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Finite groups determined by an inequality of the orders of their elements

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Abstract. In this note we introduce and characterize a class of finite groups for which the element orders satisfy a certain inequality. This is contained in some well-known classes of finite groups.

1. Introduction

Let CP_1, CP and CN be the classes of finite groups in which the centralizers of all nontrivial elements contain only elements of prime order, of prime power order and are nilpotent, respectively. Clearly, we have $\mathrm{CP}_1 \subset \mathrm{CP} \subset \mathrm{CN}$. Moreover, the classes CP_1 and CP consist of exactly those finite groups all of whose elements have prime order and prime power order, respectively. They have been studied in many papers, as [1]–[3], [5]–[7] and [13].

In the following we consider the finite groups G such that

$$o(xy) \leq \max\{o(x), o(y)\}, \quad \text{for all } x, y \in G. \tag{$*$}$$

These form another interesting subclass of CP, that will be denoted by CP_2. Its exhaustive description is the main goal of this note.

Most of our notation is standard and will not be repeated here. Basic notions and results on group theory can be found in [4], [8], [9], [11].

First of all, we observe that if a finite group G belongs to CP_2, then for every $x, y \in G$ satisfying $o(x) \neq o(y)$ we have

$$o(xy) = \max\{o(x), o(y)\},$$

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that is the order map is very close to a monoid homomorphism from $(G, \cdot)$ to $(\mathbb{N}^*, \max)$.

An immediate characterization of finite groups contained in CP_2 is indicated in the following theorem.

Theorem A. *Let G be a finite group and set $\pi_e(G) = \{o(x) \mid x \in G\}$. Then the following conditions are equivalent:*

a) *G belongs to CP_2.*

b) *For every $\alpha \in \pi_e(G)$, the set $G_\alpha = \{x \in G \mid o(x) \leq \alpha\}$ is a normal subgroup of G.*

Next, we will focus on establishing some connections between CP_2 and the previous classes CP and CP_1.

Proposition B. *The class CP_2 is properly contained in the class CP.*

On the other hand, by taking $\sigma = (12)(34), \tau = (235) \in A_5$, one obtains

$$5 = o(\sigma\tau) > 3 = \max\{o(\sigma), o(\tau)\},$$

and therefore CP_2 does not contain the alternating group A_5. Since A_5 belongs to CP_1, we conclude that CP_1 is not contained in CP_2. It is obvious that the converse inclusion also fails (for example, any abelian p-group belongs to CP_2, but not to CP_1).

Remarks.

1. Other two remarkable classes of finite p-groups, more large as the class of abelian p-groups, are contained in CP_2: regular p-groups (see Theorem 3.14 of [11], II, page 47) and p-groups whose subgroup lattices are modular (see Lemma 2.3.5 of [10]). Moreover, by the main theorem of [12], we infer that the powerful p-groups for p odd also belong to CP_2.
2. The smallest nonabelian p-group contained in CP_2 is the quaternion group Q_8, while the smallest p-group not contained in CP_2 is the dihedral group D_8. Notice that all quaternion groups Q_{2^n}, for $n \geq 4$, as well as all dihedral groups D_n, for $n \neq 1, 2, 4$, are not contained in CP_2.
3. The class CP_2 contains finite groups which are not p-groups, too. The smallest example of such a group is A_4. Remark that the groups A_n, $n \geq 5$, does not belong to CP_2, and this is also valid for the symmetric groups S_n, $n \geq 3$.

Clearly, CP_2 is closed under subgroups. On the other hand, the above results imply that CP_2 is not closed under direct products or extensions. The same thing can be said with respect to homomorphic images, as shows the following example.

Example. Let p be a prime and G be the semidirect product of an elementary abelian p-group A of order p^p by a cyclic group of order p^2, generated by an element x which permutes the elements of a basis of A cyclically. Then it is easy to see that G belongs to CP_2, $x^p \in Z(G)$ and the quotient $Q = \frac{G}{\langle x^p \rangle}$ is isomorphic to a Sylow p-subgroup of S_{p^2}. Obviously, in Q a product of two elements of order p can have order p^2, and hence it does not belong to CP_2.

The next result collects other basic properties of the finite groups contained in CP_2.

Proposition C. *Let G be a finite group contained in* CP_2*. Then:*

a) *There is a prime p dividing the order of G such that $F(G) = O_p(G)$.*

b) *Both $Z(G)$ and $\Phi(G)$ are p-groups.*

c) *$Z(G)$ is trivial if G is not a p-group.*

We are now able to present our main result, that gives a complete description of the class CP_2.

Theorem D. *A finite group G is contained in* CP_2 *if and only if one of the following statements holds:*

a) *G is a p-group and $\Omega_n(G) = \{x \in G \mid x^{p^n} = 1\}$, for all $n \in \mathbb{N}$.*

b) *G is a Frobenius group of order $p^\alpha q^\beta$, $p < q$, with kernel $F(G)$ of order p^α and cyclic complement.*

Since all p-group and all groups of order $p^\alpha q^\beta$ are solvable, Theorem D leads to the following corollary.

Corollary E. *The class* CP_2 *is properly contained in the class of finite solvable groups.*

Remark. The finite supersolvable groups and the CLT-groups constitute two important subclasses of the finite solvable groups. Since A_4 belongs to CP_2, we infer that CP_2 is not included in these classes. Conversely, a finite supersolvable group or a CLT-group does not necessarily possess the structure described above, and thus they are not necessarily contained in the class CP_2.

As we already have seen, both CP_1 and CP_2 are subclasses of CP, and each of them is not contained in the other. Consequently, an interesting problem is to find the intersection of these subclasses. This can be made by using again Theorem D.

Corollary F. *A finite group G is contained in the intersection of* CP_1 *and* CP_2 *if and only if one of the following statements holds:*

a) *G is a p-group of exponent p.*

b) *G is a Frobenius group of order $p^\alpha q$, $p < q$, with kernel $F(G)$ of order p^α and exponent p, and cyclic complement. Moreover, in this case we have $G' = F(G)$.*

Remark. A_4 is an example of a group of type b) in the above corollary. Mention that for such a group G the number of Sylow q-subgroups is p^α. It is also clear that G possesses a nontrivial partition consisting of Sylow subgroups: $F(G)$ and all conjugates of a Frobenius complement.

Finally, we indicate a natural problem concerning the class of finite groups introduced in our paper.

Open problem. Give a precise description of the structure of finite p-groups contained in CP_2.

2. Proofs of the main results

Proof of Theorem A. Assume first that G belongs to CP_2. Let $\alpha \in \pi_e(G)$ and $x, y \in G_\alpha$. Then, by $(*)$, we have

$$o(xy) \leq \max\{o(x), o(y)\} \leq \alpha,$$

which shows that $xy \in G_\alpha$. This proves that G_α is a subgroup of G. Moreover, G_α is normal in G because the order map is constant on each conjugacy class.

Conversely, let $x, y \in G$ and put $\alpha = o(x), \beta = o(y)$. By supposing that $\alpha \leq \beta$, one obtains $x, y \in G_\beta$. Since G_β is a subgroup of G, it follows that $xy \in G_\beta$. Therefore

$$o(xy) \leq \beta = \max\{o(x), o(y)\},$$

completing the proof. □

Proof of Proposition B. Let G be finite group in CP_2 and take $x \in G$. It is well-known that x can be written as a product of (commuting) elements of prime power orders, say $x = x_1 x_2 \cdots x_k$. Then the condition $(*)$ implies that

$$\prod_{i=1}^{k} o(x_i) = o(x) \leq \max\{o(x_i) \mid i = \overline{1,k}\,\},$$

and so $k = 1$. Hence x is of prime power order, i.e. G is contained in CP.

Obviously, the inclusion of CP_2 in CP is strict (we already have seen that A_5 belongs to CP, but not to CP_2). □

Proof of Proposition C.

a) We know that $F(G)$ is the product of the subgroups $O_p(G)$, where p runs over the prime divisors of $|G|$. Suppose that there are two distinct primes p and q dividing the order of $F(G)$. This leads to the existence of two elements x and y of $F(G)$ such that $o(x) = p$ and $o(y) = q$. Since $F(G)$ is nilpotent, we obtain $xy = yx$ and so $o(xy) = pq$, a contradiction. Thus $F(G) = O_p(G)$, for a prime divisor p of $|G|$.

b) It is well-known that both $Z(G)$ and $\Phi(G)$ are normal nilpotent subgroup of G. By the maximality of $F(G)$, it follows that $Z(G)$ and $\Phi(G)$ are contained in $F(G)$, and therefore they are also p-groups.

c) Assume that $Z(G)$ is not trivial and take $x \in Z(G)$ with $o(x) = p$. If G is not a p-group, it contains an element y of prime order $q \neq p$. Then $o(xy) = pq$, contradicting Proposition B. □

Proof of Theorem D. If G is a p-group, then the conclusion is obvious.

Assume now that G is not a p-group. We will proceed by induction on $|G|$. Since G belongs to CP_2, all the numbers in $\pi_e(G)$ are prime powers. Let q^n be the largest number of $\pi_e(G)$, where q is a prime, and let $N = \{g \in G \mid o(g) < q^n\}$. Then $N \trianglelefteq G$ and $\exp(G/N) = q$. Since $|N| < |G|$, by the inductive hypothesis it follows that either N is a p-group or N is a Frobenius group with kernel K of order p^α and cyclic complement H of order r^β, where p, r are distinct primes. We will prove that in both cases G is a Frobenius group whose kernel and complement are p-groups.

Case 1. N is a p-group.
Since G is not a p-group, we can take $Q \in Syl_q(G)$, where $p \neq q$. So $G = N \rtimes Q$. Since every element of N is of prime power order, we have $C_N(h) = 1$ for all $1 \neq h \in Q$. Thus, G is a Frobenius group with kernel N and complement Q.

Case 2. N is a Frobenius group.

Subcase 2.1. $q \neq p$ and $q \neq r$.
By a similar argument as that of Case 1, we know that G is a Frobenius group with kernel N. But N is not nilpotent, a contradiction.

Subcase 2.2. $q = r$.
Let $Q \in Syl_q(G)$. Then $G = K \rtimes Q$. By a similar argument as that of Case 1, we know that G is a Frobenius group with kernel K and complement Q.

Subcase 2.3. $q = p$.
We observe that all elements of $G \backslash N$ are of order q^n, and $g^q \in K$, where $g \in G \backslash N$ and $K \in Syl_q(N)$. So if $N_G(H) \cap (G \setminus N) \neq 1$, then $N_G(H) \cap K \neq 1$. But N

is a Frobenius group and $N_N(H) = H$. It follows that $N_G(H) = H$. Since H is cyclic, $N_G(H) = C_G(H)$. One obtains that G is r-nilpotent and thus $G = P \rtimes H$, where $P \in Syl_p(G)$. A similar argument as that of Case 1 shows that G is again a Frobenius group with kernel P and complement H.

Finally, we prove that H is cyclic. By Burnside's Theorem we only need to prove that H is not a 2-group. If not, let $L = \{g \in G \mid o(g) = 2\}$. Then $L \trianglelefteq G$. It follows that $K \times L \leq G$, where K is the Frobenius kernel. This contradicts the fact that all elements of G are of prime power order. □

Proof of Corollary F. The equivalence follows directly by Theorem D. In this way, we have to prove only that $G' = F(G)$ in the case b).

Obviously, $G' \subseteq F(G)$. For the converse inclusion, let $x \in F(G)$ be a nontrivial element. Then $o(x) = p$. If y is an arbitrary element of order q in G, then we have

$$o(xy) \leq \max\{o(x), o(y)\} = q,$$

and therefore $o(xy) \in \{p, q\}$. If we assume that $o(xy) = p$, it results

$$q = o(y) = o(x^{-1}xy) \leq \max\{o(x^{-1}), o(xy)\} = p,$$

a contradiction. This shows that $o(xy) = q$. Then there is $z \in G$ such that $xy \in \langle y \rangle^z$, say $xy = z^{-1}y^k z$ with $k \in \mathbb{Z}$. Since the element

$$xy^{1-k} = z^{-1}y^k zy^{-k} = [z, y^k]$$

has order p, we infer that k must be equal to 1. Hence

$$x = [z, y] \in G',$$

which completes the proof. □

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