Publ. Math. Debrecen
80/3-4 (2012), 385–404
DOI: 10.5486/PMD.2012.5072

Series with Hermite polynomials and applications

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Abstract. We obtain a series transformation formula involving the classical Hermite polynomials. We then provide a number of applications using appropriate binomial transformations. Several of the new series involve Hermite polynomials and harmonic numbers, Lucas sequences, exponential and geometric numbers. We also obtain a series involving both Hermite and Laguerre polynomials, and a series with Hermite polynomials and Stirling numbers of the second kind.

1. Introduction and main result

Let $H_n(x)$ be the Hermite polynomials defined by the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n\left(x\right) \frac{t^n}{n!} \tag{1.1}$$

and satisfying the Rodrigues formula

$$H_n\left(x\right) = \left(-1\right)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \tag{1.2}$$

In this paper we present and discuss a series transformation formula for the series

$$\sum_{n=0}^{\infty} a_n H_n\left(x\right) \frac{t^n}{n!}$$

Mathematics Subject Classification: 11B83, 33C45, 05A19.
Key words and phrases: Hermite polynomials, Laguerre polynomials, harmonic numbers, Fibonacci numbers, exponential numbers, geometric numbers, Stirling numbers of the second kind, binomial transformation, Euler series transformation, generating functions.
Research of the second author is supported by Akdeniz University Scientific Research Project Unit.

where

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \tag{1.3}$$

is an arbitrary function, analytical in a neighborhood of zero. Most of our results are based on the theorem:

Theorem 1.1. *With $f(t)$ as above the following series transformation formula holds*

$$\sum_{n=0}^{\infty} a_n H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t) \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\}. \tag{1.4}$$

For example, when $a_k = 1$ for all $k = 0, 1, \ldots$ then $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0$ except when $n = 0$, and (1.4) turns into (1.1) .

For the proof of the theorem we need the following lemma.

Lemma 1.2. *Let*

$$g(t) = \sum_{k=0}^{\infty} b_k t^k$$

be another analytical function like $f(t)$. Then

$$\sum_{n=0}^{\infty} a_n b_n t^n = \sum_{n=0}^{\infty} \frac{(-1)^n g^{(n)}(t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\}. \tag{1.5}$$

in some neighborhood of zero where both sides are convergent.

The proof of the lemma can be found in [5]. This result originates in the works of Euler. In a modified form (1.5) appears in [20, Chapter 6, Problem 19, p. 245].

Now we are ready to give the proof of the Theorem 1.1.

Proof. We use (1.5) with $g(t) = e^{2xt-t^2}$. From Rodrigues' formula it follows that

$$\left(\frac{d}{dt}\right)^n e^{2xt-t^2} = \left(\frac{d}{dt}\right)^n e^{x^2} e^{-(x-t)^2} = e^{x^2} (-1)^n \left(\frac{d}{dx}\right)^n e^{-(x-t)^2}$$
$$= e^{x^2} (-1)^n \left(\frac{d}{d(x-t)}\right)^n e^{-(x-t)^2} = e^{2xt-t^2} H_n(x-t).$$

That is,

$$\left(\frac{d}{dt}\right)^n e^{2xt-t^2} = e^{2xt-t^2} H_n(x-t) \tag{1.6}$$

from which (1.4) follows in view of Lemma 1.2. The proof is completed. $\square$

In the next section we present several corollaries resulting from the theorem. For these corollaries we use appropriate binomial transforms ([13], [20]).

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k, \quad n = 0, 1, \ldots \tag{1.7}$$

We also note that the binomial transform (1.7) can be inverted, with inversion sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k, \quad n = 0, 1, \ldots \tag{1.8}$$

It is good to notice that such binomial transforms can be computed conveniently by using the Euler series transformation formula (see [16], [18]).

$$\frac{1}{1-\lambda t} f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} \mu^k \lambda^{n-k} a_k\right\}$$

where $f(n)$ is as in (1.3) and λ, μ are parameters. With $\lambda = 1$ and $\mu = 1$, $\mu = -1$ we have correspondingly

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\}, \tag{1.9}$$

$$\frac{1}{1-t} f\left(\frac{-t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k\right\}. \tag{1.10}$$

In the sequel we shall use (1.9) or (1.10).

Lastly, we also introduce two transformation formulas which we need later.

Suppose we are given an entire function f and a function g, analytic in a region containing the disk $K = \{z : r < |z| < R\}$. Hence we have the following transformation formula ([3]),

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k g^{(k)}(x) \tag{1.11}$$

where $\left\{ {n \atop k} \right\}$ are the the Stirling numbers of the second kind.

A generalization of (1.11) is given in [12] as:

$$\sum_{n=r}^{\infty} \frac{g^{(n)}(0)}{n!} \binom{n}{r} \frac{r!}{n^r} f_r(n) t^n = \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r t^k g^{(k)}(t) \tag{1.12}$$

where $f_r(x)$ denotes the Maclaurin series of $f(x)$ exclude the first r terms and $\left\{ {n \atop k} \right\}_r$ are the the r-Stirling numbers of the second kind ([6]).

These two transformation formulas tightly coupled with the derivative operator $\left(t\frac{d}{dt}\right)$ which defined as:

$$\left(t\frac{d}{dt}\right) f(t) = tf'(t). \tag{1.13}$$

From (1.13) it is easy to see that

$$\left(t\frac{d}{dt}\right)^n f(t) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^k f^{(k)}(t). \tag{1.14}$$

2. Applications

This section consists of two parts. The first part depends on the Theorem 1.1 and transformation formulas (1.9), (1.10). Second part depends on transformation formulas (1.11) and (1.12).

In all series below the variable is taken in some neighborhood of zero, small enough to ensure convergence of both sides.

2.1. Transformation formulas (1.9) **and** (1.10)**.** Our first application is the following.

Corollary 2.1. *We have*

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{(n+1)!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t) \frac{t^n}{(n+1)!}. \tag{2.1}$$

Proof. Take $a_k = \frac{1}{k+1}$ in (1.4) and use the fact that

$$\frac{1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{k+1}. \tag{2.2}$$

In this case

$$f(t) = \frac{-\ln(1-t)}{t} \tag{2.3}$$

which is invariant under the transformation

$$f(t) \to \frac{1}{1-t} f\left(\frac{-t}{1-t}\right) \tag{2.4}$$

and (2.2) follows from (1.10). $\square$

For the rest of the section we shall use subtitles for the convenience of the reader.

2.1.1. *Harmonic numbers.*

Remark 2.2. Since the notation $H_n(x)$ for Hermite polynomials and the standard notation H_n for harmonic numbers are very much alike, in order to avoid confusion we shall use here the notation h_n for harmonic numbers.

Now let

$$h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},\ h_0 = 0 \tag{2.5}$$

be the usual harmonic numbers. They have the representation as binomial transform

$$-h_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1}{k}. \tag{2.6}$$

Formula (2.6) corresponds to $f(t) = -\ln(1-t)$ and to the generating function

$$\frac{1}{1-t} f\left(\frac{-t}{1-t}\right) = \frac{\ln(1-t)}{1-t} = -\sum_{n=0}^{\infty} h_n t^n. \tag{2.7}$$

Therefore, we obtain the following corollary (with $a_k = \frac{1}{k}$).

Corollary 2.3.

$$\sum_{n=1}^{\infty} H_n(x) \frac{t^n}{n!n} = e^{2xt-t^2} \sum_{n=1}^{\infty} (-1)^{n-1} H_n(x-t) h_n \frac{t^n}{n!}. \tag{2.8}$$

Next we have an identity "symmetric" to (2.8).

Corollary 2.4.

$$\sum_{n=1}^{\infty} H_n(x) h_n \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=1}^{\infty} (-1)^{n-1} H_n(x-t) \frac{t^n}{n!n} \tag{2.9}$$

Proof. This series results from (1.4) and the inversion of (2.6)

$$-\frac{1}{n} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k h_k. \tag{2.10}$$

□

Corollary 2.5.

$$\sum_{n=0}^{\infty} H_n(x) h_n \frac{t^n}{(n+1)!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t) \frac{t^n}{(n+1)!}. \tag{2.11}$$

Proof. The representation follows from (1.4) and the binomial identity,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{h_k}{k+1}=\frac{-h_n}{n+1}. \tag{2.12}$$

The generating function here is

$$f(t)=\frac{1}{2t}\ln^2(1-t)=\sum_{k=0}^{\infty}\frac{h_k}{k+1}t^k \tag{2.13}$$

with $a_k=\frac{h_k}{k+1}$. This function has the property

$$-f(t)=\frac{1}{1-t}f\left(\frac{-t}{1-t}\right) \tag{2.14}$$

and we apply (1.10) to obtain (2.12). □

The next corollary uses the "square" harmonic numbers

$$h_n^{(2)}=1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}. \tag{2.15}$$

Lemma 2.6. *The numbers $h_n^{(2)}$ have the binomial representation*

$$-h_n^{(2)}=\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{h_k}{k}. \tag{2.16}$$

Proof. It is easy to see that the generating function for the sequence $a_k=\frac{h_k}{k}$ is

$$f(t)=Li_2(t)+\frac{1}{2}\ln^2(1-t) \tag{2.17}$$

where

$$Li_2(t)=\sum_{n=1}^{\infty}\frac{t^n}{n^2} \tag{2.18}$$

is the dilogarithm function. This verification can be done by differentiating in (2.17) and using (2.8). The dilogarithm satisfies the Landen identity

$$Li_2(t)+\frac{1}{2}\ln^2(1-t)=-Li_2\left(\frac{-t}{1-t}\right),$$

that is, $f(t)=-Li_2\left(\frac{-t}{1-t}\right)$, and therefore

$$\frac{1}{1-t}f\left(\frac{-t}{1-t}\right)=\frac{-1}{1-t}Li_2(t)=-\sum_{n=1}^{\infty}h_n^{(2)}t^n \tag{2.19}$$

so (2.16) follows from (1.10). The proof is completed. □

Lemma 2.6 yields the following.

$$\sum_{n=1}^{\infty} H_n(x)\, h_n \frac{t^n}{n!n} = e^{2xt-t^2} \sum_{n=1}^{\infty} (-1)^{n-1} H_n(x-t)\, h_n^{(2)} \frac{t^n}{n!}. \tag{2.20}$$

The symmetric version of (2.20) based on the inversion of (2.16) is left to the reader.

2.1.2. *Bilinear series with Hermite and Laguerre polynomials.* An interesting and curious two-variable series exists involving the Hermite polynomials together with the Laguerre polynomials, $L_n(x)$, $n = 0, 1, \ldots$. The Laguerre polynomials have the representation

$$L_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{z^k}{k!} \tag{2.21}$$

which follows immediately by comparing their generating function

$$\frac{1}{1-t} \exp\left(\frac{-zt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(z)\, t^n \tag{2.22}$$

to (1.10) with $f(t) = e^{zt}$. From (2.21) we obtain via (1.4) .

Corollary 2.7.

$$\sum_{n=0}^{\infty} H_n(x) \frac{(zt)^n}{(n!)^2} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t)\, L_n(z) \frac{t^n}{n!}. \tag{2.23}$$

Even more interesting is the "symmetric" series resulting from the inversion of (2.21).

$$\frac{z^n}{n!} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k(z).$$

Namely, we have the following.

Corollary 2.8.

$$\sum_{n=0}^{\infty} H_n(x)\, L_n(z) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t) \frac{(zt)^n}{(n!)^2}. \tag{2.24}$$

2.1.3. *Bilinear series with Hermite polynomials.* The next result includes pairs of Hermite polynomials. We start with a well-known property

$$H_n(z+y) = \sum_{k=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(z). \tag{2.25}$$

This identity can be written as a binomial transform (dividing both sides by $(2y)^n$ and replacing y by $-y$),

$$(-1)^n \frac{H_n(z-y)}{(2y)^n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{H_k(z)}{(2y)^k}. \tag{2.26}$$

Applying now (1.4) with $a_k = \frac{H_k(z)}{(2y)^k}$ we obtain,

$$\sum_{n=0}^{\infty} H_n(x) H_n(z) \frac{1}{n!} \frac{t^n}{(2y)^n} = e^{2xt-t^2} \sum_{n=0}^{\infty} H_n(x-t) H_n(z-y) \frac{1}{n!} \frac{t^n}{(2y)^n}. \tag{2.27}$$

Replacing here t by $2ty$ we obtain our next result.

Corollary 2.9. *For all x, y, z and for t small enough we have*

$$\sum_{n=0}^{\infty} H_n(x) H_n(z) \frac{t^n}{n!} = e^{4xyt-4y^2t^2} \sum_{n=0}^{\infty} H_n(x-2yt) H_n(z-y) \frac{t^n}{n!}. \tag{2.28}$$

Note that the variable y does not appear in the left-hand side.

We can compare this expansion to the well-known bilinear series [[18], p. 198], [[22], p. 167],

$$\sum_{n=0}^{\infty} H_n(x) H_n(z) \frac{t^n}{n!} = \frac{1}{\sqrt{1-4t^2}} \exp\left\{ x^2 - \frac{(x-2zt)^2}{1-4t^2} \right\}, \tag{2.29}$$

i.e. Mehler's formula, to derive the equation

$$\frac{1}{\sqrt{1-4t^2}} \exp\left\{ x^2 - \frac{(x-2zt)^2}{1-4t^2} \right\} = \exp\left\{4xyt - 4y^2t^2\right\} \sum_{n=0}^{\infty} H_n(x-2yt) H_n(z-y) \frac{t^n}{n!}. \tag{2.30}$$

When $y = 0$, this turns into (2.29). In fact, (2.30) follows directly from (2.29) when replacing x by $x - 2yt$ and z by $z - y$ in the series on the left hand side.

2.1.4. *Binomial coefficients.* For the next corollary we use the binomial transform,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{k} = (-1)^n \binom{p}{n} \tag{2.31}$$

which is a version of the Vandermonde convolution formula ([13], [20]). Here p can be any complex number. The generating function for $a_k = \binom{p+k}{k}$ is

$$f(t) = (1-t)^{-p-1} = \sum_{k=0}^{\infty} \binom{p+k}{k} t^k \tag{2.32}$$

with

$$\frac{1}{1-t} f\left(\frac{-t}{1-t}\right) = (1-t)^p = \sum_{n=0}^{\infty} \binom{p}{n} (-1)^n t^n. \tag{2.33}$$

According to (1.4) we obtain.

Corollary 2.10. *For any complex* p*,*

$$\sum_{n=0}^{\infty} \binom{p+n}{n} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} \binom{p}{n} H_n(x-t) \frac{t^n}{n!}. \tag{2.34}$$

It is interesting that when p is a positive integer, the right-hand side is finite, i.e. we have the closed-form evaluation

$$\sum_{n=0}^{\infty} \binom{p+n}{n} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{p} \binom{p}{n} H_n(x-t) \frac{t^n}{n!}. \tag{2.35}$$

2.1.5. *Stirling numbers of the second kind.* We have the following equation for Stirling numbers of the second kind ([10], [15]):

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} = \left\{ {n+1 \atop m+1} \right\} \tag{2.36}$$

and inverse binomial transformation of (2.36) is

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left\{ {k+1 \atop m+1} \right\} = \left\{ {n \atop m} \right\} \tag{2.37}$$

which can equally well be written

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \left\{ {k+1 \atop m+1} \right\} = (-1)^n \left\{ {n \atop m} \right\}. \tag{2.38}$$

Corollary 2.11. *We have*

$$\sum_{n=0}^{\infty} \left\{ {n+1 \atop m+1} \right\} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} H_n(x-t) \frac{t^n}{n!}. \tag{2.39}$$

Proof. By setting $a_k = \left\{ {k+1 \atop m+1} \right\}$ in (1.4) and considering (2.38) we obtain (2.39). □

For our next corollary we use the Stirling numbers of the second kind extended for complex argument. Butzer et al. ([7]) defined the generalized Stirling numbers (Stirling functions of the second kind) by

$$\left\{ {\alpha \atop n} \right\} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^{\alpha} \tag{2.40}$$

for any complex number $\alpha \neq 0$. Obviously, this equation can be regarded as a binomial transform. In order to match (1.7) we rewrite (2.40) in the form (see also [1]).

$$(-1)^n n! \left\{ {\alpha \atop n} \right\} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{\alpha}. \tag{2.41}$$

This binomial identity with $a_k = k^{\alpha}$ leads to the following.

Corollary 2.12. *For every complex number $\alpha \neq 0$ we have*

$$\sum_{k=0}^{\infty} k^{\alpha} H_k(x) \frac{t^k}{k!} = e^{2xt-t^2} \sum_{n=0}^{\infty} \left\{ {\alpha \atop n} \right\} H_n(x-t) t^n. \tag{2.42}$$

In the case when $\alpha = m$ is a positive integer, $\left\{ {\alpha \atop n} \right\} = \left\{ {m \atop n} \right\}$ are the usual Stirling numbers of the second kind ([14]). These numbers have the property $\left\{ {m \atop n} \right\} = 0$ when $m < n$. Therefore, we obtain the closed form evaluation

$$\sum_{k=0}^{\infty} k^m H_k(x) \frac{t^k}{k!} = e^{2xt-t^2} \sum_{n=0}^{m} \left\{ {m \atop n} \right\} H_n(x-t) t^n \tag{2.43}$$

for every positive integer m. This formula was obtained independently in [14] and [21] by different means.

2.1.6. *Exponential numbers.* We have the following equation for exponential numbers ([8], [10])

$$\phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k \tag{2.44}$$

and inverse binomial transformation of (2.44) is

$$\phi_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \phi_{k+1} \tag{2.45}$$

which can equally well be written

$$(-1)^n \phi_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_{k+1}. \tag{2.46}$$

Then we have the following corollary.

Corollary 2.13. *We have*

$$\sum_{n=0}^{\infty} \phi_{n+1} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} \phi_n H_n(x-t) \frac{t^n}{n!}. \tag{2.47}$$

Proof. By setting $a_k = \phi_{k+1}$ in (1.4) and considering (2.46) we obtain (2.47). □

2.1.7. *Geometric numbers.* We have the following equation for geometric numbers ([8], [10])

$$2w_n = \sum_{k=0}^{n} \binom{n}{k} w_k \tag{2.48}$$

and inverse binomial transformation of (2.48) is

$$w_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2w_k \tag{2.49}$$

which can equally well be written

$$(-1)^n w_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2w_k. \tag{2.50}$$

Then we have the following corollary.

Corollary 2.14. *We have*

$$\sum_{n=0}^{\infty} 2w_n H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} w_n H_n(x-t) \frac{t^n}{n!}. \tag{2.51}$$

Proof. Setting $a_k = 2w_k$ in (1.4) and considering (2.50) gives (2.51). □

2.1.8. *Fibonacci numbers.* Let us consider the generating function of Fibonacci numbers

$$F(t) = \frac{t}{1-t-t^2} = \sum_{n=0}^{\infty} F_n t^n. \tag{2.52}$$

On the other hand we have,

$$\frac{1}{1-t} F\left(\frac{t}{1-t}\right) = \frac{t}{1-3t+t^2}. \tag{2.53}$$

But we know that ([9], [17]):

$$\frac{t}{1-3t+t^2} = \sum_{n=0}^{\infty} F_{2n} t^n. \tag{2.54}$$

Then from (2.54) and formula (1.9) we get

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k. \tag{2.55}$$

Here using inverse binomial transformation we get

$$F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{2k}. \tag{2.56}$$

Equation (2.56) may also be put in the form

$$(-1)^n F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2k}. \tag{2.57}$$

Corollary 2.15. *We have*

$$\sum_{n=0}^{\infty} F_{2n} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} F_n H_n(x-t) \frac{t^n}{n!}. \tag{2.58}$$

Proof. By setting $a_k = F_{2k}$ in (1.4) and considering (2.57) we obtain (2.58). □

Let us consider the function

$$\overline{F}(t) = \frac{-t}{1+t-t^2} = \sum_{n=0}^{\infty} (-1)^n F_n t^n. \tag{2.59}$$

Then we have

$$\frac{1}{1-t}\overline{F}\left(\frac{t}{1-t}\right)=\frac{-t}{1-t-t^2}=-\sum_{n=0}^{\infty}F_nt^n. \tag{2.60}$$

Now using (1.9) we get

$$-F_n=\sum_{k=0}^{n}\binom{n}{k}\left(-1\right)^kF_k. \tag{2.61}$$

Corollary 2.16. *We have*

$$\sum_{n=0}^{\infty}F_nH_n\left(x\right)\frac{t^n}{n!}=e^{2xt-t^2}\sum_{n=0}^{\infty}\left(-1\right)^{n+1}F_nH_n\left(x-t\right)\frac{t^n}{n!}. \tag{2.62}$$

Proof. By setting $a_k=F_k$ in (1.4) and considering (2.61) we obtain (2.62). □

Corollary 2.17. *We have*

$$\sum_{n=0}^{\infty}\left(-1\right)^nF_nH_n\left(x\right)\frac{t^n}{n!}=e^{2xt-t^2}\sum_{n=0}^{\infty}\left(-1\right)^nF_{2n}H_n\left(x-t\right)\frac{t^n}{n!}. \tag{2.63}$$

Proof. By setting $a_k=\left(-1\right)^kF_k$ in (1.4) and considering (2.55) we obtain (2.63). □

Similar transformation formulas can be obtain for Lucas numbers.

2.1.9. *Lucas numbers.* Let us consider the generating function of Lucas numbers

$$L\left(t\right)=\frac{2-t}{1-t-t^2}=\sum_{n=0}^{\infty}L_nt^n \tag{2.64}$$

Then we have

$$\frac{1}{1-t}L\left(\frac{-t}{1-t}\right)=\frac{2-t}{1-t-t^2}=\sum_{n=0}^{\infty}L_nt^n. \tag{2.65}$$

(2.65) combines with (1.10) to give

$$L_n=\sum_{k=0}^{n}\binom{n}{k}\left(-1\right)^kL_k. \tag{2.66}$$

Now we can state the following corollary.

Corollary 2.18. *We have*

$$\sum_{n=0}^{\infty} L_n H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n L_n H_n(x-t) \frac{t^n}{n!}. \tag{2.67}$$

Proof. By setting $a_k = L_k$ in (1.4) and considering (2.66) we obtain (2.67). □

Due to giving more applications let us consider the following generating function

$$\overline{L}(t) = \frac{2+t}{1+t-t^2} = \sum_{n=0}^{\infty} (-1)^n L_n t^n. \tag{2.68}$$

Then we have

$$\frac{1}{1-t}\overline{L}\left(\frac{-t}{1-t}\right) = \frac{2-3t}{1-3t+t^2}. \tag{2.69}$$

But we know that ([9], [17]):

$$\frac{2-3t}{1-3t+t^2} = \sum_{n=0}^{\infty} L_{2n} t^n. \tag{2.70}$$

From (2.70) and (1.10) it follows that

$$L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_k. \tag{2.71}$$

Now using inverse binomial transformation we get

$$L_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} L_{2k} \tag{2.72}$$

which can equally well be written

$$(-1)^n L_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2k}. \tag{2.73}$$

Now we have the following corollaries.

Corollary 2.19. *We have*

$$\sum_{n=0}^{\infty} L_{2n} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} L_n H_n(x-t) \frac{t^n}{n!}. \tag{2.74}$$

Proof. By setting $a_k = L_{2k}$ in (1.4) and considering (2.73) we obtain (2.74). □

Corollary 2.20. *We have*

$$\sum_{n=0}^{\infty} (-1)^n L_n H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n L_{2n} H_n(x-t) \frac{t^n}{n!}. \tag{2.75}$$

Proof. By setting $a_k = (-1)^k L_k$ in (1.4) and considering (2.71) we obtain (2.75). □

Now we give some results obtained by using transformation formulas (1.11) and (1.12).

2.2. Transformation formulas (1.11) **and** (1.12)**.** Applying the transformation formula (1.11) to the equation (1.1) we get the following formula which is a generalization of (2.43):

$$\sum_{n=0}^{\infty} H_n(x) f(n) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^k H_k(x-t). \tag{2.76}$$

If we set $f(t) = t^m$ in (2.76) we get (2.43).

It is possible to obtain more general results than (2.43) by setting $f(t)$ as an arbitrary polynomial of order m as

$$f(t) = p_m t^m + p_{m-1} t^{m-1} + \cdots + p_1 t + p_0$$

where $p_0, p_1, \ldots, p_{m-1}, p_m$ are any complex numbers. Hence we get following equation,

$$\sum_{n=0}^{\infty} \left(p_m n^m + p_{m-1} n^{m-1} + \cdots + p_1 n + p_0\right) H_n(x) \frac{t^n}{n!}$$
$$= e^{2xt-t^2} \sum_{n=0}^{m} p_n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^k H_k(x-t). \tag{2.77}$$

To obtain more general results, let us set $g(t) = e^{2xt-t^2}$ in the generalized transformation formula (1.12). Then we have

$$\sum_{n=r}^{\infty} H_n(x) \frac{f_r(n)}{(n-r)! n^r} t^n = e^{2xt-t^2} \sum_{n=r}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r t^k H_k(x-t). \tag{2.78}$$

If we set $f(t) = t^m$ such that $m \geq r$ in (2.78) we obtain

$$\sum_{n=r}^{\infty} n^{m-r} H_n(x) \frac{t^n}{(n-r)!} = e^{2xt-t^2} \sum_{k=0}^{m} \begin{Bmatrix} m \\ k \end{Bmatrix}_r t^k H_k(x-t) \tag{2.79}$$

which is a generalization of (2.43) .

Again to obtain more general formula than (2.79) we set $f(t) = p_m t^m + p_{m-1}t^{m-1} + \cdots + p_1 t + p_0$ in (2.78). Hence we get

$$\sum_{n=r}^{\infty} H_n(x) \frac{\left(p_m n^m + p_{m-1} n^{m-1} + \cdots + p_r n^r\right)}{(n-r)! n^r} t^n$$
$$= e^{2xt-t^2} \sum_{n=r}^{m} p_n \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_r t^k H_k(x-t).$$

2.2.1. *Results using transformation formula* (1.11). The generating function of Hermite polynomials is an entire function. Therefore we can consider $f(t) = e^{2xt-t^2}$ in (1.11). Then we have

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} e^{2nx-n^2} t^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} t^k g^{(k)}(t). \tag{2.80}$$

i) If we set $g(t) = t^m$ in the formula (2.80) we get

$$e^{2mx-m^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (m)_k \tag{2.81}$$

where $(m)_k$ is the Pochhammer symbol, i.e.

$$(m)_k = m(m-1)(m-2)\ldots(m-k+1). \tag{2.82}$$

Now comparison of the coefficients of the both sides in (2.81) gives the following well known equation:

$$m^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (m)_k \tag{2.83}$$

Later we generalize (2.83).

ii) If we set $g(t) = e^t$ in the formula (2.80) we get

$$\sum_{m=0}^{\infty} e^{2mx-m^2} \frac{t^m}{m!} = e^t \sum_{n=0}^{\infty} \frac{H_n(x)\phi_n(t)}{n!} \tag{2.84}$$

where $\phi_n(t)$ is nth exponential polynomial ([3], [4], [10]). This can equally well be written by means generating function of Hermite polynomials as:

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}m^n\right)\frac{t^m}{m!}=e^t\sum_{n=0}^{\infty}\frac{H_n(x)\phi_n(t)}{n!}.$$

Then we have

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}\sum_{m=0}^{\infty}m^n\frac{t^m}{m!}=e^t\sum_{n=0}^{\infty}\frac{H_n(x)\,\phi_n(t)}{n!}.$$

Hence we get following equation

$$\left(t\frac{d}{dt}\right)^n e^t=e^t\phi_n(t)\,. \tag{2.85}$$

The equation (2.85) can be found in [3].

iii) If we set $g(t)=\frac{1}{1-t}$ in the formula (2.80) we get

$$\sum_{k=0}^{\infty}e^{2kx-k^2}t^k=\frac{1}{1-t}\sum_{n=0}^{\infty}\frac{H_n(x)\,w_n\left(\frac{t}{1-t}\right)}{n!}$$

where $w_n(t)$ is nth geometric polynomial ([3], [10]). Now this can equally well be written

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}k^n\right)t^k=\frac{1}{1-t}\sum_{n=0}^{\infty}\frac{H_n(x)w_n\left(\frac{t}{1-t}\right)}{n!}.$$

By rearranging we get

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}\sum_{k=0}^{\infty}k^nt^k=\frac{1}{1-t}\sum_{n=0}^{\infty}\frac{H_n(x)\,w_n\left(\frac{t}{1-t}\right)}{n!}.$$

This can equally well be written by means of $\left(t\frac{d}{dt}\right)$ operator as

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}\left(t\frac{d}{dt}\right)^n\frac{1}{1-t}=\frac{1}{1-t}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}w_n\left(\frac{t}{1-t}\right)$$

Then we have

$$\left(t\frac{d}{dt}\right)^n\frac{1}{1-t}=\frac{1}{1-t}w_n\left(\frac{t}{1-t}\right). \tag{2.86}$$

The equation (2.86) also can be found in [3].

2.2.2. *Results using transformation formula* (1.12). We can generalize previous results by considering the generalized transformation formula (1.12).

Let us take $f(t) = e^{2xt-t^2}$ in (1.12). Then we have

$$\sum_{n=r}^{\infty} \frac{g^{(n)}(0)}{n!} \binom{n}{r} \frac{r!}{n^r} \left(\sum_{s=r}^{\infty} H_s(x) \frac{n^s}{s!} \right) t^n = \sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r t^k g^{(k)}(t). \quad (2.87)$$

i) If we set $g(t) = t^m$, $(m \geq r)$ in the formula (2.87) we get

$$\sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \binom{m}{r} r! m^{n-r} = \sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r \binom{m}{k} k!. \quad (2.88)$$

From (2.88) we get a generalization of (2.83) as follows:

$$(m)_r m^{n-r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r (m)_k . \quad (2.89)$$

In (2.89) we use the Pochhammer symbol from (2.82).

ii) If we set $g(t) = e^t$ in the formula (2.87) we get

$$\sum_{s=r}^{\infty} \frac{H_s(x)}{s!} \sum_{n=r}^{\infty} \frac{n^{s-r} t^n}{(n-r)!} = e^t \sum_{n=r}^{\infty} \frac{H_n(x)_r \phi_n(t)}{n!}, \quad (2.90)$$

where ${}_r\phi_n(t)$ is the nth r-exponential polynomial ([12]). After rearranging (2.90) we get

$$\sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \left(t\frac{d}{dt} \right)^{n-r} t^r e^t = e^t \sum_{n=r}^{\infty} \frac{H_n(x)_r \phi_n(t)}{n!}. \quad (2.91)$$

Equation (2.91) gives a generalization of (2.85) as

$$\left(t\frac{d}{dt} \right)^{n-r} t^r e^t =_r \phi_n(t) e^t \quad (2.92)$$

iii) If we set $g(t) = \frac{1}{1-t}$ in the formula (2.87) we get

$$\sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \sum_{m=r}^{\infty} \binom{m}{r} r! m^{n-r} t^m = \sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \frac{{}_r w_n\left(\frac{t}{1-t}\right)}{1-t} \quad (2.93)$$

where ${}_r w_n(t)$ is the nth r-geometric polynomial ([12]). By rearranging (2.93) we get

$$\sum_{n=r}^{\infty} \frac{H_n(x)}{n!} r! \left(t\frac{d}{dt} \right)^{n-r} \frac{t^r}{(1-t)^{r+1}} = \sum_{n=r}^{\infty} \frac{H_n(x)}{n!} \frac{{}_r w_n\left(\frac{t}{1-t}\right)}{1-t}. \quad (2.94)$$

From (2.94) we see that

$$r! \left(t\frac{d}{dt} \right)^{n-r} \frac{t^r}{(1-t)^{r+1}} = \frac{1}{1-t} \, {}_r w_n\left(\frac{t}{1-t} \right) \quad (2.95)$$

which is a generalization of (2.86).

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(Received December 20, 2010; revised May 9, 2011)