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# Inner product space and circle power

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**Abstract.** In this paper we present a norm equality which, in its most general form with a free parameter, characterizes an inner-product space with no use of triangle inequality and homogenity. For any given fixed parameter the equality characterizes an inner-product space with no use of triangle inequality. The proof of the main theorem reduces to the consideration of the system of functional equations.

## 1. Introduction

Let $X$ be a real vector space. Usually, a norm $\|.\| : X \longrightarrow \mathbb{R}$ is considered to be a real functional satisfying the following conditions:

$$\|x\| \geq 0,\ x \in X,\ \text{and } \|x\| = 0 \Leftrightarrow x = 0 \qquad \text{positive definiteness} \qquad (1)$$

$$\|tx\| = |t|\,\|x\|,\ t \in \mathbb{R},\ x \in X \qquad \text{homogenity} \qquad (2)$$

$$\|x + y\| \leq \|x\| + \|y\|,\ x, y \in X \qquad \text{triangle inequality} \qquad (3)$$

It is well known that the norm is generated by an inner-product if and only if the norm satisfies the parallelogram equality [2].

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,\ x, y \in X \quad \text{parallelogram equality} \qquad (4)$$

Many equalities similar to (4) were proven to characterize an inner-product space. For example (see [4] and [3]):

$$\|x\|^2 + 3\|x + 2y\|^2 = 3\|x + y\|^2 + \|x + 3y\|^2, \quad x, y \in X \qquad (5)$$

$$\|x + 2y\|^2 + \|2x + y\|^2 + \|x - y\|^2 = 3\|x + y\|^2 + 3\|x\|^2 + 3\|y\|^2, \quad x, y \in X \qquad (6)$$

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More precisely, $(X, \|.\|)$ is an inner-product space if and only if (1), (2) and (3) and any of the conditions (4), (5) or (6) are satisfied. Many other characterizations of inner-product spaces are presented in [1]. It was proved by KUREPA [6] that it is enough to assume only (1), (2) and (4), to conclude that $(X, \|.\|)$ is an inner-product space. Thus, Kurepa showed that the triangle inequality is redundant in the classical Jordan-von Neumann characterization of inner-product space. Motivated by Kurepa's result ŠEMRL [5] proved that (1), (2) and either (5) or (6) yield that $X$ is an inner-product space.

It is clear that in an inner-product space $(X, \|.\|)$ the equality

$$\|x + \lambda y\|^2 = \lambda\|x + y\|^2 + (1 - \lambda)\|x\|^2 + \lambda(\lambda - 1)\|y\|^2, \tag{7}$$

holds for all $x, y \in X$ and for all $\lambda \in \mathbb{R}$. This equation (in its abstract algebraic form) is derived from elementary geometric properties of a 'circle power' in a classical Euclidean geometry.

In this paper we show that besides (7) it is enough to assume (1) to obtain an inner-product space. The assumption that (7) holds for every $\lambda \in \mathbb{R}$ is very strong, so it is not surprising that we can derive all the properties of an inner-product space and also, that the proofs are rather simple and straightforward.

More interesting, assuming (7) for any fixed $\lambda$, we obtain a generalization of a parallelogram equality and an according analog of Kurepa's and Šemrl's results.

## 2. An alternative definition of inner-product space

We will see, that a real vector space equipped with a positive definite function $\|.\| : X \to \mathbb{R}$, for which (7) holds for all $x, y \in X$ and for all $\lambda \in \mathbb{R}$, is an inner-product space. Furthermore, the equation (7) can be replaced by the equation

$$\|x + \lambda y\|^2 = \lambda\|x + y\|^2 + (\lambda - 1)^2 \tag{8}$$

which holds for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and for all $\lambda \in \mathbb{R}$.

Let $X$ be a real vector space. A mapping $\|.\| : X \to \mathbb{R}$ satisfying only (1) is a *positive definite function.*

**Proposition 2.1** (Homogenity)**.** *In a vector space $X$ with a positive definite function $\|.\| : X \to \mathbb{R}$, either of the two conditions* (7) *and* (8) *implies* (2).

PROOF OF PROPOSITION. Equation (7) immediately implies the homogenity of the function $\|.\| : X \to \mathbb{R}$: we just put $x = 0$. To obtain the homogenity of the positive definite function $\|.\| : X \to \mathbb{R}$ from the equation (8), we may consider the following two variations of equation (8) for any $\lambda, \mu \in \mathbb{R}$.

$$\|x + \lambda y\|^2 = \lambda\|x + y\|^2 + (\lambda - 1)^2$$
$$\|x + \mu y\|^2 = \mu\|x + y\|^2 + (\mu - 1)^2$$

We calculate $\mu\|x+\lambda y\|^2+\lambda(\mu-1)^2 = \lambda\|x+\mu y\|^2+\mu(\lambda-1)^2$. Now take $\mu = -1$ and $y = x$ to get $\|(1 + \lambda)x\|^2 = (1 + \lambda)^2$. To see that $\|\lambda x\| = |\lambda|\,\|x\|$ for any $x \in X$ we write $\lambda x = \lambda\|x\|\frac{x}{\|x\|}$. □

**Corollary 2.2.** *In a vector space $X$ with a positive definite function $\|.\| : X \to \mathbb{R}$, the two conditions* (7) *and* (8) *are equivalent.*

Proof of corollary. It is obvious that (7) implies (8). To show the opposite we plug $\frac{x}{\|x\|}$, $\frac{y}{\|y\|}$ and $\lambda = \mu\frac{\|y\|}{\|x\|}$ into (8) and multiply the equality by $\|x\|^2$ to obtain

$$\|x + \mu y\|^2 = \mu\|x\|\|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^2 + (\mu\|y\| - \|x\|)^2$$

Since this equation holds for every $\mu$, we can put $\mu = 1$ and multiply the equality by $\mu$ to get

$$\mu\|x + y\|^2 = \mu\|x\|\,\|y\|\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|^2 + \mu(\|y\| - \|x\|)^2$$

Subtracting the two equations gives (7). □

**Proposition 2.3** (Triangle inequality)**.** *In a vector space $X$ with a positive definite function $\|.\| : X \to \mathbb{R}$, either of the two conditions* (7) *and* (8) *implies* (3).

Proof of proposition. It suffices to prove that triangle inequality follows from the condition (8). Let $x, y \in X$ be any two fixed vectors with $\|x\| = \|y\| = 1$. We define $\varphi(\lambda) = \|x + \lambda y\|^2$. By (8), $\varphi(\lambda)$ is a quadratic function of $\lambda$ (see (1.7) in [1]). Since from (8)

$$\varphi(\lambda) = \|x + \lambda y\|^2 = \lambda\|x + y\|^2 + (\lambda - 1)^2$$

we can write

$$\varphi(\lambda) = \lambda^2 + \lambda(\|x + y\|^2 - 2) + 1$$

Can $\varphi(\lambda) = \|x + \lambda y\|^2$ have two different zeros? By (1) this would imply $x+\lambda_1 y = x+\lambda_2 y = 0$ and since $\|y\| = 1$ also $\lambda_1 = \lambda_2$. Therefore $\varphi(\lambda)$ has at most one zero and the discriminant of $\varphi(\lambda)$ must not be positive. Thus $(\|x+y\|^2-2)^2-4 \leq 0$ and $\|x+y\|^2 \leq 4$. Now let $u, v \in X$ be arbitrary and we set $x = \frac{u}{\|u\|}$, $y = \frac{v}{\|v\|}$ and $\nu = \frac{\|v\|}{\|u\|}$. Starting with $\|x+y\|^2 \leq 4$, the following calculation proves the triangle inequality:

$$
\begin{aligned}
\|x+y\|^2 &\leq 4 \\
\nu^2 + \nu\|x+y\|^2 + 1 &\leq \nu^2 + 4\nu + 1 \\
\nu^2 - 2\nu + 1 + \nu\|x+y\|^2 &\leq \nu^2 + 2\nu + 1 \\
(\nu-1)^2 + \nu\|x+y\|^2 &\leq (\nu+1)^2 \\
\|x+\nu y\|^2 &\leq (\nu+1)^2 \\
\|x+\nu y\| &\leq \nu + 1 \\
\left\| \frac{u}{\|u\|} + \frac{\|v\|}{\|u\|}\frac{v}{\|v\|} \right\| &\leq \frac{\|v\|}{\|u\|} + 1 \\
\|u+v\| &\leq \|u\| + \|v\|
\end{aligned}
$$

□

**Corollary 2.4.** *A vector space $X$ with a positive definite function $\|.\| : X \to \mathbb{R}$, fulfilling either of the two conditions* (7) *and* (8), *is a normed space.*

**Theorem 2.5.** *A vector space $X$ with a positive definite function $\|.\| : X \to \mathbb{R}$ fulfilling either of the two conditions* (7) *and* (8), *is an inner-product space.*

Proof of theorem. It suffices to put $\lambda = -1$ into the equation (7) to get the well known parallelogram identity. And it is well known that a norm yielding the parallelogram identity is generated by an inner-product. □

*Remark.* We gave self-contained proofs of 2.3, 2.4 and 2.5, as they are simple and short. Of course, it would be possible to deduce this statements from proposition 2.1 and corollary 2.2. Namely, for $\lambda = -1$ the equation (7) is the parallelogram equality and thus, $X$ is an inner-product space by Kurepa's result.

## 3. The generalization of parallelogram equality

So far we have seen that positive definiteness of a function $\|.\| : X \to \mathbb{R}$ together with equality (7) or (8) suffices for the function $\|.\| : X \to \mathbb{R}$ to be a norm generated by inner-product. Even though the result seems nice, the requirements of the two conditions (7) or (8) are very strong. We assumed a whole family of equations (with a free parameter) and obtained homogenity, triangle inequality and parallelogram equality.

Focusing on the equation (7), we see, that if we take $\lambda = -1$, we get parallelogram equality. Furthermore, if we take $\lambda = \frac{4}{3}$ and replace $y := -\frac{3}{2}(x+y)$ in (7), we get (5). Could therefore the equation (7) for any other fixed parameter be equivalent to parallelogram equality? We shall see that the answer is affirmative.

It is obvious that in vector space with positive definite function $\|.\| : X \to \mathbb{R}$, the equality (7) holds for $\lambda = 0, 1$ and for any $x, y \in X$.

**Theorem 3.1.** *Let a positive definite function $\|.\| : X \to \mathbb{R}$ satisfy* (2) *(homogenity) and suppose that there exists $\lambda \neq 0, 1$, such that the equality* (7) *holds for every $x, y \in X$. Then* (7) *holds for every $\lambda \in \mathbb{R}$, for every $x, y \in X$, and $X$ is an inner product space.*

Proof of theorem. Let $\lambda \neq 0, 1$, for which the equality (7) holds for every $x, y \in X$ be denoted by $\alpha$. Since

$$\|x + \alpha y\|^2 = \alpha\|x + y\|^2 + (1 - \alpha)\|x\|^2 + (\alpha^2 - \alpha)\|y\|^2 \tag{9}$$

for any $x, y \in X$, we can apply the equality for $x$ and $u = \frac{y}{\alpha}$ to get

$$\|\alpha x + y\|^2 = \alpha\|x + y\|^2 + (\alpha^2 - \alpha)\|x\|^2 + (1 - \alpha)\|y\|^2. \tag{10}$$

Next we choose any pair of linearly independent vectors $x, y \in X$ and define

$$\begin{aligned} f(t) &= \|x + ty\|^2 - t\|x + y\|^2 + (t - 1)\|x\|^2 + (t - t^2)\|y\|^2, \\ g(t) &= \|tx + y\|^2 - t\|x + y\|^2 + (t - t^2)\|x\|^2 + (t - 1)\|y\|^2. \end{aligned}$$

It is obvious that

$$f(0) = f(1) = g(0) = g(1) = f(\alpha) = g(\alpha) = 0.$$

Furthermore

$$f(t) = t^2 g\left(\frac{1}{t}\right) \quad \text{and} \quad g(t) = t^2 f\left(\frac{1}{t}\right). \tag{11}$$

In the following calculations we assume that $\alpha \neq -1$. We may do so without loss of generality, since $\alpha = -1$ yields the well known parallelogram equality. Let $t, s \in \mathbb{R}$ be any two numbers. By lengthy calculations the expression

$$(1 + \alpha)^2 f\left(\frac{t + \alpha s}{1 + \alpha}\right) - 4\alpha f\left(\frac{t + s}{2}\right) + (\alpha - 1)f(t) + (\alpha - \alpha^2)f(s)$$

simplifies into

$$
\begin{aligned}
&(1+\alpha)^2\left\|x+\frac{t+\alpha s}{1+\alpha}y\right\|^2-4\alpha\left\|x+\frac{t+s}{2}y\right\|^2+(\alpha-1)\|x+ty\|^2\\
&\quad+(\alpha-\alpha^2)\|x+sy\|^2=\|(x+ty)+\alpha(x+sy)\|^2-\alpha\|(x+ty)+(x+sy)\|^2\\
&\quad+(\alpha-1)\|x+ty\|^2+(\alpha-\alpha^2)\|x+sy\|^2,
\end{aligned}
$$

which equals 0 by (9). Similar calculations transform the expression

$$(1+\alpha)^2 g\left(\frac{\alpha t+s}{1+\alpha}\right)-4\alpha g\left(\frac{t+s}{2}\right)+(\alpha-\alpha^2)g(t)+(\alpha-1)g(s)$$

into

$$
\begin{aligned}
&(1+\alpha)^2\left\|\frac{\alpha t+s}{1+\alpha}x+y\right\|^2-4\alpha\left\|\frac{t+s}{2}x+y\right\|^2+(\alpha-\alpha^2)\|tx+y\|^2\\
&\quad+(\alpha-1)\|sx+y\|^2=\|\alpha(tx+y)+(sx+y)\|^2-\alpha\|(tx+y)+(sx+y)\|^2\\
&\quad+(\alpha-\alpha^2)\|tx+y\|^2+(\alpha-1)\|sx+y\|^2,
\end{aligned}
$$

which equals 0 by (10). Therefore,

$$(1+\alpha)^2 f\left(\frac{t+\alpha s}{1+\alpha}\right)-4\alpha f\left(\frac{t+s}{2}\right)+(\alpha-1)f(t)+(\alpha-\alpha^2)f(s)=0$$

and

$$(1+\alpha)^2 g\left(\frac{\alpha t+s}{1+\alpha}\right)-4\alpha g\left(\frac{t+s}{2}\right)+(\alpha-\alpha^2)g(t)+(\alpha-1)g(s)=0$$

for any $t,s\in\mathbb{R}$.

If we take $s=-t$ we obtain

$$(1+\alpha)^2 f\left(\frac{1-\alpha}{1+\alpha}t\right)+(\alpha-1)f(t)+(\alpha-\alpha^2)f(-t)=0 \tag{12}$$

$$(1+\alpha)^2 g\left(\frac{\alpha-1}{1+\alpha}t\right)+(\alpha-\alpha^2)g(t)+(\alpha-1)g(-t)=0. \tag{13}$$

We split $f(x)$ into even and odd components $f(x)=f_e(x)+f_o(x)$ where $f_e(x)=\frac{1}{2}(f(x)+f(-x))$ and $f_o(x)=\frac{1}{2}(f(x)-f(-x))$. Similarly $g(x)=g_e(x)+g_o(x)$. It is easy to verify that the above relations hold also for odd and even components. Therefore

$$(1+\alpha)^2 f_e\left(\frac{1-\alpha}{1+\alpha}t\right) + (\alpha-1)f_e(t) + (\alpha-\alpha^2)f_e(-t) = 0$$
$$(1+\alpha)^2 g_e\left(\frac{\alpha-1}{1+\alpha}t\right) + (\alpha-\alpha^2)g_e(t) + (\alpha-1)g_e(-t) = 0$$

and

$$f_e\left(\frac{1-\alpha}{1+\alpha}t\right) = \frac{(1-\alpha)^2}{(1+\alpha)^2}f_e(t)$$
$$g_e\left(\frac{1-\alpha}{1+\alpha}t\right) = \frac{(1-\alpha)^2}{(1+\alpha)^2}g_e(t).$$

A simple substitution for $t$ yields also

$$g_e\left(\frac{1+\alpha}{1-\alpha}t\right) = \frac{(1+\alpha)^2}{(1-\alpha)^2}g_e(t)$$

and similarly for $f(t)$. Property (11) implies

$$f_e(t) = t^2 g_e\left(\frac{1}{t}\right).$$

We calculate

$$\begin{aligned} f_e\left(\frac{1-\alpha}{1+\alpha}\,t\right) &= \frac{(1-\alpha)^2}{(1+\alpha)^2}\,t^2\cdot g_e\left(\frac{1+\alpha}{(1-\alpha)}\frac{1}{t}\right) \\ &= \frac{(1-\alpha)^2}{(1+\alpha)^2}\,t^2\cdot\frac{(1+\alpha)^2}{(1-\alpha)^2}\cdot g_e\left(\frac{1}{t}\right) = f_e(t). \end{aligned}$$

But we know that

$$f_e\left(\frac{1-\alpha}{1+\alpha}t\right) = \frac{(1-\alpha)^2}{(1+\alpha)^2}f_e(t)$$

and we have

$$\left[1-\frac{(1-\alpha)^2}{(1+\alpha)^2}\right]\cdot f_e(t) = \frac{4\alpha}{(1+\alpha)^2}\cdot f_e(t) = 0,$$

which implies that $f_e(t)\equiv 0$. Therefore, $f = f_o$ is an odd function.

We now know that $f(t)$ is an odd function and recall its definition. It is dependent on the choice of $x$ and $y$, and we can write $f_{x,y}(-t) = -f_{x,y}(t)$ for every $x, y \in X$ to get

$$\begin{aligned} &\|x-ty\|^2 + t\|x+y\|^2 - (t+1)\|x\|^2 - (t+t^2)\|y\|^2 \\ &\qquad = -\|x+ty\|^2 + t\|x+y\|^2 - (t-1)\|x\|^2 - (t-t^2)\|y\|^2. \end{aligned}$$

Thus, for $t = 1$ and for every $x, y \in X$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

which means that the parallelogram equality holds in $X$. Therefore, $X$ is an inner-product space and for the above defined $f(t)$ and $g(t)$, we have $f(t) \equiv g(t) \equiv 0$. $\square$

*Remark.* Equation (7) is equivalent to the parallelogram equality for any given $\lambda \neq 0, 1$. As stated, $\lambda = -1$, gives the parallelogram equality; $\lambda = \frac{4}{3}$ (and $y := -\frac{3}{2}(x+y)$) yields (5). Equation (6) contains the squares of the norms of six vectors, of which any pair are linearly independent. So (6) is not directly equivalent to (7), which only contains the squares of the norms of four different vectors. But if we write

$$E(\lambda, x, y) = \|x+\lambda y\|^2 - \lambda\|x+y\|^2 - (1-\lambda)\|x\|^2 - \lambda(\lambda-1)\|y\|^2,$$

we obtain (6) as $3E(2,x,y) + 3E(2,y,x) + E(-2,x,y) + E(-2,-y,x) = 0$.

Therefore, assuming (7) for every $\lambda \in \mathbb{R}$ together with only (1) yields an inner-product space. Assuming (7) for only one specific $\lambda \in \mathbb{R}$ requires the assumption (2) to be added to (1) in order to obtain the same result.

*Questions.* Is it possible that the assumption of (7) for all $\lambda \in B$ together with (1) can yield an inner-product space for a set $B \subset \mathbb{R}$ smaller than $\mathbb{R}$? Possibly for a set $B = \{\lambda_1, \lambda_2\}$ of only two elements? What if the assumption (7) is replaced by the weaker condition (8)?

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