{2,\mathcal{U}_\pm}_\pm \oplus \Lambda^{2,\mathcal{U}_\pm}_\pm.$$

PROOF. We have

$$\Lambda_\pm^{2,\mathcal{U}_\pm}\otimes\Lambda_\pm^{2,\mathcal{U}_\pm}=\{\theta\in\otimes^4V^*:\theta(x,y,z,w)=-\theta(y,x,z,w)=-\theta(x,y,w,z)$$
$$\text{and}\quad\theta(x,y,z,w)=\pm\theta(J_\pm x,J_\pm y,z,w)=\pm\theta(x,y,J_\pm z,J_\pm w)\}.$$

It follows from these symmetries and from Lemma 3.3 that there is only one $\mathcal{U}_\pm^\star$ invariant of $\Lambda_\pm^{2,\mathcal{U}_\pm}\otimes\Lambda_\pm^{2,\mathcal{U}_\pm}$ given by $h^{ik}h^{jl}\theta(e_i,e_j,e_k,e_l)$. Thus

$$\dim\{\mathcal{I}^{\mathcal{U}_\pm^*}(\Lambda_\pm^{2,\mathcal{U}_\pm}\otimes\Lambda_\pm^{2,\mathcal{U}_\pm})\}\leq 1. \tag{4.c}$$

Let $\operatorname{Hom}^{\mathcal{U}_\pm^*}(\Lambda_\pm^{2,\mathcal{U}_\pm})$ be the set of all linear maps $T:\Lambda_\pm^{2,\mathcal{U}_\pm}\to\Lambda_\pm^{2,\mathcal{U}_\pm}$ with $Tg=gT$ for all $g\in\mathcal{U}_\pm^*$. Let $\Xi_T(\theta_1\otimes\theta_2):=h(\theta_1,T\theta_2)$ be the linear invariant defined by $T\in\operatorname{Hom}^{\mathcal{U}_\pm^*}(\Lambda_\pm^{2,\mathcal{U}_\pm})$; for example, $\Xi_1=\Xi_{\mathrm{id}}$. equation (4.c) then shows

$$\operatorname{Hom}^{\mathcal{U}_\pm^*}(\Lambda_\pm^{2,\mathcal{U}_\pm^\star})=\mathrm{Id}\cdot\mathbb{R}. \tag{4.d}$$

Let ξ be a proper $\mathcal{U}_\pm^*$ submodule of $\Lambda_\pm^{2,\mathcal{U}_\pm}\oplus\Lambda_\pm^{2,\mathcal{U}_\pm}$. Let π_1 (resp. π_2) be projection on the first (resp. on the second) factor. Since ξ is non-trivial, we may assume without loss of generality that $\pi_1\xi\neq\{0\}$; since ξ is a proper submodule, ξ is necessarily irreducible and hence π_1 is an isomorphism. If $\pi_2=0$, then $\xi=\xi(1,0)$. Thus we may assume that $\pi_2\neq 0$ and hence $\pi_2^{-1}\pi_1=T$ is a non-trivial $\mathcal{U}_\pm^*$ equivariant map of $\Lambda_\pm^{2,\mathcal{U}_\pm}$. equation (4.d) then shows $T=b\,\mathrm{id}$ and $\xi=\xi(1,b)$. □

5. The proof of Theorem 1.5

Let $(V,h,J_\pm)$ be a para/pseudo-Hermitian vector space. Let

$$\mathfrak{K}_{\pm,\mathfrak{W}}:=\{A\in\mathfrak{W}:A(x,y,z,w)=\mp A(x,y,J_\pm z,J_\pm w)\ \forall\ x,y,z,w\}$$

be the space of all Weyl tensors satisfying the (para)-Kähler identity of equation (1.i). We use the decomposition of Theorem 4.7 and set

$$\mathfrak{K}^1_{\pm,\mathfrak{W}}:=\{\oplus_{4\leq i\leq 13}W_{\pm,i}\}\cap\mathfrak{K}_{\pm,\mathfrak{W}}.$$

We use Theorem 4.5 to see $\mathfrak{R}\cap\mathfrak{K}_{\pm,\mathfrak{W}}=W_{\pm,1}\oplus W_{\pm,2}\oplus W_{\pm,3}$. Consequently

$$\mathfrak{K}_{\pm,\mathfrak{W}}=W_{\pm,1}\oplus W_{\pm,2}\oplus W_{\pm,3}\oplus\mathfrak{K}^1_{\pm,\mathfrak{W}}.$$

We prove Theorem 1.5 by showing $\mathfrak{K}^1_{\pm,\mathfrak{W}}=\{0\}$. Suppose that $4\leq i\leq 13$ and $i\neq 9,13$. Since $W_{\pm,i}$ appears with multiplicity 1 in $\oplus_{4\leq i\leq 13}W_{\pm,i}$, Lemma 3.1 shows that either $W_{\pm,i}\subset\mathfrak{K}^1_{\pm,\mathfrak{W}}$ or $W_{\pm,i}\perp\mathfrak{K}^1_{\pm,\mathfrak{W}}$. By Theorem 4.5,
$W_{\pm,i}\cap\mathfrak{K}_{\pm,\mathfrak{R}}=\{0\}$ for $4\leq i\leq 10$. Consequently

$$\mathfrak{K}^1_{\pm,\mathfrak{W}}=\{W_{\pm,9}\oplus W_{\pm,11}\oplus W_{\pm,12}\oplus W_{\pm,13}\}\cap\mathfrak{K}_{\pm,\mathfrak{W}}.$$

5.1. The module $W_{\pm,11}$. We use equation (4.a) to see:

(1) $\sigma(\Omega_\pm)(e_1,e_4,e_3,e_1) = -h(e_4,J_\pm e_3)h(e_1,e_1) = -h_{11}h_{44}$,

(2) $\mp\sigma(\Omega_\pm)(e_1,e_4,J_\pm e_3,J_\pm e_1) = \pm h(e_1,J_\pm J_\pm e_1)h(e_4,J_\pm e_3) = h_{11}h_{44}$,

(3) Thus $\sigma(\mathbb{R}\cdot\Omega_\pm) \not\subset \mathfrak{K}^1_{\pm,\mathfrak{W}}$ if $n \geq 4$.

5.2. The module $W_{\pm,12}$. Let $\psi_{0,\pm} := e^1\otimes e^2 - e^2\otimes e^1 + \delta_\pm\{e^3\otimes e^4 - e^4\otimes e^3\}$ where $\delta_\pm$ is chosen to ensure $\psi_{0,\pm} \perp \Omega_\pm$. We have $J_\pm^*\psi_{0,\pm} = \mp\psi_{0,\pm}$ and thus $\psi_{0,\pm} \in \Lambda^{2,\mathcal{U}_\pm}_{0,\pm}$. We use equation (4.a) to verify:

(1) $\sigma(\psi_{0,\pm})(e_5,e_1,e_2,e_5) = -\psi_{0,\pm}(e_1,e_2)h(e_5,e_5) = -h_{55}$.

(2) $\mp\sigma(\psi_{0,\pm})(e_5,e_1,J_\pm e_2,J_\pm e_5) = \pm\psi_{0,\pm}(e_5,J_\pm e_5)h(e_1,J_\pm e_2) = 0$.

(3) $W_{\pm,12} \not\subset \mathfrak{K}^1_{\pm,\mathfrak{W}}$ if $n \geq 6$.

5.3. The module $W_{\pm,9}\oplus W_{\pm,13}$. Let $\psi_\pm := e^1\otimes e^3 - e^3\otimes e^1 \pm e^2\otimes e^4 \mp e^4\otimes e^2$. Then $J_\pm^*\psi_\pm = \pm\psi_\pm$ so $\psi_\pm \in \Lambda^{2,\mathcal{U}_\pm}_\pm$. By equation (4.a) and equation (4.b):

(1) $\sigma(\psi_\pm)(e_5,e_1,e_3,e_5) = -\psi_\pm(e_1,e_3)h(e_5,e_5) = -h_{55}$.

(2) $\sigma(\psi_\pm)(e_5,e_1,e_4,e_6) = 0$.

(3) $\Psi(\psi_\pm)(e_5,e_1,e_3,e_5) = 0$.

(4) $\Psi(\psi_\pm)(e_5,e_1,e_4,e_6) = -\psi_\pm(e_1,J_\pm e_4)h(e_5,J_\pm e_6) = -h_{55}$.

(5) $\sigma(\psi_\pm)(e_5,e_6,e_1,e_4) = 0$.

(6) $\sigma(\psi_\pm)(e_5,e_6,J_\pm e_1,J_\pm e_4) = 0$.

(7) $\Psi(\psi_\pm)(e_5,e_6,e_1,e_4) = 2h(e_5,J_\pm e_6)\psi_\pm(e_1,J_\pm e_4) = 2h_{55}$.

(8) $\Psi(\psi_\pm)(e_5,e_6,J_\pm e_1,J_\pm e_4) = 2h(e_5,J_\pm e_6)\psi_\pm(J_\pm e_1,J_\pm J_\pm e_4) = \pm 2h_{55}$.

For $(a,b) \neq (0,0)$, let $\xi(a,b) := \operatorname{Range}\{a\sigma + b\Psi\} \subset W_{\pm,9}\oplus W_{\pm,13}$. We suppose $\xi(a,b)\cap\mathfrak{K}^1_{\pm,\mathfrak{W}} \neq \{0\}$ and thus $\xi(a,b) \subset \mathfrak{K}^1_{\pm,\mathfrak{W}}$. Assertions (1)-(4) then yield $a = \mp b$ while Assertions (5)-(8) yield $b = 0$. We apply Lemma 4.9 to see that every non-trivial proper submodule of $W_{\pm,9}\oplus W_{\pm,13}$ is isomorphic to $\xi(a,b)$ for some $(a,b) \neq 0$. Thus

$$\{W_{\pm,9}\oplus W_{\pm,13}\}\cap\mathfrak{K}^1_{\pm,\mathfrak{W}} = \{0\}.$$

and consequently $\mathfrak{K}^1_{\pm,\mathfrak{W}} = \{0\}$. This completes the proof of Theorem 1.5. $\square$

Acknowledgments. Research of P. Gilkey partially supported by DFG PI 158/4-6 (Germany) and by project MTM2009-07756 (Spain). Research of S. Nikčević partially supported by a research grant of the TU Berlin, by project MTM2009-07756 (Spain), and by 144032 (Serbia). It is a pleasant task to acknowledge useful conversations with E. García-Río of the Universidad de Santiago (Spain), with U. Simon of the Technische Universität (Berlin), and with A. Swann of the University of Southern Denmark.

References

[1] M. Brozos-Vázquez, P. Gilkey, H. Kang, S. Nikčević and G. Weingart, Differential Geom. Appl., Vol. 27, 2009, 696–701.

[2] M. Brozos-Vázquez, P. Gilkey and S. Nikčević, Geometric Realizations of Curvature Tensors, *Imperial College Press*, May 2011.

[3] D. Calderbank and H. Pedersen, Selfdual spaces with complex structures, Einstein–Weyl geometry and geodesics, *Ann. Inst. Fourier (Grenoble)* **50** (2000), 921–963.

[4] T. Fukami, Invariant tensors under the real representation of unitary groups and their application, *J. Math. Soc. Japan* **10** (1958), 135–144.

[5] G. Ganchev and S. Ivanov, Semi-symmetric W-metric connections and the W-conformal group, *God. Sofij. Univ. Fak. Mat. Inform.* **81** (1994), 181–193.

[6] A. Ghosh, Einstein–Weyl structures on contact manifolds, *Ann. Global Anal. Geom.* **35** (2009), 431–441.

[7] P. Gilkey and S. Nikčević, Geometrical representations of equiaffine curvature operators, *Results Math.* **52** (2008), 281–287.

[8] P. Gilkey, S. Nikčević and U. Simon, Geometric realizations, curvature decompositions, and Weyl manifolds, *J. Geom. and Physics* **61** (2011), 270–275.

[9] P. Gilkey, S. Nikčević and D. Westerman, Geometric realizations of generalized algebraic curvature operators, *J. Math. Phys.* **50** (2009), 013515.

[10] H. Hayden, Sub-spaces of a space with torsion, *Proc. Lond. Math. Soc. II* **34** (1932), 27–50.

[11] T. Higa, Weyl manifolds and Einstein–Weyl manifolds, *Comm. Math. Univ. St. Pauli* **42** (1993), 143–160.

[12] T. Higa, Curvature tensors and curvature conditions in Weyl geometry, *Comm. Math. Univ. St. Pauli* **43** (1994), 139–153.

[13] G. Kokarev and D. Kotschick, Fibrations and fundamental groups of Kähler–Weyl manifolds, *Proc. Am. Math. Soc.* **138** (2010), 997–1010.

[14] M. Itoh, Affine locally symmetric structures and finiteness theorems for Einstein–Weyl manifolds, *Tokyo J. Math.* **23** (2000), 37–40.

[15] N. Iwahori, Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$, *J. Math. Soc. Japan* **10** (1958), 146–160.

[16] F. Nakata, A construction of Einstein–Weyl spaces via Lebrun–Mason type twistor correspondence, *Commun. Math. Phys.* **289** (2009), 663–699.

[17] J. Park, Projectively flat Yang-Mills connections, *Kyushu J. Math.* **64** (2010), 49–58.

[18] H. Pedersen, Y. Poon and A. Swann, The Einstein–Weyl equations in complex and quaternionic geometry, *Differential Geom. Appl.* **3** (1993), 309–321.

[19] H. Pedersen and A. Swann, Riemannian submersions, 4-dimensional manifolds, and Einstein–Weyl geometry, *Proc. London Math. Soc.* **66** (1991), 381–399.

[20] H. Pedersen and K. Tod, Three-dimensional Einstein–Weyl geometry, *Adv. Math.* **97** (1993), 74–109.

[21] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, 1969 Global Analysis (Papers in Honor of K. Kodaira), *Univ. Tokyo Press, Tokyo*, 355–365.

[22] F. Tricerri and L. Vanhecke, Curvature tensors on almost Hermitian manifolds, *Trans. Amer. Math. Soc.* **267** (1981), 365–397.

[23] I. Vaisman, Generalized Hopf manifolds, *Geom. Dedicata* **13** (1982), 231–255.

[24] I. Vaisman, A survey of generalized Hopf manifolds, Differential Geometry on Homogeneous Spaces, Proc. Conf. Torino Italy (1983), Rend. Semin. Mat. Torino, Fasc. Spec. 205–221.

[25] G. Vassal, Asymptotically flat conformal structures, *Commun. Math. Physics.* **295** (2010), 503–529.

[26] H. Weyl, Space-Time-Matter, *Dover Publ. 1922.*

[27] H. Weyl, The classical groups, *Princeton Univ. Press, Princeton*, 1946, (8th printing).

PETER GILKEY
MATHEMATICS DEPARTMENT
UNIVERSITY OF OREGON
EUGENE OR 97403
USA

E-mail: gilkey@uoregon.edu

STANA NIKČEVIĆ
MATHEMATICAL INSTITUTE
SANU
KNEZ MIHAILOVA 36, P.P. 367
11001 BELGRADE
SERBIA

E-mail: stanan@mi.sanu.ac.rs

(Received November 23, 2010; revised November 14, 2011)