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Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators

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Abstract. The main objective of this paper is an improvement of the original weighted operator arithmetic-geometric mean inequality in Hilbert space. We define the difference operator between the arithmetic and geometric means, and investigate its properties. Due to the derived properties, we obtain a refinement and a converse of the observed operator mean inequality. As an application, we establish one significant operator mean, which interpolates the arithmetic and geometric means, that is, the Heinz operator mean. We also obtain an improvement of this interpolation.

1. Introduction

Let H be a Hilbert space and let $\mathcal{B}_h(H)$ be the semi-space of all bounded linear self-adjoint operators on H. Further, let $\mathcal{B}^+(H)$ and $\mathcal{B}^{++}(H)$, respectively, denote the sets of all positive and positive invertible operators in $\mathcal{B}_h(H)$. The weighted operator arithmetic mean ∇_ν and geometric mean $\sharp_\nu$, for $\nu \in [0,1]$ and $A, B \in \mathcal{B}^{++}(H)$, are defined as follows:

$$A\nabla_\nu B = (1-\nu)A + \nu B, \tag{1}$$

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}. \tag{2}$$

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If $\nu = 1/2$, we denote the arithmetic and geometric means, respectively, by ∇ and $\sharp$.

Like in the classical case, the arithmetic-geometric mean inequality holds

$$A \sharp_\nu B \leq A \nabla_\nu B, \quad \nu \in [0,1], \tag{3}$$

with respect to operator order. The above definitions and properties will be valid throughout the whole paper. For more details, see [3].

Both the classical and the operator arithmetic-geometric mean inequalities lie in the fields of interest of numerous mathematicians. For example, Kittaneh and Manasrah (see [5], [6]) obtained the following improvement of the classical arithmetic-geometric mean inequality:

$$\begin{aligned} &a^\nu b^{1-\nu} + \max\{\nu, 1-\nu\}\big(\sqrt{a} - \sqrt{b}\,\big)^2 \geq \nu a + (1-\nu) b \\ &\qquad \geq a^\nu b^{1-\nu} + \min\{\nu, 1-\nu\}\big(\sqrt{a} - \sqrt{b}\,\big)^2, \quad a, b \geq 0,\ \nu \in [0,1]. \end{aligned} \tag{4}$$

The first inequality in (4) can be regarded as a converse, and the second one as a refinement of the arithmetic-geometric mean inequality.

On the other hand, one remarkable mean, which interpolates the arithmetic and geometric means is the so called Heinz mean H_ν, defined by

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}, \quad a, b \geq 0,\ \nu \in [0,1]. \tag{5}$$

It is obvious that $\sqrt{ab} \leq H_\nu(a,b) \leq (a+b)/2$, and Kittaneh and Manasrah [5], obtained the following improvement:

$$H_\nu(a,b) + \min\{\nu, 1-\nu\}\big(\sqrt{a} - \sqrt{b}\,\big)^2 \leq \frac{a+b}{2}, \quad a, b \geq 0,\ \nu \in [0,1]. \tag{6}$$

Further, they (see [5]) gave some matrix variants of inequalities (4) and (6), including trace and norms, that are also improvements of previously known matrix inequalities, from the literature. For more details about matrix variants of the arithmetic, geometric, Heinz means, and related inequalities, the reader is referred to [1], [2], and [4].

In [7], one can find a matrix form of (3) which asserts that if $B, C \in M_n(\mathbb{C})$ are such that B is positive definite, C is invertible, $A = C^*C$, and $\nu \in [0,1]$,then

$$C^* \left(C^{*-1} B C^{-1}\right)^\nu C \leq A \nabla_\nu B. \tag{7}$$

Clearly, if $C = A^{\frac{1}{2}}$, then $C^* \left(C^{*-1}BC^{-1}\right)^{\nu} C = A\sharp_{\nu}B$, so we may regard the operator $C^* \left(C^{*-1}BC^{-1}\right)^{\nu} C$ as a generalization of the geometric mean.

In [6], the authors obtained matrix extensions of relations (4) and (6) in view of (3) and (7). For example, a matrix variant of (4), obtained in [6], claims that if $B, C \in M_n(\mathbb{C})$ are such that B is positive definite, C is invertible, $A = C^*C$, and $\nu \in [0, 1]$, then

$$\begin{aligned} 2\max\{\nu, 1-\nu\} \left[A\nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right] &\geq A\nabla_{\nu}B - C^* \left(C^{*-1}BC^{-1}\right)^{\nu} C \\ &\geq 2\min\{\nu, 1-\nu\} \left[A\nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right]. \end{aligned} \tag{8}$$

As distinguished from matrices in [5] and [6], we shall consider more general approach in extensions of inequalities (4) and (6), described at the beginning of this Introduction. Namely, we shall get operator extensions of all presented inequalities in the Hilbert space setting.

For example, in [10], the reader can find several refinements and converses of the operator arithmetic-geometric mean inequality. The same problem area is also considered in [8], [9], and [11]. In addition, for a comprehensive inspection of the recent results about inequalities for bounded self-adjoint operators on Hilbert space, the reader is referred to [3].

The paper is organized in the following way: After this Introduction, In Section 2 we deduce some auxiliary results, concerning interpolation of the classical weighted arithmetic-geometric mean inequality. Such interpolating inequalities will be used in obtaining our main results. Afterwards, in Section 3 we define the operator difference between the arithmetic and geometric means in Hilbert space, and prove its important properties of superadditivity and monotonicity. These properties enable us to establish improvements, that is, a refinement and a converse of operator arithmetic-geometric mean inequality (3). As an application, in Section 4 we define the Heinz operator mean and obtain an extension of relation (6) in Hilbert space.

The techniques that will be used in the proofs are mainly based on classical real and functional analysis, especially on the well known monotonicity property for operator functions.

2. Auxiliary results

To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions: If $X \in$

$\mathcal{B}_h(H)$ with a spectrum $\mathrm{Sp}(X)$ and f, g are continuous real-valued functions on $\mathrm{Sp}(X)$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(X) \implies f(X) \geq g(X). \tag{9}$$

For more details about this property, the reader is referred to [3].

In the sequel, we have to obtain a refinement, that is, an interpolating inequality for the classical weighted arithmetic-geometric mean inequality. For that sake, we define the functional $J : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$J(\mathbf{x}, \mathbf{p}) = (p_1 + p_2) \left[A(\mathbf{x}, \mathbf{p}) - G(\mathbf{x}, \mathbf{p}) \right], \tag{10}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{p} = (p_1, p_2)$, and

$$A(\mathbf{x}, \mathbf{p}) = \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}, \quad G(\mathbf{x}, \mathbf{p}) = (x_1^{p_1} x_2^{p_2})^{\frac{1}{p_1+p_2}}.$$

Obviously, functional (10) is non-negative. Its properties are contained in the following result.

Lemma 1. *If* $\mathbf{x}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^2_+$, *then functional* (10) *possess the following properties:*
(i) $J(\mathbf{x}, \cdot)$ *is superadditive on* $\mathbb{R}^2_+$, *i.e.,*

$$J(\mathbf{x}, \mathbf{p} + \mathbf{q}) \geq J(\mathbf{x}, \mathbf{p}) + J(\mathbf{x}, \mathbf{q}). \tag{11}$$

(ii) *If* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2_+$ *with* $\mathbf{p} \geq \mathbf{q}$ *(i.e.,* $p_1 \geq q_1$, $p_2 \geq q_2$*), then*

$$J(\mathbf{x}, \mathbf{p}) \geq J(\mathbf{x}, \mathbf{q}) \geq 0, \tag{12}$$

i.e., $J(\mathbf{x}, \cdot)$ *is increasing on* $\mathbb{R}^2_+$.

Proof. (i) The expression $J(\mathbf{x}, \mathbf{p} + \mathbf{q})$ can be rewritten in the following form:

$$\begin{aligned} J(\mathbf{x}, \mathbf{p} + \mathbf{q}) &= (p_1 + q_1) x_1 + (p_2 + q_2) x_2 \\ &\quad - (p_1 + q_1 + p_2 + q_2) \left(x_1^{p_1+q_1} x_2^{p_2+q_2} \right)^{\frac{1}{p_1+q_1+p_2+q_2}} \\ &= (p_1 x_1 + p_2 x_2) + (q_1 x_1 + q_2 x_2) \\ &\quad - (p_1 + p_2 + q_1 + q_2) \left(x_1^{\frac{p_1}{p_1+p_2}} x_2^{\frac{p_2}{p_1+p_2}} \right)^{\frac{p_1+p_2}{p_1+p_2+q_1+q_2}} \\ &\quad \times \left(x_1^{\frac{q_1}{q_1+q_2}} x_2^{\frac{q_2}{q_1+q_2}} \right)^{\frac{q_1+q_2}{p_1+p_2+q_1+q_2}}. \end{aligned} \tag{13}$$

On the other hand, by the arithmetic-geometric mean inequality, we have

$$\left(x_1^{\frac{p_1}{p_1+p_2}} x_2^{\frac{p_2}{p_1+p_2}}\right)^{\frac{p_1+p_2}{p_1+p_2+q_1+q_2}} \left(x_1^{\frac{q_1}{q_1+q_2}} x_2^{\frac{q_2}{q_1+q_2}}\right)^{\frac{q_1+q_2}{p_1+p_2+q_1+q_2}}$$
$$\leq \frac{(p_1+p_2)x_1^{\frac{p_1}{p_1+p_2}} x_2^{\frac{p_2}{p_1+p_2}} + (q_1+q_2)x_1^{\frac{q_1}{q_1+q_2}} x_2^{\frac{q_2}{q_1+q_2}}}{p_1+p_2+q_1+q_2}. \tag{14}$$

Thus, relation (13), together with inequality (14), yields the inequality

$$\begin{aligned} J(\mathbf{x}, \mathbf{p}+\mathbf{q}) \geq (p_1+p_2) &\left[\frac{p_1x_1+p_2x_2}{p_1+p_2} - (x_1^{p_1}x_2^{p_2})^{\frac{1}{p_1+p_2}}\right] \\ &+ (q_1+q_2)\left[\frac{q_1x_1+q_2x_2}{q_1+q_2} - (x_1^{q_1}x_2^{q_2})^{\frac{1}{q_1+q_2}}\right] = J(\mathbf{x}, \mathbf{p}) + J(\mathbf{x}, \mathbf{q}), \end{aligned}$$

that is, the superadditivity of $J(\mathbf{x}, \cdot)$ on $\mathbb{R}_+^2$.
(ii) Monotonicity follows easily from superadditivity. Since $\mathbf{p} \geq \mathbf{q}$, we can represent the ordered pair $\mathbf{p} \in \mathbb{R}_+^2$ as the sum of two ordered pairs in $\mathbb{R}_+^2$, namely $\mathbf{p} = (\mathbf{p}-\mathbf{q}) + \mathbf{q}$. Now, from relation (11) we get

$$J(\mathbf{x}, \mathbf{p}) = J(\mathbf{x}, \mathbf{p}-\mathbf{q}+\mathbf{q}) \geq J(\mathbf{x}, \mathbf{p}-\mathbf{q}) + J(\mathbf{x}, \mathbf{q}).$$

Finally, since $J(\mathbf{x}, \mathbf{p}-\mathbf{q}) \geq 0$, it follows that $J(\mathbf{x}, \mathbf{p}) \geq J(\mathbf{x}, \mathbf{q})$, which completes the proof. □

As an immediate consequence of the monotonicity property (12), we obtain lower and upper bounds for functional (10).

Corollary 1. *If* $\mathbf{x}, \mathbf{p} \in \mathbb{R}_+^2$, *then*

$$\max\{p_1, p_2\}\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq J(\mathbf{x}, \mathbf{p}) \geq \min\{p_1, p_2\}\left(\sqrt{x_1} - \sqrt{x_2}\right)^2. \tag{15}$$

Proof. We can compare the ordered pair $\mathbf{p} \in \mathbb{R}_+^2$ with the constant ordered pairs

$$\mathbf{p}_{\max} = \left(\max\{p_1, p_2\}, \max\{p_1, p_2\}\right) \text{ and } \mathbf{p}_{\min} = \left(\min\{p_1, p_2\}, \min\{p_1, p_2\}\right).$$

Clearly, $\mathbf{p}_{\max} \geq \mathbf{p} \geq \mathbf{p}_{\min}$, hence, yet another use of property (12) yields an interpolating series of inequalities:

$$J(\mathbf{x}, \mathbf{p}_{\max}) \geq J(\mathbf{x}, \mathbf{p}) \geq J(\mathbf{x}, \mathbf{p}_{\min}).$$

Now,

$$\begin{aligned}J(\mathbf{x},\mathbf{p}_{\max}) &= 2\max\{p_1,p_2\}\left[A(\mathbf{x},\mathbf{p}_{\max})-G(\mathbf{x},\mathbf{p}_{\max})\right]\\ &= 2\max\{p_1,p_2\}\left[\frac{x_1+x_2}{2}-\sqrt{x_1x_2}\right]=\max\{p_1,p_2\}\left(\sqrt{x_1}-\sqrt{x_2}\right)^2,\end{aligned}$$

and similarly, $J(\mathbf{x},\mathbf{p}_{\min})=\min\{p_1,p_2\}\left(\sqrt{x_1}-\sqrt{x_2}\right)^2$, from which we get relation (15). $\square$

Remark 1. It is obvious that the first inequality in (15), from the left, yields a converse of the arithmetic-geometric mean inequality, while the second one provides an appropriate refinement. In addition, if $p_1+p_2=1$, then inequality (15) becomes relation (4) from Introduction.

3. Main results

With the help of the results from Lemma 1 and Corollary 1, in this section, we obtain a refinement and a converse of the operator arithmetic-geometric mean inequality (3). Therefore, we have to establish the difference operator between the arithmetic and geometric means in Hilbert space. In a more general manner, we define the operator $\mathcal{L}:\mathcal{B}^{++}(H)\times\mathcal{B}^{-1}(H)\times\mathbb{R}^2_+\to\mathcal{B}^+(H)$ as

$$\mathcal{L}(B,C,\mathbf{p})=(p_1+p_2)\left[A\nabla_{\frac{p_1}{p_1+p_2}}B-C^*\left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}}C\right], \tag{16}$$

where $A=C^*C$. Here, $\mathcal{B}^{-1}(H)$ denotes the set of bounded linear invertible operators in the Hilbert space H. We have to justify the previous definition, i.e., to conclude that $\mathcal{L}(B,C,\mathbf{p})$ is positive with respect to operator order. That will be clarified in the sequel. Basic properties of the operator $\mathcal{L}$ are listed in our main result.

Theorem 1. *Suppose H is a Hilbert space, $A,B\in\mathcal{B}^{++}(H)$, $C\in\mathcal{B}^{-1}(H)$, $A=C^*C$, and $\mathbf{p},\mathbf{q}\in\mathbb{R}^2_+$. Then, operator (16) satisfies the following properties:*
(i) *$\mathcal{L}(B,C,\cdot)$ is superadditive on $\mathbb{R}^2_+$, that is,*

$$\mathcal{L}(B,C,\mathbf{p}+\mathbf{q})\geq\mathcal{L}(B,C,\mathbf{p})+\mathcal{L}(B,C,\mathbf{q}). \tag{17}$$

(ii) *If $\mathbf{p},\mathbf{q}\in\mathbb{R}^2_+$ with $\mathbf{p}\geq\mathbf{q}$ (i.e., $p_1\geq q_1$, $p_2\geq q_2$), then*

$$\mathcal{L}(B,C,\mathbf{p})\geq\mathcal{L}(B,C,\mathbf{q})\geq 0, \tag{18}$$

i.e., $\mathcal{L}(B,C,\cdot)$ is increasing on $\mathbb{R}^2_+$.

Proof. (i) We use the monotonicity property (9) for operator functions and property (11) from Lemma 1. Namely, if we substitute $x_1 = x$ and $x_2 = 1$ in (11), we get inequality

$$j(x, \mathbf{p} + \mathbf{q}) \geq j(x, \mathbf{p}) + j(x, \mathbf{q}), \tag{19}$$

where

$$j(x, \mathbf{p}) = p_1 x + p_2 - (p_1 + p_2) x^{\frac{p_1}{p_1 + p_2}}. \tag{20}$$

On the other hand, since $B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$ the operator $C^{*-1}BC^{-1}$ is well defined and strictly positive. This means that spectrum $\mathrm{Sp}\big(C^{*-1}BC^{-1}\big)$ is positive.

Now, since inequality (19) holds for all $x \in \mathbb{R}_+$, according to the monotonicity property (9), we can replace x in (19) with $C^{*-1}BC^{-1}$. We get the inequality

$$j\big(C^{*-1}BC^{-1}, \mathbf{p} + \mathbf{q}\big) \geq j\big(C^{*-1}BC^{-1}, \mathbf{p}\big) + j\big(C^{*-1}BC^{-1}, \mathbf{q}\big), \tag{21}$$

where $j\big(C^{*-1}BC^{-1}, \mathbf{p}\big) = p_1 C^{*-1}BC^{-1} + p_2 1_H - (p_1 + p_2)\big(C^{*-1}BC^{-1}\big)^{\frac{p_1}{p_1+p_2}}$. Here, 1_H denotes the identity operator on the Hilbert space H.

In addition, if we multiply inequality (21) by C^* on the left, and C on the right, we get

$$C^* j\big(C^{*-1}BC^{-1}, \mathbf{p} + \mathbf{q}\big) C \geq C^* j\big(C^{*-1}BC^{-1}, \mathbf{p}\big) C + C^* j\big(C^{*-1}BC^{-1}, \mathbf{q}\big) C. \tag{22}$$

Finally, since

$$\begin{aligned} C^* j\big(C^{*-1}BC^{-1}, \mathbf{p}\big) C &= p_1 B + p_2 C^* C - (p_1 + p_2) C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C \\ &= (p_1 + p_2) \left[A \nabla_{\frac{p_1}{p_1+p_2}} B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C \right] \\ &= \mathcal{L}\left(B, C, \mathbf{p}\right), \end{aligned} \tag{23}$$

relation (22) becomes (17), that is, $\mathcal{L}\left(B, C, \cdot\right)$ is superadditive on $\mathbb{R}^2_+$.
(ii) Monotonicity follows easily from the deduced superadditivity property. Since $\mathbf{p} \geq \mathbf{q}$, the ordered pair $\mathbf{p} \in \mathbb{R}^2_+$ can be represented as the sum of two ordered pairs in $\mathbb{R}^2_+$, that is, $\mathbf{p} = (\mathbf{p} - \mathbf{q}) + \mathbf{q}$. Now, from the superadditivity property (17), we get

$$\mathcal{L}\left(B, C, \mathbf{p}\right) = \mathcal{L}\left(B, C, \mathbf{p} - \mathbf{q} + \mathbf{q}\right) \geq \mathcal{L}\left(B, C, \mathbf{p} - \mathbf{q}\right) + \mathcal{L}\left(B, C, \mathbf{q}\right).$$

At last, relation (23), together with property (9), ensures the positivity of the operator $\mathcal{L}$, that is, $\mathcal{L}\left(B, C, \mathbf{p} - \mathbf{q}\right) \geq 0$. It follows that $\mathcal{L}(B, C, \mathbf{p}) \geq \mathcal{L}(B, C, \mathbf{q})$. The proof of Theorem 1 is now completed. $\square$

Remark 2. If we consider the function j, defined by (20), for $x \in \mathbb{R}_+$ and $p_1, p_2 > 0$, we easily conclude that $j(x, \mathbf{p}) = 0$ if and only if $x = 1$. Now, taking into account the proof of Theorem 1, we see that $\mathcal{L}(B, C, \mathbf{p}) = 0$ for $p_1, p_2 > 0$, if and only if $C^{*-1}BC^{-1} = 1_H$, that is, $B = C^*C = A$.

As an operator analogue of Corollary 1, we also get lower and upper bounds for operator (16). That result is contained in the following statement.

Corollary 2. *Suppose H is a Hilbert space, $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $\mathbf{p} \in \mathbb{R}^2_+$. Then, operator (16) satisfies the following series of inequalities:*

$$\begin{aligned} 2\max\{p_1, p_2\} \left[A \nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right] &\geq \mathcal{L}\left(B, C, \mathbf{p}\right) \\ &\geq 2\min\{p_1, p_2\} \left[A \nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right]. \end{aligned} \tag{24}$$

Proof. We use the same procedure as in the proof of Corollary 1, but on the level of operators. We have, $\mathbf{p}_{\max} \geq \mathbf{p} \geq \mathbf{p}_{\min}$, where $\mathbf{p}_{\max}$ and $\mathbf{p}_{\min}$ are constant ordered pairs composed of the minimum and maximum numbers in $\mathbf{p}$. Now, the monotonicity property (18) implies the relation

$$\mathcal{L}(B, C, \mathbf{p}_{\max}) \geq \mathcal{L}(B, C, \mathbf{p}) \geq \mathcal{L}(B, C, \mathbf{p}_{\min}).$$

In addition,

$$\begin{aligned} \mathcal{L}(B, C, \mathbf{p}_{\max}) &= 2\max\{p_1, p_2\} \left[A \nabla_{\frac{\max\{p_1, p_2\}}{2\max\{p_1, p_2\}}} B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{\max\{p_1, p_2\}}{2\max\{p_1, p_2\}}} C\right] \\ &= 2\max\{p_1, p_2\} \left[A \nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right], \end{aligned}$$

and similarly,

$$\mathcal{L}(B, C, \mathbf{p}_{\min}) = 2\min\{p_1, p_2\} \left[A \nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right],$$

which completes the proof. □

Remark 3. Note that relations (18) and (24) are proved in the operator setting. These proofs could be established in another way, like the proof of property (17), where we have used the monotonicity property for operator functions. More precisely, we could apply property (9) to relations (12) and (15). Of course, as a result, we would get relations (18) and (24) again. Moreover, we may regard relations (17), (18), (24) as operator extensions of inequalities (11), (12), (15) in Hilbert space. In addition, if $H = \mathbb{C}^n$ and $p_1 + p_2 = 1$, then relation (24) becomes relation (8) from Introduction.

It is easy to see that Corollary 2 provides an improvement of the operator arithmetic-geometric mean inequality.

Corollary 3. *Let H be a Hilbert space, let $A, B \in \mathcal{B}^{++}(H)$, and let $\mathbf{p} \in \mathbb{R}_+^2$. Then,*

$$2\max\{p_1, p_2\}[A\nabla B - A\sharp B] \geq \mathcal{J}(A, B, \mathbf{p}) \geq 2\min\{p_1, p_2\}[A\nabla B - A\sharp B], \tag{25}$$

where the operator $\mathcal{J} : \mathcal{B}^{++}(H) \times \mathcal{B}^{++}(H) \times \mathbb{R}_+^2 \to \mathcal{B}^+(H)$ is defined by

$$\mathcal{J}(A, B, \mathbf{p}) = (p_1 + p_2)\left[A\nabla_{\frac{p_1}{p_1+p_2}} B - A\sharp_{\frac{p_1}{p_1+p_2}} B\right]. \tag{26}$$

Proof. Follows immediately from relation (24) by replacing C with $A^{\frac{1}{2}}$. □

Remark 4. Since $\mathcal{J}(A, B, \mathbf{p})$ represents the difference between the arithmetic and geometric operator means, the first inequality in (25) yields a converse of inequality (3). On the other hand, the second inequality in (25) provides a refinement of the operator inequality (3).

Remark 5. Note also that operator (26) possess the superadditivity and monotonicity properties. Furthermore, according to Remark 2 and provided that $p_1, p_2 > 0$, we see that $\mathcal{J}(A, B, \mathbf{p}) = 0$, i.e., $A\nabla_{\frac{p_1}{p_1+p_2}} B = A\sharp_{\frac{p_1}{p_1+p_2}} B$, if and only if $A = B$.

4. Applications to Heinz means

In this section we apply our main results from Section 3 to the operator Heinz means. According to the classical definition (5), the operator Heinz mean is defined by

$$H_\nu(A, B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}, \tag{27}$$

where $A, B \in \mathcal{B}^{++}(H)$, and $\nu \in [0, 1]$. It is easy to show that, like in the classical case, the Heinz mean interpolates the arithmetic and geometric means, with respect to operator order.

Proposition 1. *If $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, $\nu \in [0, 1]$, then*

$$\begin{aligned} &2C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C \\ &\qquad \leq C^*\left(C^{*-1}BC^{-1}\right)^{\nu}C + C^*\left(C^{*-1}BC^{-1}\right)^{1-\nu}C \leq 2A\nabla B. \end{aligned} \tag{28}$$

In particular,

$$A\sharp B \leq H_\nu(A, B) \leq A\nabla B. \tag{29}$$

Proof. By using Theorem 1, we have

$$C^*(C^{*-1}BC^{-1})^{\nu}C + C^*(C^{*-1}BC^{-1})^{1-\nu}C \leq A\nabla_\nu B + A\nabla_{1-\nu}B = A + B,$$

which implies the second inequality in (28).

To obtain the other inequality in (28), we consider the inequality

$$x^{\nu} + x^{1-\nu} \geq 2\sqrt{x},$$

which holds for all $x \in \mathbb{R}_+$. On the other hand, since the operator $C^{*-1}BC^{-1}$ has a positive spectrum, according to rule (9), we can insert $C^{*-1}BC^{-1}$ in above inequality, i.e., we have

$$\big(C^{*-1}BC^{-1}\big)^{\nu} + \big(C^{*-1}BC^{-1}\big)^{1-\nu} \geq 2\big(C^{*-1}BC^{-1}\big)^{\frac{1}{2}}. \tag{30}$$

Finally, if we multiply inequality (30) by C^* on the left and C on the right, we get the first inequality in (28).

At last, if we replace C with $A^{\frac{1}{2}}$ in (28), we obtain the series of inequalities (29), which completes the proof. □

Remark 6. If $\nu \in \langle 0, 1\rangle$, then, according to Remark 2, it follows that equality in the second inequality in (28) holds if and only if $A = B$. Similarly, from the proof of Proposition 1, we see that equality in first inequality in (28) holds if and only if $A = B$, provided that $\nu \in \langle 0, 1\rangle$ and $\nu \neq \frac{1}{2}$. Of course, the same discussion is valid for (29).

Concerning relation (6) from Introduction, we want to decrease the difference between the arithmetic and Heinz means. For that sake, we have to establish the operator that measures the difference between the arithmetic and Heinz means in Hilbert space. We define the operator $\mathcal{M} : \mathcal{B}^{++}(H) \times \mathcal{B}^{-1}(H) \times \mathbb{R}_+^2 \to \mathcal{B}^+(H)$ as

$$\begin{aligned}\mathcal{M}(B, C, \mathbf{p}) = {} & \frac{1}{2}(p_1 + p_2) \\ & \times \left[2A\nabla B - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C - C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_2}{p_1+p_2}} C\right],\end{aligned} \tag{31}$$

where $A = C^*C$. It follows immediately from (28) that $\mathcal{M}(B, C, \mathbf{p})$ is positive with respect to operator order. Further, it is interesting that the operator $\mathcal{M}$ possess the same properties as operators (16) and (26) from the previous section. These properties are listed in the following theorem.

Theorem 2. *Suppose H is a Hilbert space, $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^2$. Then, operator (31) satisfies the following properties:*
(i) *$\mathcal{M}(B, C, \cdot)$ is superadditive on $\mathbb{R}_+^2$, that is,*

$$\mathcal{M}(B, C, \mathbf{p} + \mathbf{q}) \geq \mathcal{M}(B, C, \mathbf{p}) + \mathcal{M}(B, C, \mathbf{q}). \tag{32}$$

(ii) *If $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^2$ with $\mathbf{p} \geq \mathbf{q}$ (i.e,. $p_1 \geq q_1$, $p_2 \geq q_2$), then*

$$\mathcal{M}(B, C, \mathbf{p}) \geq \mathcal{M}(B, C, \mathbf{q}) \geq 0, \tag{33}$$

i.e,. $\mathcal{M}(B, C, \cdot)$ is increasing on $\mathbb{R}_+^2$.

Proof. (i) We use the superadditivity property of operator $\mathcal{L}$, defined by (16), for ordered pairs $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$:

$$\mathcal{L}(B, C, \mathbf{p} + \mathbf{q}) \geq \mathcal{L}(B, C, \mathbf{p}) + \mathcal{L}(B, C, \mathbf{q}). \tag{34}$$

In addition, let us rewrite relation (34) with the ordered pairs $\widetilde{\mathbf{p}} = (p_2, p_1)$ and $\widetilde{\mathbf{q}} = (q_2, q_1)$:

$$\mathcal{L}(B, C, \widetilde{\mathbf{p}} + \widetilde{\mathbf{q}}) \geq \mathcal{L}(B, C, \widetilde{\mathbf{p}}) + \mathcal{L}(B, C, \widetilde{\mathbf{q}}). \tag{35}$$

Now, since

$$\begin{aligned}
\mathcal{L}(B, C, \mathbf{p}) + \mathcal{L}(B, C, \widetilde{\mathbf{p}}) &= (p_1 + p_2)\left[A \nabla_{\frac{p_1}{p_1+p_2}} B - C^*\left(C^{*-1} B C^{-1}\right)^{\frac{p_1}{p_1+p_2}} C\right] \\
&\quad + (p_1 + p_2)\left[A \nabla_{\frac{p_2}{p_1+p_2}} B - C^*\left(C^{*-1} B C^{-1}\right)^{\frac{p_2}{p_1+p_2}} C\right] \\
&= (p_1 + p_2)\Bigg[2A \nabla B - C^*\left(C^{*-1} B C^{-1}\right)^{\frac{p_1}{p_1+p_2}} C \\
&\quad - C^*\left(C^{*-1} B C^{-1}\right)^{\frac{p_2}{p_1+p_2}} C\Bigg] = 2\mathcal{M}(B, C, \mathbf{p}),
\end{aligned}$$

we get the superadditivity of operator (31) by adding inequalities (34) and (35).
(ii) Monotonicity follows in the same way as in Lemma 1 or Theorem 1. Since $\mathbf{p} \geq \mathbf{q}$, the ordered pairs $\mathbf{p} - \mathbf{q}$ and $\mathbf{q}$ have non-negative coordinates, which implies the inequality

$$\mathcal{M}(B, C, \mathbf{p}) = \mathcal{M}(B, C, \mathbf{p} - \mathbf{q} + \mathbf{q}) \geq \mathcal{M}(B, C, \mathbf{p} - \mathbf{q}) + \mathcal{M}(B, C, \mathbf{q}).$$

Finally, since $\mathcal{M}(B, C, \mathbf{p} - \mathbf{q}) \geq 0$, it follows that $\mathcal{M}(B, C, \mathbf{p}) \geq \mathcal{M}(B, C, \mathbf{q})$, which completes the proof. $\square$

In the sequel, we apply the monotonicity property (33) to get lower and upper bounds for the operator $\mathcal{M}$.

Corollary 4. *Suppose H is a Hilbert space, $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $\mathbf{p} \in \mathbb{R}_+^2$. Then, operator (31) satisfies the following series of inequalities:*

$$\begin{aligned}(p_1+p_2)\left[A\nabla B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right] &\geq \mathcal{M}\left(B, C, \mathbf{p}\right)\\ &\geq 2\min\{p_1,p_2\}\left[A\nabla B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right]. \qquad (36)\end{aligned}$$

Proof. The first inequality in (36) holds trivially, due to inequality (28). To get the other inequality in (36), we compare the ordered pair $\mathbf{p}$ with constant ordered pair $\mathbf{p}_{\min}$ composed of the minimum number in $\mathbf{p}$. Namely, since $\mathbf{p} \geq \mathbf{p}_{\min}$, the monotonicity property of the operator $\mathcal{M}$ implies the inequality

$$\mathcal{M}\left(B, C, \mathbf{p}\right) \geq \mathcal{M}\left(B, C, \mathbf{p}_{\min}\right).$$

Finally, since

$$\begin{aligned}\mathcal{M}\left(B, C, \mathbf{p}_{\min}\right) &= \min\{p_1,p_2\}\left[2A\nabla B - 2C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right]\\ &= 2\min\{p_1,p_2\}\left[A\nabla B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right],\end{aligned}$$

we get the other inequality in (36), as well. □

Corollary 4 provides an improvement of the inequality between the arithmetic and Heinz means. More precisely, we have the following result.

Corollary 5. *Let H be a Hilbert space, let $A, B \in \mathcal{B}^{++}(H)$, and let $\mathbf{p} \in \mathbb{R}_+^2$. Then,*

$$(p_1+p_2)\left[A\nabla B - A\sharp B\right] \geq \mathcal{R}\left(A, B, \mathbf{p}\right) \geq 2\min\{p_1,p_2\}\left[A\nabla B - A\sharp B\right], \qquad (37)$$

where the operator $\mathcal{R} : \mathcal{B}^{++}(H) \times \mathcal{B}^{++}(H) \times \mathbb{R}_+^2 \to \mathcal{B}^{+}(H)$ is defined by

$$\mathcal{R}(A, B, \mathbf{p}) = (p_1+p_2)\left[A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A, B)\right]. \qquad (38)$$

Proof. Follows immediately from relation (36) by replacing C with $A^{\frac{1}{2}}$. □

Remark 7. It is obvious that operator (38) also possess the superadditivity and monotonicity properties.

Remark 8. The first inequality in (37) yields a converse of the arithmetic-Heinz mean inequality, while the second one yields a refinement of that inequality. Furthermore, we can regard the second inequality in (37) as an operator extension of the classical inequality (6) in Hilbert space.

Remark 9. Note that in Corollaries 4 and 5 we do not consider the constant ordered pair $\mathbf{p}_{\max}$. Let us explain why on the example of relation (37). Due to the monotonicity property of the operator R, the relation $\mathbf{p}_{\max} \geq \mathbf{p}$ implies the inequality

$$2\max\{p_1,p_2\}\left[A\nabla B - A\sharp B\right] \geq \mathcal{R}\left(A,B,\mathbf{p}\right). \tag{39}$$

Furthermore, if we denote $r = 2\max\{p_1,p_2\}/(p_1+p_2)$, inequality (39) can be rewritten in the form

$$H_{\frac{p_1}{p_1+p_2}}(A,B) \geq A\sharp B - (r-1)\left[A\nabla B - A\sharp B\right]. \tag{40}$$

Now, since $r \geq 1$ and $A\nabla B - A\sharp B \geq 0$, inequality (40) is worse than the original relationship between the Heinz and geometric means, i.e., $H_{\frac{p_1}{p_1+p_2}}(A,B) \geq A\sharp B$. That is, the reason why inequality (39) is omitted in the statement of Corollary 5.

Remark 10. It is interesting that the relationship between the geometric and Heinz means can be deduced from the superadditivity property of the operator $\mathcal{R}$. Namely, if we use the mentioned property for the ordered pairs $\mathbf{p} = (p_1,p_2)$ and $\widetilde{\mathbf{p}} = (p_2,p_1)$, we get

$$\mathcal{R}\left(A,B,\mathbf{p}+\widetilde{\mathbf{p}}\right) \geq \mathcal{R}\left(A,B,\mathbf{p}\right) + \mathcal{R}\left(A,B,\widetilde{\mathbf{p}}\right),$$

i.e.,

$$\begin{aligned}2(p_1+p_2)\left[A\nabla B - H_{\frac{1}{2}}(A,B)\right] \geq (p_1+p_2)&\left[A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B)\right]\\ &+ (p_1+p_2)\left[A\nabla B - H_{\frac{p_2}{p_1+p_2}}(A,B)\right].\end{aligned}$$

Now, since $H_{\frac{1}{2}}(A,B) = A\sharp B$ and $H_{\frac{p_1}{p_1+p_2}}(A,B) = H_{\frac{p_2}{p_1+p_2}}(A,B)$, the previous inequality reduces to the original relationship between the Heinz and geometric means.

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