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Invariance of Bajraktarevič mean with respect to quasi arithmetic means

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Abstract. Without any regularity conditions, we determine all Bajraktarevič means $B^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right)$ which are invariant respect to the mean-type mapping $\left(A^{[f]}, A^{[g]}\right)$ where $A^{[f]}$ denotes the quasi-arithmetic mean generated by f. The case of weighted quasi-arithmetic means is also considered. Applications in iteration theory and functional equation are included. A relation of the invariance of means and their "harmony" is mentioned.

1. Introduction

A function $M : I^2 \to \mathbb{R}$ is called a *mean* in an interval $I \subseteq \mathbb{R}$, if

$$\min(x,y) \leq M(x,y) \leq \max(x,y), \quad x,y \in I.$$

A mean M is called *strict*, if for all $x,y \in I$, $x \neq y$, these inequalities are sharp; and *symmetric,* if $M(x,y) = M(y,x)$ for all $x,y \in I$ (cf. Bullen–Mitrinović–Vasić [4]).

If M is a mean in I then $M(J^2) = J$ for every subinterval $J \subseteq I$. Moreover M is *reflexive*, i.e.

$$M(x,x) = x, \quad x \in I.$$

Every reflexive function $M : I^2 \to \mathbb{R}$ which is increasing with respect to each variable is a mean in I.

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Let $M, N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is called *invariant with respect to the mean-type mapping* $(M,N) : I^2 \to I^2$ (briefly, K is (M,N)-*invariant*), if

$$K(M(x,y), N(x,y)) = K(x,y), \quad x, y \in I.$$

The mean K is also referred to as the Gauss composition of the means M and N (cf. [5]). The invariant mean is useful when we are looking for the limits of the sequence of iterates of the mean-type mapping $(M,N) : I^2 \to I^2$ (cf. Borwein–Borwein [3], also [12], [13], [14]).

Let us note that the proportion $x : \frac{x+y}{2} = \frac{2xy}{x+y} : y$, the base of the theory of harmony made by Pythagorean school, can be written in the form

$$\sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} = \sqrt{xy}.$$

Setting $A(x,y) = \frac{x+y}{2}$, $H(x,y) = \frac{2xy}{x+y}$ and $G(x,y) = \sqrt{xy}$ for the arithmetic, harmonic and geometric mean, respectively, we hence get $G \circ (A,H) = G$ which says that the geometric mean G is (A,H)-invariant. This fact allows to determine effectively the limit of the sequence of iterates $((A,H)^n)_{n\in\mathbb{N}}$ of the mean-type mapping $(A,H) : (0,\infty)^2 \to (0,\infty)^2$, namely (cf. [12], [14]), we have

$$\lim_{n\to\infty} (A,H)^n = (G,G),$$

that is useful in the theory of functional equations.

To find other triples of means satisfying the invariance (or "harmony") condition, consider the following problem. Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{K}$ be three classes of means in the interval I. Determine all means means $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $K \in \mathcal{K}$ such that K is (M,N)-invariant. Recently this problem has been completely solved in the case when $\mathcal{M} = \mathcal{N} = \mathcal{K} = \mathcal{A}$ where $\mathcal{A}$ is the class of quasi-arithmetic means. Recall that for every continuous and strictly monotonic function $f : I \to \mathbb{R}$ and $p \in (0,1)$ the function $M = A_p^{[f]} : I^2 \to I$,

$$A_p^{[f]}(x,y) := f^{-1}\left(pf(x) + (1-p)f(y)\right), \quad x, y \in I,$$

is a mean, and it is called *quasi-arithmetic weighted mean.* The function f is called a *generator* of the mean and p its *weight.* Of course, every quasi-arithmetic weighted mean is increasing and continuous. If $p = \frac{1}{2}$ the mean $A^{[f]} := A_{1/2}^{[f]}$, that is

$$A^{[f]}(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \quad x, y \in I,$$

is called *quasi-arithmetic* (cf. [4]). In the class $\mathcal{A}$ the invariance problem reduces to functional equation

$$A^{[f]}(x,y) + A^{[g]}(x,y) = x + y.$$

The analytic solutions f, g of this equation were examined by Sutô [15], twice continuously differentiable solutions by the present author [13] and continuously differentiable solutions by Daróczy and Maksa [5]. A complete solution was done by Daróczy and Páles [6]. The invariance problem in the case when $\mathcal{M} = \mathcal{N} = \mathcal{K}$ is the class of weighted quasi-arithmetic means was first treated in Jarczyk and Matkowski [10] in the class C^2 and it has been completely solved by Jarczyk [8].

Let the functions $f, g : I \to \mathbb{R}$ be continuous, $g(x) \neq 0$ for $x \in I$, and such that $\frac{f}{g}$ is one-to-one. Then the function $B^{[f,g]} : I^2 \to I$ defined by

$$B^{[f,g]}(x,y) = \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right), \quad x, y \in I,$$

is a mean in I and it is called Bajraktarevič mean of generators f and g ([3]). $B^{[f,g]}$ is a strict mean, and it is a generalization of quasi-arithmetic mean. If g is constant then $B^{[f,g]} = A^{[f]}$ and, if f is constant then $B^{[f,g]} = A^{[g]}$. Thus the class $\mathcal{B}$ of all Bajraktarevič means in I is essentially larger than $\mathcal{A}$.

In the present paper we solve the invariance problem $K \circ (M,N) = K$ in the case when $K \in \mathcal{B}$ and $M, N \in \mathcal{A}$.

Let us add that the invariance equation $K \circ (M,N) = K$ where K is a quasi-arithmetic mean with weight-functions and $M, N \in \mathcal{B}$, under C^4 regularity condition of some of the involved functions, was considered by Jarczyk [10]. A special case, under the same regularity conditions, was considered earlier by Domsta and Matkowski [7].

2. Main result

We begin with (cf. [1], p. 246, Corollary 5):

Remark 1. If $f, g : I \to \mathbb{R}$ are continuous and one-to-one and $p \in (0,1)$, then $A_p^{[f]} = A_p^{[g]}$ if, and only if, for some $a.b \in \mathbb{R}$, $a \neq 0$,

$$f(x) = ag(x) + b, \quad x \in I.$$

The following facts are easy to verify.

Remark 2. Assume that $f, g : I \to \mathbb{R}$ are continuous and one-to-one and $g(x) \neq 0$ for all $x \in I$. Then, for arbitrary $a, b \in \mathbb{R}$, $ab \neq 0$,

$$B^{[af,bg]} = B^{[f,g]}.$$

Remark 3. Assume that $f, g : I \to \mathbb{R}$ are continuous and one-to-one and $f(x)g(x) \neq 0$. Then

$$B^{[f,g]} = B^{[g,f]}$$

The main results reads as follows:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval. Suppose that the functions $f, g : I \to \mathbb{R}$ are one-to-one, continuous, $f(x)g(x) \neq 0$ for $x \in I$, and $\frac{f}{g}$ is one-to-one. Then the mean $B^{[f,g]}$ is $(A^{[f]}, A^{[g]})$-invariant, i.e.,*

$$B^{[f,g]} \circ (A^{[f]}, A^{[g]}) = B^{[f,g]}, \tag{1}$$

if, and only if, either there are $a, b \in \mathbb{R}$, $a \neq 0 \neq b$, such that

$$f(x) = ag(x) + b, \quad x \in I,$$

and

$$B^{[f,g]} = A^{[g]} = A^{[f]}.$$

or there is $c \in \mathbb{R}$, $c \neq 0$, such that

$$f(x) = \frac{c}{g(x)}, \quad x \in I,$$

and

$$B^{[f,g]}(x, y) = g^{-1}\left(\sqrt{g(x)g(y)}\right), \quad A^{[f]}(x, y) = g^{-1}\left(\frac{2g(x)g(y)}{g(x) + g(y)}\right), \quad x, y \in I.$$

Proof. Assume that equation (1) holds. Hence, by the definitions of $B^{[f,g]}$, $A^{[f]}$ and $A^{[g]}$, we get

$$\frac{f\left(A^{[f]}(x, y)\right) + f\left(A^{[g]}(x, y)\right)}{g(A^{[f]}(x, y)) + g\left(A^{[g]}(x, y)\right)} = \frac{f(x) + f(y)}{g(x) + g(y)}, \quad x, y \in I, \tag{2}$$

whence, of course,

$$\frac{\frac{f(x)+f(y)}{2} + f\left(A^{[g]}(x, y)\right)}{g(A^{[f]}(x, y)) + \frac{g(x)+g(y)}{2}} = \frac{f(x) + f(y)}{g(x) + g(y)}, \quad x, y \in I,$$

which reduces to the equation

$$f\left(g^{-1}\left(\frac{g(x)+g(y)}{2}\right)\right)(g(x)+g(y))=g\left(f^{-1}\left(\frac{f(x)+f(y)}{2}\right)\right)(f(x)+f(y)),$$

for all $x,y\in I$. Setting

$$\varphi(u):=f\left(g^{-1}(u)\right),\quad u\in g(I),$$

we can write this equation in the form

$$\varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right)(\varphi(u)+\varphi(v))=\varphi\left(\frac{u+v}{2}\right)(u+v),\quad u,v\in g(I).$$

Taking $t=\frac{u+v}{2}$ for $u,v\in g(I)$ we hence get

$$\varphi^{-1}\left(\frac{\varphi(u)+\varphi(2t-u)}{2}\right)(\varphi(u)+\varphi(2t-u))=\varphi(t)\,t \tag{3}$$

for all $t,u\in g(I)$ such that $2t-u\in g(I)$.

The properties of means imply that equations (2) and (3) remain true if we replace the interval I by an arbitrary subinterval $J\subset I$.

Assume that there is a nontrivial subinterval $J_1\subset I$ such that for every $t\in g(J_1)$ the function

$$g(J_1)\ni u\to\varphi(u)+\varphi(2t-u) \tag{4}$$

is a constant (depending on $t\in J_1$). Then

$$\varphi(u)+\varphi(2t-u)=\varphi(v)+\varphi(2t-v)$$

for all $t,u,v\in g(J_1)$ such that $2t-u,2t-v\in g(J_1)$. Taking $t=\frac{w+v}{2}$ for $v,w\in g(J_1)$ we hence get

$$\varphi(u)+\varphi(w+v-u)=\varphi(v)+\varphi(w)$$

for all $u,v,w\in g(J_1)$ such that $w+v-u\in g(J_1)$. For $u=\frac{v+w}{2}$ we hence obtain

$$\varphi\left(\frac{v+w}{2}\right)=\frac{\varphi(u)+\varphi(v)}{2},\quad v,w\in g(J_1).$$

Applying a well-known result (cf. Aczél [1], p. 43, Kuczma [11], p. 316) there are $a, b \in \mathbb{R}$ such that $\varphi(u) = g \circ f^{-1}(u) = au + b$ for $u \in g(J_1)$, and $a \neq 0$, as $f \circ g^{-1}$ is invertible. Thus

$$f(x) = ag(x) + b, \quad x \in J_1, \tag{5}$$

and, consequently,

$$A^{[f]}(x, y) = A^{[g]}(x, y) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right), \quad x, y \in J_1. \tag{6}$$

Since

$$\frac{f(x)}{g(x)} = \frac{ag(x) + b}{g(x)}, \quad x \in J_1,$$

the injectivity of $\frac{f}{g}$ implies that $b \neq 0$ as, in the opposite case, $\frac{f}{g}$ would be constant. Consequently,

$$\left(\frac{f}{g}\right)^{-1}(u) = g^{-1}\left(\frac{b}{u - a}\right), \quad u \in g(J_1).$$

Hence, by the definition of $B^{[f,g]}$, for all $x, y \in J_1$,

$$\begin{aligned} B^{[f,g]}(x, y) &= g^{-1}\left(b\left(\frac{f(x) + f(y)}{g(x) + g(y)} - a\right)^{-1}\right) \\ &= g^{-1}\left(b\left(\frac{ag(x) + b + ag(y) + b}{g(x) + g(y)} - a\right)\right) \\ &= g^{-1}\left(\frac{g(x) + g(y)}{2}\right) = A^{[g]}(x, y), \end{aligned}$$

that is

$$B^{[f,g]}(x, y) = A^{[g]}(x, y), \quad x, y \in J_1. \tag{7}$$

Of course we can assume that J_1 is a maximal subinterval of I such that formula (5) holds true.

If for some $t \in I$ the function (4) is not constant then, by (3), the function $z \to \varphi^{-1}(\frac{z}{2})z$ is constant on the range of the function (4). Consequently there is a nontrivial subinterval $J_2 \subset I$ such that the function $u \to \varphi(u)u$ is constant on $g(J_2)$, that is $\varphi(u)u = c$ for some $c \in \mathbb{R}$, $c \neq 0$ and all $u \in g(J_2)$. Hence, by the definition of φ,

$$f(x) = \frac{c}{g(x)}, \quad x \in J_2, \tag{8}$$

whence, making easy calculations, we get

$$B^{[f,g]}(x,y) = g^{-1}\left(\sqrt{g(x)g(y)}\right),$$

$$A^{[f]}(x,y) = g^{-1}\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right), \quad x,y \in J_2. \tag{9}$$

In this case we also may assume that J_2 is a maximal subinterval of I such that f has the form (8). Clearly, we have

$$\operatorname{int} J_1 \cap \operatorname{int} J_2 = \emptyset.$$

Thus we have proved that every $x \in I$ belongs either to the interval of the type J_1 or to the interval of type J_2.

To end the proof it is enough to show that if $\operatorname{int} J_1 \neq \emptyset$ then $\operatorname{int} J_2 = \emptyset$, and, consequently, $J_1 = I$.

Assume, on the contrary, that this implication is false. Then, putting $\alpha := \inf J_1$ and $\beta := \sup J_1$ we would have either $\inf I < \alpha$ or $\beta < \sup I$.

Assume that $\beta < \sup I$.

Take $x_0 \in \operatorname{int} J_1$. Since

$$f^{-1}\left(\frac{f(x_0)+f(\beta)}{2}\right) \in \operatorname{int} J_1 \quad \text{and} \quad g^{-1}\left(\frac{g(x_0)+g(\beta)}{2}\right) \in \operatorname{int} J_1$$

the continuity of f and g implies that there exists $\delta > 0$ such that

$$f^{-1}\left(\frac{f(x)+f(y)}{2}\right) \in \operatorname{int} J_1 \quad \text{and} \quad g^{-1}\left(\frac{g(x)+g(y)}{2}\right) \in \operatorname{int} J_1$$

for all $x \in (x_0-\delta, x_0+\delta)$, $y \in [\beta, \beta+\delta)$. Moreover, the maximality of J_1 implies that there exists an open interval $J \subset (\beta, \beta+\delta)$ such that $\operatorname{int} J \neq \emptyset$ and J is contained in an interval of the type J_2.

Then, by (5) and (8), for all $x \in (x_0-\delta, x_0+\delta)$, $y \in [\beta, \beta+\delta)$, we have

$$\begin{aligned} A^{[f]}(x,y) &= f^{-1}\left(\frac{f(x)+f(y)}{2}\right) = f^{-1}\left(\frac{(ag(x)+b)+\frac{c}{g(y)}}{2}\right) \\ &= g^{-1}\left(\frac{\frac{(ag(x)+b)+\frac{c}{g(y)}}{2}-b}{a}\right) = g^{-1}\left(\frac{ag(x)-b+\frac{c}{g(y)}}{2a}\right) \end{aligned}$$

for some real $c \neq 0$.

By (6) and (7) it follows that, for all $x \in (x_0 - \delta, x_0 + \delta)$, $y \in [\beta, \beta + \delta)$,

$$\begin{aligned}
B^{[f,g]}(A^{[f]}(x,y), A^{[g]}(x,y)) &= A^{[g]}(A^{[f]}(x,y), A^{[g]}(x,y)) \\
&= g^{-1}\left(\frac{g\big(g^{-1}\big(\frac{ag(x)-b+\frac{c}{g(y)}}{2a}\big)\big) + g\big(g^{-1}\big(\frac{g(x)+g(y)}{2}\big)\big)}{2}\right) \\
&= g^{-1}\left(\frac{2ag(x) - b + \frac{c}{g(y)} + ag(y)}{4a}\right).
\end{aligned}$$

On the other hand, according to the first part of the proof, for each $(x, y) \in (x_0 - \delta, x_0 + \delta) \times [\beta, \beta + \delta)$, we have either

$$\begin{aligned}
B^{[f,g]}(x,y) &= \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right) = g^{-1}\left(\frac{B}{\frac{(ag(x)+b)+\frac{c}{g(y)}}{g(x)+g(y)} - A}\right) \\
&= g^{-1}\left(\frac{B(g(x)+g(y))}{(a-A)g(x) + b + \frac{c}{g(y)} - Ag(y)}\right)
\end{aligned}$$

for some $A, B \in \mathbb{R}$ such that $B \neq 0$, or

$$\begin{aligned}
B^{[f,g]}(x,y) &= \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right) = g^{-1}\left(\sqrt{\frac{C}{\frac{(ag(x)+b)+\frac{c}{g(y)}}{g(x)+g(y)}}}\right) \\
&= g^{-1}\left(\sqrt{\frac{C(g(x)+g(y))}{(ag(x)+b) + \frac{c}{g(y)}}}\right)
\end{aligned}$$

for some real $C \neq 0$.

Hence, from (1), we have that either

$$g^{-1}\left(\frac{2ag(x) - b + \frac{c}{g(y)} + ag(y)}{4a}\right) = g^{-1}\left(\frac{B(g(x)+g(y))}{(a-A)g(x) + b + \frac{c}{g(y)} - Ag(y)}\right)$$

or

$$g^{-1}\left(\frac{2ag(x) - b + \frac{c}{g(y)} + ag(y)}{4a}\right) = g^{-1}\left(\sqrt{\frac{C(g(x)+g(y))}{(ag(x)+b) + \frac{c}{g(y)}}}\right)$$

for all $(x, y) \in (x_0 - \delta, x_0 + \delta) \times [\beta, \beta + \delta)$. As each of these equalities implies that g is a constant function on a nontrivial interval, we obtain the desired contradiction.

Thus, if there exists a nontrivial interval J_1, then $I = J_1$ and if there is a nontrivial interval J_2, then $I = J_2$.

Since in the case when $\inf I < \alpha$ the reasoning is similar, we omit it.

Applying, respectively, either the formulas (6) and (7) or (8) and (9), we obtain "the only if" part of our theorem. Since the "if" part is easy to check, the proof is complete. □

Remark 4. Theorem 1 can be proved without using functional equation (3).

To show it note that the invariance equation (1), equivalent to (2), can be written in the form

$$\frac{f\left(A^{[f]}(x,y)\right)+f\left(A^{[g]}(x,y)\right)}{g(A^{[f]}(x,y))+g\left(A^{[g]}(x,y)\right)}=\frac{f(A^{[f]}(x,y))}{g(A^{[g]}(x,y))},\quad x,y\in I,$$

which simplifies to

$$f\left(A^{[f]}(x,y)\right)g(A^{[f]}(x,y))=f(A^{[g]}(x,y))g(A^{[g]}(x,y)),\quad x,y\in I. \tag{10}$$

Suppose first that there exists a nontrivial subinterval $J_1\subset I$ such that

$$A^{[f]}(x,y)=A^{[g]}(x,y),\quad x,y\in J_1.$$

By Remark 1 it follows that there are $a,b\in\mathbb{R}$, $a\neq 0$, such that

$$f(x)=ag(x)+b,\quad x\in J_1. \tag{11}$$

To consider the opposite case, assume that $A^{[f]}$ and $A^{[g]}$ do not coincide on any nontrivial subinterval of I.

Take arbitrary $x_0,y_0\in I$, $x_0<y_0$, and define $\gamma:[x_0,y_0]\to[x_0,y_0]$ by

$$\gamma(u):=\begin{cases}[c]cg^{-1}\left(\dfrac{g(x_0)+g\circ f^{-1}\left(2f(u)-f(x_0)\right)}{2}\right) & \text{for } u\in[x_0,A^{[f]}(x_0,y_0)]\\ g^{-1}\left(\dfrac{g\circ f^{-1}\left(2f(u)-f(y_0)+g(y_0)\right)}{2}\right) & \text{for } u\in[A^{[f]}(x_0,y_0),y_0].\end{cases}$$

Of course, γ is continuous. It is not difficult to verify that the mean type mapping $(A^{[f]},A^{[g]})$ maps $[x_0,y_0]^2$ onto the set contained between the diagonal $\{(u,u):u\in[x_0,y_0]\}$ and the graph of the function γ. Moreover, replacing, if necessary, x_0 and y_0 by arbitrary close x_0' and y_0', we can guarantee that the equality $\gamma(u)=u$ does not hold true in the whole interval $[x_0,y_0]$. Indeed, in the opposite case, we would have $f=ag+b$ on a nontrivial subinterval of $[x_0,y_0]$.

As in the above reasoning the numbers $x_0,y_0\in I$, $x_0<y_0$, has been chosen arbitrarily, we conclude that the set $W:=(A^{[f]},A^{[g]})(I^2)$, the range of the mean-type mapping $(A^{[f]},A^{[g]})$, has the following property: for every $w_0\in I$ and

$\varepsilon > 0$ there are $u_0 \in I$ and $\delta > 0$ such that $|u_0 - w_0| < \varepsilon$ and for all $u, v \in I \cap (u_0 - \delta, u_0 + \delta)$, $u \neq v$, there are $x, y \in I$ such that

$$(A^{[f]}(x,y), A^{[g]}(x,y)) = (u,v).$$

Hence, by (10), we get

$$f(u)\, g(u) = f(v) g(v)$$

for all $u, v \in I \cap (u_0 - \delta, u_0 + \delta)$, $u \neq v$, which means that the function fg is constant in $J_2 := I \cap (u_0 - \delta, u_0 + \delta)$.

Now, applying (11), we can repeat the suitable argument of the proof of Theorem 1.

3. The case of weighted quasi-arithmetic means

The harmony proportion $x : A_p(x,y) = H_p(x,y) : y$, where

$$A_p(x,y) := px + (1-p)y, \quad H_p(x,y) = \frac{1}{\frac{1-p}{x} + \frac{p}{y}}, \quad x, y > 0,$$

remains true for arbitrary $p \in (0,1)$, so the geometric mean G is invariant with respect to the mean-type mapping (A_p, H_p). To examine the possibility of extension of the results of the previous section to the weighted quasi-arithmetic means, we begin with the following

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval, $f, g : I \to (0, \infty)$ be continuously differentiable, one-to-one, $\frac{f}{g}$ one-to-one, and $p, r \in (0,1)$ fixed. If*

$$B^{[f,g]} \circ (A_p^{[f]}, A_r^{[g]}) = B^{[f,g]}, \tag{12}$$

then $p + r = 1$.

Proof. From (12), by the definitions of the means $B^{[f,g]}$, $A_p^{[f]}$, $A_r^{[g]}$, we have

$$\frac{pf(x) + (1-p)f(y) + f\left(g^{-1}(rg(x) + (1-r)g(y)\right)}{g\left(f^{-1}(pf(x) + (1-p)f(y)\right) + rg(x) + (1-r)g(y)} = \frac{f(x) + f(y)}{g(x) + g(y)}, \quad x, y \in I,$$

whence, for all $x, y \in I$,

$$\begin{aligned}[g(x) + g(y)] &\left[pf(x) + (1-p)f(y) + f\left(g^{-1}(rg(x) + (1-r)g(y))\right)\right] \\ &= [f(x) + f(y)] \left[g\left(f^{-1}(pf(x) + (1-p)f(y)\right) + rg(x) + (1-r)g(y)\right].\end{aligned}$$

Differentiating the functions of both sides with respect to x we get, for all $x, y \in I$,

$$\begin{aligned} g'(x) & \left[pf(x) + (1-p)f(y) + f\left(A_r^{[g]}\right)\right] \\ & + [g(x)+g(y)]\left[pf'(x) + \frac{f'\left(A_r^{[g]}\right)}{g'\left(A_r^{[g]}\right)} rg'(x)\right] \\ & = f'(x)\left[g\left(A_p^{[f]}\right) + rg(x) + (1-r)g(y)\right] \\ & + [f(x)+f(y)]\left[\frac{g'\left(A_p^{[f]}\right)}{f'\left(A_p^{[f]}\right)} pf'(x) + rg'(x)\right], \end{aligned} \tag{13}$$

where $A_p^{[f]} = A_p^{[f]}(x,y)$, $A_r^{[g]} = A_r^{[g]}(x,y)$. Hence, letting $y \to x$, after simple calculations, we get

$$(p+r-1)\left[g'(x)f(x) - f'(x)g(x)\right] = 0, \quad x \in I.$$

For an indirect argument assume that $p + r \neq 1$. Then

$$g'(x)f(x) - f'(x)g(x) = 0, \quad x \in I,$$

whence

$$f(x) = cg(x), \quad x \in I,$$

and, consequently, the function $\frac{f}{g}$ would be constant, contrary to the assumption. □

This lemma suggests the following

Problem 1. *Let $I \subset \mathbb{R}$ be an interval and $p \in (0,1)$ fixed. Determine all continuous and strictly monotonic functions $f, g : I \to (0,\infty)$ such that $g(x) \neq 0$ for $x \in I$, $\frac{f}{g}$ is one-to-one, and*

$$B^{[f,g]} \circ (A_p^{[f]}, A_{1-p}^{[g]}) = B^{[f,g]}.$$

Note that this equation can be written in the form

$$\begin{aligned} & [f(x)+f(y)]\left[g\left(f^{-1}(pf(x)+(1-p)f(y))\right] + (2p-1)\right) f(y)g(y) \\ & \quad = [g(x)+g(y)]\left[f\left(g^{-1}((1-p)g(x)+pg(y))\right] + (2p-1)\right) f(x)g(x) \end{aligned}$$

for all $x, y \in I$. Setting

$$\varphi := f \circ g^{-1}$$

we hence get, for all $u, v \in g(I)$,

$$[\varphi(u) + \varphi(v)]\,\varphi^{-1}\left(p\varphi(u) + (1-p)\varphi(v)\right) \\ = (u+v)\varphi\left((1-p)u + pv\right) + (2p-1)\left[\varphi(u)u - \varphi(v)v\right]$$

that is

$$[\varphi(u)+\varphi(v)]A_p^{[\varphi]}(u,v) = (u+v)A_{1-p}^{[\mathrm{id}]}(u,v) + (2p-1)[\varphi(u)u - \varphi(v)v], \quad u, v \in g(I).$$

Interchanging here u and v we get

$$[\varphi(u)+\varphi(v)]A_{1-p}^{[\varphi]}(u,v) = (u+v)A_p^{[\mathrm{id}]}(u,v) + (2p-1)[\varphi(v)v - \varphi(u)u], \quad u, v \in g(I).$$

Adding the respective sides of the last two equations we obtain the functional equation

$$[\varphi(u)+\varphi(v)]\left[A_p^{[\varphi]}(u,v) + A_{1-p}^{[\varphi]}(u,v)\right] = (u+v)\left[A_{1-p}^{[\mathrm{id}]}(u,v) + A_p^{[\mathrm{id}]}(u,v)\right], \\ u, v \in g(I).$$

Assuming additionally some regularity conditions on the functions f and g we can prove the following

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $p \in (0,1)$ fixed. Suppose that one-to-one functions $f, g : I \to (0,\infty)$ are twice continuously differentiable, $f'(x)$ $g'(x) \neq 0$ for $x \in I$, and $\frac{f}{g}$ one-to-one. Then $B^{[f,g]}$ is $(A_p^{[f]}, A_{1-p}^{[g]})$-invariant, i.e.*

$$B^{[f,g]} \circ (A_p^{[f]}, A_{1-p}^{[g]}) = B^{[f,g]}, \tag{14}$$

if, and only if, either there are $a, b \in \mathbb{R}$, $a \neq 0 \neq b$, such that

$$f(x) = ag(x) + b, \quad x \in I,$$

and

$$B^{[f,g]} = A^{[g]},$$

or fg is constant and, for all $x, y \in I$,

$$B^{[f,g]}(x,y) = g^{-1}\left(\sqrt{g(x)g(y)}\right), \quad A_p^{[f]}(x,y) = g^{-1}\left(\frac{2g(x)g(y)}{(1-p)g(x) + pg(y)}\right).$$

Proof. Taking $\frac{\partial^2}{\partial x \partial y}$ of both sides of (14) (that is calculating $\frac{\partial}{\partial y}$ of both sides of (13)) and then letting $y \to x$ we get the differential equation

$$(f'g + fg')(f''g' - f'g'') = 0.$$

As this procedure is easy, we omit writing too long formulas. Assume that there is nontrivial subinterval $J \subset I$ such that $f'(x)g(x) + f(x)g'(x) \neq 0$ for all $x \in J$. Then

$$f''(x)g'(x) - f'(x)g''(x) = 0, \quad x \in J,$$

whence, for some $a, b \in \mathbb{R}$, $a \neq 0$,

$$f(x) = ag(x) + b, \quad x \in J.$$

Similarly, if for a nontrivial subinterval $J \subset I$ we have $f''(x)g'(x) - f'(x)g''(x) \neq 0$ for all $x \in J$, then

$$f'(x)g(x) + f(x)g'(x) = 0, \quad x \in J,$$

whence, for some $c \neq 0$,

$$f(x) = \frac{c}{g(x)}, \quad x \in J.$$

Now, repeating the suitable argument applied in the proof of Theorem 1, we complete the proof. □

4. Applications

Let us quote the following ([12], [14]):

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval. If $M, N : I^2 \to I$ are continuous means such that at least one of them is strict, then*

(1) *the sequence of iterates of the mean-type mapping $(M, N) : I^2 \to I^2$ converges to a continuous mean-type mapping $(K, K) : I^2 \to I^2$ where $K : I^2 \to I$ is a continuous mean;*
(2) *K is (M, N)-invariant, i.e. $K \circ (M, N) = K$;*
(3) *a continuous (M, N)-invariant mean is unique;*
(4) *if the means M, N are strict, then so is K.*

Applying this result and Theorem 1 we obtain the following:

Theorem 4. *Let $g : I \to \mathbb{R}$ be continuous, one-to-one and $g(x) \neq 0$ for $x \in I$. Let $M, N, K : I^2 \to I$ be defined by*

$$M(x,y) := g^{-1}\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right), \quad N(x,y) := g^{-1}\left(\frac{g(x)+g(y)}{2}\right),$$
$$K(x,y) := g^{-1}\left(\sqrt{g(x)g(y)}\right).$$

Then

(1) *the mean K is (M,N)-invariant*

(2) *the sequence $((M,N)^n)_{n\in\mathbb{N}}$ of iterates of the mean-type mapping (M,N) converges pointwise in I^2 and $\lim_{n\to\infty}(M,N)^n = (K,K)$.*

(3) *a function $\Phi : I^2 \to \mathbb{R}$, continuous on the diagonal $\{(x,x) : x \in I\}$, satisfies the functional equation*

$$\Phi\left(g^{-1}\left(\frac{g(x)+g(y)}{2}\right), g^{-1}\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right)\right) = \Phi(x,y), \quad x,y \in I,$$

if, and only if, there is a continuous function in a single variable $\varphi : I \to \mathbb{R}$ such that

$$\Phi(x,y) = \varphi\left(g^{-1}\left(\sqrt{g(x)g(y)}\right)\right), \quad x,y \in I.$$

Proof. The first part is easy to verify. The second is a consequence of Theorem 3. To prove part 3 write the functional equation in the form

$$\Phi(M(x,y), N(x,y)) = \Phi(x,y), \quad x,y \in I,$$

and note that, by induction,

$$\Phi((M(x,y), N(x,y))^n) = \Phi(x,y), \quad n \in \mathbb{N}, x,y \in I,$$

Letting $n \to \infty$ and making use of part 2, we obtain

$$\Phi(x,y) = \Phi((K(x,y), K(x,y))), \quad x,y \in I,$$

whence, setting $\varphi(u) := \Phi(u,u)$ for $u \in I$, we get the desired form of Φ.The converse implication is easy to verify. □

Remark 5. Applying Theorems 2 and 3 one gets a more general result for the weighted quasi-arithmetic means.

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