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The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain

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Abstract. The study of the semigroups $\mathcal{OP}_n$, of all orientation-preserving transformations on an n-element chain, and $\mathcal{OR}_n$, of all orientation-preserving or orientation-reversing transformations on an n-element chain, has began in [17] and [5]. In order to bring more insight into the subsemigroup structure of $\mathcal{OP}_n$ and $\mathcal{OR}_n$, we characterize their maximal subsemigroups.

Introduction and preliminaries

For $n \in \mathbb{N}$, let $X_n = \{1 < 2 < \cdots < n\}$ be a finite chain with n elements. As usual, we denote by $\mathcal{T}_n$ the monoid (under composition) of all full transformations of X_n.

We say that a transformation $\alpha \in \mathcal{T}_n$ is *order-preserving* [respectively, *order-reversing*] if $x \leq y$ implies that $x\alpha \leq y\alpha$ [respectively, $x\alpha \geq y\alpha$], for all $x, y \in X_n$. As usual, $\mathcal{O}_n$ denotes the submonoid of $\mathcal{T}_n$ of all order-preserving transformations of X_n. This monoid has been extensively studied, for instance in [1], [7], [13], [15], [20].

Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 1$) elements from the chain X_n. We say that a is *cyclic* [respectively, *anti-cyclic*] if there exists no more than

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one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [respectively, $a_i < a_{i+1}$], where a_{t+1} denotes a_1. Notice that the sequence a is cyclic [respectively, anti-cyclic] if and only if a is empty or there exists $i \in \{0, 1, \ldots, t-1\}$ such that $a_{i+1} \leq a_{i+2} \leq \cdots \leq a_t \leq a_1 \leq \cdots \leq a_i$ [respectively, $a_{i+1} \geq a_{i+2} \geq \cdots \geq a_t \geq a_1 \geq \cdots \geq a_i$] (the index $i \in \{0, 1, \ldots, t-1\}$ is unique unless a is constant and $t \geq 2$). We say that a transformation $\alpha \in \mathcal{T}_n$ is *orientation-preserving* [respectively, *orientation-reversing*] if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ of its images is cyclic [respectively, anti-cyclic]. The notion of an orientation-preserving transformation was introduced by McAlister in [17] and, independently, by Catarino and Higgins in [5]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving, and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing (see [5]). We denote by $\mathcal{OP}_n$ [respectively, $\mathcal{OR}_n$] the monoid of all orientation-preserving [respectively, orientation-preserving or orientation-reversing] full transformations. It is clear that $\mathcal{OP}_n$ is a submonoid of $\mathcal{OR}_n$.

Regarding the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$, presentations for them were exhibited by Catarino in [4] and by Arthur and Ruškuc in [2], the Green's relations, their sizes and ranks, among other properties, were determined by Catarino and Higgins in [5] and a description of their congruences were given in [10] by Fernandes, Gomes and Jesus. In [22], Zhao, Bo and Mei characterized the locally maximal idempotent-generated subsemigroups of $\mathcal{OP}_n$ (excluding the permutations).

In this paper, we aim to give more insight into the subsemigroup structure of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ by characterizing the maximal subsemigroups of these monoids and of their ideals. By a maximal subsemigroup of a semigroup S we mean a maximal element, under set inclusion, of the family of all proper subsemigroups of S. In Section 1, we study the monoid $\mathcal{OP}_n$ and its ideals. First, we describe all maximal subsemigroups of $\mathcal{OP}_n$ (some of them are associated with the maximal subsemigroups of the additive group $\mathbb{Z}_n$). The main result of this section is the characterization of the maximal subsemigroups of the ideals of $\mathcal{OP}_n$. In Section 2, we study the monoid $\mathcal{OR}_n$ and its ideals. Again, first we describe all maximal subsemigroups of $\mathcal{OR}_n$ (some of them are associated with the maximal subsemigroups of the dihedral group $\mathcal{D}_n$ of order $2n$). The main result of this section is the characterization of the maximal subsemigroups of the ideals of $\mathcal{OR}_n$, which are associated with the maximal subsemigroups of the ideals of $\mathcal{OP}_n$.

The maximal subsemigroups of the monoid $\mathcal{T}_n$ were described by Bayramov [3] in 1966. Much more recently (2001), Yang [21] classified the maximal subsemigroups of the semigroup $Sing_n$ of all singular full transformations of X_n. In 1985, Todorov and Kračolova [18] constructed four types of maximal subsemigroups of the ideals of $\mathcal{T}_n$. A complete description of the maximal subsemigroups of the ideals of $\mathcal{T}_n$ was given in 2004 by Yang and Yang [19]. The maximal subsemigroups of the semigroup $\mathcal{O}_n$ were characterized by Yang [20] in 2000. Gyudzhenov and Dimitrova (2006) completely described in [14] the maximal subsemigroups of the semigroup $\mathcal{OD}_n$ of all order-preserving or order-reversing full transformations of X_n. In 2008, Dimitrova and Koppitz [6] classified the maximal subsemigroups of the ideals of $\mathcal{O}_n$ as well as of the ideals of $\mathcal{OD}_n$.

On the other hand, Ganyushkin and Mazorchuk [11] in 2003 gave a description of the maximal subsemigroups of the semigroup $\mathcal{POI}_n$ of all order-preserving partial injections of X_n and, in 2009, Dimitrova and Koppitz [7] characterized the maximal subsemigroups of the ideals of the semigroup $\mathcal{POI}_n$ and of the ideals of the semigroup $\mathcal{PODI}_n$ of all order-preserving or order-reversing partial injections of X_n.

For every transformation $\alpha \in \mathcal{T}_n$, we denote by $\ker \alpha$ and $\operatorname{im} \alpha$ the kernel and the image of α, respectively. The number $\operatorname{rank} \alpha = |\ker \alpha| = |\operatorname{im} \alpha|$ is called the *rank* of α. Given a subset U of $\mathcal{T}_n$, we denote by $E(U)$ its set of idempotents. The weight of an equivalence relation π on X_n is the number $|X_n/\pi|$. Let $A \subseteq X_n$ and let π be an equivalence relation on X_n of weight $|A|$. We say that A is a *transversal* of π (denoted by $A\#\pi$) if $|A \cap \bar{x}| = 1$ for every equivalence class $\bar{x}$ of π.

Since $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$ are regular submonoids of $\mathcal{T}_n$, the definition of the Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$ follow immediately from well known results on regular semigroups and from their descriptions on $\mathcal{T}_n$. We have $\alpha\mathcal{L}\beta \iff \operatorname{im} \alpha = \operatorname{im} \beta$ and $\alpha\mathcal{R}\beta \iff \ker \alpha = \ker \beta$, for every transformations α and β. Recall also that for the Green's relation $\mathcal{J}$, we have (on $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$) $\alpha\mathcal{J}\beta \iff \operatorname{rank} \alpha = \operatorname{rank} \beta$, for every transformations α and β.

Given a semigroup S, we denote by L_s^S, R_s^S and H_s^S (or, if not ambiguous, simply by L_s, R_s and H_s) the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class, respectively, of an element $s \in S$.

For general background on Semigroup Theory, we refer the reader to Howie's book [16]. Regarding notions on Group Theory, the book [8] by Dummit and Foote is our reference.

1. Maximal subsemigroups of the ideals of $\mathcal{OP}_n$

Let $n \in \mathbb{N}$. The semigroup $\mathcal{OP}_n$ is the union of its $\mathcal{J}$-classes $J_1, J_2, \ldots, J_n$, where

$$J_k = \{\alpha \in \mathcal{OP}_n \mid \operatorname{rank} \alpha = k\},$$

for $k = 1, \ldots, n$. It follows that the ideals of the semigroup $\mathcal{OP}_n$ are unions of the $\mathcal{J}$-classes $J_1, J_2, \ldots, J_k$, i.e. the sets

$$OP(n,k) = \{\alpha \in \mathcal{OP}_n \mid \operatorname{rank} \alpha \le k\},$$

with $k = 1, \ldots, n$. See [9, Note of page 181].

Now, notice that for $\alpha \in OP(n,k)$, with $k = 1, \ldots, n$, we have $L_\alpha^{OP(n,k)} = L_\alpha^{\mathcal{OP}_n}$, $R_\alpha^{OP(n,k)} = R_\alpha^{\mathcal{OP}_n}$ and $H_\alpha^{OP(n,k)} = H_\alpha^{\mathcal{OP}_n}$. Moreover, for $\alpha, \beta \in J_k$, with $k = 1, \ldots, n$, the product $\alpha\beta$ belongs to J_k (if and only if $\alpha\beta \in R_\alpha \cap L_\beta$) if and only if $\operatorname{im} \alpha \# \ker \beta$. Thus, it is easy to show:

Lemma 1.1. *Let $k \in \{1, 2, \ldots, n\}$ and let $\alpha, \beta \in J_k$ be such that $\operatorname{im} \alpha \# \ker \beta$. Then $\alpha R_\beta^{\mathcal{OP}_n} = R_{\alpha\beta}^{\mathcal{OP}_n} = R_\alpha^{\mathcal{OP}_n}$, $L_\alpha^{\mathcal{OP}_n} \beta = L_{\alpha\beta}^{\mathcal{OP}_n} = L_\beta^{\mathcal{OP}_n}$, $\alpha H_\beta^{\mathcal{OP}_n} = H_\alpha^{\mathcal{OP}_n} \beta = H_{\alpha\beta}^{\mathcal{OP}_n}$ and $L_\alpha^{\mathcal{OP}_n} R_\beta^{\mathcal{OP}_n} = J_k$.*

Next, recall that Catarino and Higgins [5] proved:

Proposition 1.2. *Let $k \in \{1, 2, \ldots, n\}$ and let $\alpha \in \mathcal{OP}_n$ be an element of rank k. Then $|H_\alpha| = k$. Moreover, if α is an idempotent, then H_α is a cyclic group of order k.*

Let G be a cyclic group of order k, with $k \in \mathbb{N}$. It is well known that there exists an one-to-one correspondence between the subgroups of G and the (positive) divisors of k.

Let us consider the following elements of $\mathcal{OP}_n$:

$$g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \in J_n$$

and

$$u_i = \left(\begin{array}{cccc|c|ccc} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i+1 & \cdots & n \end{array}\right) \in J_{n-1},$$

for $i = 1, \ldots, n$ (with $i = n$ we take $i + 1 = 1$).

Notice that the group of units of $\mathcal{OP}_n$ is the cyclic group $J_n = H_g^{\mathcal{OP}_n}$ of order n.

We will use the following well known result (see [4], [17]).

Proposition 1.3. $\mathcal{OP}_n = \langle u_1, g \rangle$.

Next, we present alternative generating sets of the monoid $\mathcal{OP}_n$.

Proposition 1.4. *Let $\alpha, \gamma \in \mathcal{OP}_n$. If $\alpha \in J_{n-1}$ and γ is a permutation of order n then $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$.*

Proof. Since $\gamma \in J_n$ has order n, we have $\langle \gamma \rangle = J_n$ and so $g \in \langle \gamma \rangle$. From $\alpha \in J_{n-1}$, it follows that there exist $1 \leq i, j \leq n$ such that $\operatorname{im} \alpha = X_n \setminus \{j\}$ and $(i, i+1) \in \ker \alpha$ (by taking $i+1 = 1$, if $i = n$). Put $s = i - j$, if $j < i$, and $s = n + i - j$, otherwise. Then, it is easy to show that $\beta = \alpha g^s \in H_{u_i}$. Now, as u_i is an idempotent of $\mathcal{OP}_n$, by Proposition 1.2, it follows that u_i is a power of β. On the other hand, it is a routine matter to show that $u_1 = g^{n+i-1} u_i g^{n-i+1}$. Thus, by Proposition 1.3, we deduce that $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$. $\square$

For a prime divisor p of n, we put $W_p = \langle g^p \rangle = \{1, g^p, g^{2p}, \ldots, g^{(\frac{n}{p}-1)p}\}$, which is a cyclic group of order $\frac{n}{p}$. Furthermore, from well known results regarding finite cyclic groups, we have:

Lemma 1.5. *The groups W_p, with p a prime divisor of n, are the maximal subsemigroups of J_n.*

Now, we can describe the maximal subsemigroups of $\mathcal{OP}_n$.

Theorem 1.6. *Let S be a subsemigroup of the semigroup $\mathcal{OP}_n$. Then S is maximal if and only if $S = OP(n, n-2) \cup J_n$ or $S = OP(n, n-1) \cup W_p$, for a prime divisor p of n.*

Proof. Let S be a maximal subsemigroup of $\mathcal{OP}_n$. Then, it is clear that $OP(n, n-2) \subseteq S$ and thus $S = OP(n, n-2) \cup T$, for some subset T of $J_{n-1} \cup J_n$. By Proposition 1.4, we have $T \cap J_{n-1} = \emptyset$ or T does not contain any element of J_n of order n. In the latter case, we must have $J_{n-1} \subseteq T$, by the maximality of S. This shows that $S = OP(n, n-1) \cup T'$, for some subset T' of J_n, whence T' must be a maximal subsemigroup of J_n. Thus, by Lemma 1.5, we have $T' = W_p$, for some prime divisor p of n. On the other hand, if $T \cap J_{n-1} = \emptyset$ then $S \subseteq OP(n, n-2) \cup J_n$, whence $S = OP(n, n-2) \cup J_n$, by the maximality of S.

The converse part follows immediately from Proposition 1.4 and Lemma 1.5. $\square$

Let $n \geq 3$ and $1 \leq k \leq n-1$. In the remaining of this section, we consider the ideal $OP(n, k)$ of $\mathcal{OP}_n$.

Clearly, the maximal subsemigroups of $OP(n, 1)$ are the sets of the form $OP(n, 1) \setminus \{\alpha\}$, for $\alpha \in OP(n, 1)$. Therefore, in what follows, we consider $k \geq 2$.

Notice that any element $\alpha \in \mathcal{O}_n$ of rank $k-1$, for $2 \le k \le n-1$, is expressible as a product of elements of $\mathcal{O}_n$ of rank k (see [12]). On the other hand, any element $\beta \in \mathcal{OP}_n$ admits a decomposition $\beta = g^t\alpha$, for some $1 \le t \le n$ and $\alpha \in \mathcal{O}_n$ (see [5]). Then, it is easy to deduce that any element of J_{k-1} is a product of elements of J_k, for $2 \le k \le n-1$. Thus, we have:

Lemma 1.7. $OP(n,k) = \langle J_k \rangle$.

Next, observe that for a transformation $\alpha \in \mathcal{OP}_n$, it is easy to show that if $(1,n) \notin \ker\alpha$ then all kernel classes of α are intervals of X_n and, on the other hand, if $(1,n) \in \ker\alpha$ then all kernel classes of α are intervals of X_n, except the class containing 1 and n which is a union of two intervals of X_n (one containing 1 and the other n). Moreover, if α is not a constant, then $H_\alpha^{\mathcal{OP}_n} \cap \mathcal{O}_n = \emptyset$ if and only if $(1,n) \in \ker\alpha$.

Proposition 1.8. *Let C be any subset of J_k containing $J_k \cap \mathcal{O}_n$ and at least one element from each $\mathcal{R}$-class of $\mathcal{OP}_n$ of rank k. Then $OP(n,k) = \langle C \rangle$.*

Proof. First, let $\alpha \in C$ with kernel $\{\{1, k+1, \ldots, n\}, \{2\}, \ldots, \{k\}\}$. Let β be an order-preserving transformation with image $\{1, \ldots, k\}$ such that $\operatorname{im}\alpha \,\#\, \ker\beta$. Then, $\beta \in C$, $\ker(\alpha\beta) = \ker\alpha$ and $\operatorname{im}(\alpha\beta) = \operatorname{im}\beta$, from which it follows that the idempotent power of $\alpha\beta$ is the element $\left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & 1 & \cdots & 1 \end{array}\right) \in \langle C \rangle$. Therefore,

$$\gamma = \left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 2 & \cdots & k+1 & k+1 & \cdots & k+1 \end{array}\right)\left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & 1 & \cdots & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|ccc} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 2 & \cdots & k & 1 & 1 & \cdots & 1 \end{array}\right) \in \langle C \rangle,$$

since the first element of the second member of these equalities is an order-preserving transformation of rank k and so an element of C. Furthermore, as γ generates a cyclic group of order k, then the $\mathcal{H}$-class H_γ of $\mathcal{OP}_n$ is contained in $\langle C \rangle$.

Now, $\varepsilon = \gamma^k = \left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & k & \cdots & k \end{array}\right)$ is the idempotent of H_γ and let H be any $\mathcal{H}$-class of $\mathcal{OP}_n$ contained in the $\mathcal{R}$-class $R_\varepsilon = R_\gamma$ of $\mathcal{OP}_n$. Since the elements of H have the same kernel as $\varepsilon \in \mathcal{O}_n$, then H has an order-preserving element τ. From $\varepsilon\mathcal{R}\tau$ it follows that $\varepsilon\tau = \tau$, whence $\operatorname{im}\varepsilon \,\#\, \ker\tau$ and so, by Lemma 1.1, we have $H_\varepsilon\tau = H_\tau$. As $\tau \in C$ and $H_\varepsilon \subseteq \langle C \rangle$, we also have $H = H_\tau \subseteq \langle C \rangle$. Hence $R_\varepsilon \subseteq \langle C \rangle$.

Next, let $\theta \in J_k$.

Suppose first that $(1,n) \notin \ker\theta$. Then, there exists an order-preserving transformation $\tau \in L_\varepsilon \cap R_\theta$. Since $\varepsilon \in L_\varepsilon \cap R_\varepsilon = L_\tau \cap R_\varepsilon$, we have $\tau\varepsilon = \tau$, whence $\operatorname{im}\tau \,\#\, \ker\varepsilon$ and so, by Lemma 1.1, we obtain $\tau R_\varepsilon = R_\tau = R_\theta$. As $\tau \in C$ and $R_\varepsilon \subseteq \langle C\rangle$, then the $\mathcal{R}$-class R_θ of $\mathcal{OP}_n$ is contained in $\langle C\rangle$.

Finally, suppose that $(1,n) \in \ker\theta$ and let $\tau \in C \cap R_\theta$. Take an order-preserving idempotent ε' such that $\operatorname{im}\varepsilon' = \operatorname{im}\tau$. Then, $\varepsilon' \in L_{\varepsilon'} \cap R_{\varepsilon'} = L_\tau \cap R_{\varepsilon'}$, whence $\tau\varepsilon' = \tau$ and so $\operatorname{im}\tau \,\#\, \ker\varepsilon'$. Thus, by Lemma 1.1, we have $\tau R_{\varepsilon'} = R_\tau = R_\theta$. As $\tau \in C$ and $R_{\varepsilon'} \subseteq \langle C\rangle$ (by the previous case), then the $\mathcal{R}$-class R_θ is also contained in $\langle C\rangle$.

Hence, we have proved that $J_k \subseteq \langle C\rangle$ and so, by Lemma 1.7, we obtain $OP(n,k) = \langle C\rangle$, as required. □

Since $J_k \cap \mathcal{O}_n \subseteq \langle E(J_k \cap \mathcal{O}_n)\rangle$ (see [12]) and each $\mathcal{R}$-class of $\mathcal{OP}_n$ contains at least one idempotent, we have:

Corollary 1.9. $OP(n,k) = \langle E(J_k)\rangle$.

Notice that it is easy to show that, in fact, each $\mathcal{R}$-class of $\mathcal{OP}_n$ contained in J_k has at least two idempotents. Moreover, as $2 \le k \le n-1$, it also is easy to show that each $\mathcal{L}$-class of $\mathcal{OP}_n$ contained in J_k also has at least two idempotents.

Next, we define a fundamental concept (first considered by Yang and Yang in [19]) for our description of the maximal subsemigroups of $OP(n,k)$.

Let Im_k be any non-empty family of subsets of X_n of cardinality k. Let Ker_k be any non-empty collection of equivalence relations on X_n of weight k. Let $\mathcal{I}$ be a non-empty proper subset of Im_k and let $\mathcal{K}$ be a non-empty proper subset of Ker_k. The pair $(\mathcal{I},\mathcal{K})$ is called a *coupler* of $(\mathrm{Im}_k, \mathrm{Ker}_k)$ if the following three conditions are satisfied:

(1) For every $A \in \mathcal{I}$ and $\pi \in \mathcal{K}$, A is not a transversal of π;

(2) For every $B \in \mathrm{Im}_k \setminus \mathcal{I}$, there exists $\pi \in \mathcal{K}$ such that $B \,\#\, \pi$;

(3) For every $\rho \in \mathrm{Ker}_k \setminus \mathcal{K}$, there exists $A \in \mathcal{I}$ such that $A \,\#\, \rho$.

Now, let

$$\mathrm{Im}_k(\mathcal{OP}_n) = \{\operatorname{im}\alpha \mid \alpha \in \mathcal{OP}_n \text{ and } \operatorname{rank}\alpha = k\}$$

(i.e. $\mathrm{Im}_k(\mathcal{OP}_n) = \binom{X_n}{k}$, the family of all subsets of X_n of cardinality k) and let

$$\mathrm{Ker}_k(\mathcal{OP}_n) = \{\ker\alpha \mid \alpha \in \mathcal{OP}_n \text{ and } \operatorname{rank}\alpha = k\}.$$

Then, to a coupler of $(\mathrm{Im}_k(\mathcal{OP}_n), \mathrm{Ker}_k(\mathcal{OP}_n))$ we also call *k-coupler* of $\mathcal{OP}_n$. Analogously, being $\mathrm{Im}_k(\mathcal{O}_n) = \{\operatorname{im}\alpha \mid \alpha \in \mathcal{O}_n \text{ and } \operatorname{rank}\alpha = k\}$ and $\mathrm{Ker}_k(\mathcal{O}_n) =$

$\{\ker\alpha \mid \alpha \in \mathcal{O}_n \text{ and } \operatorname{rank}\alpha = k\}$, we also call *$k$-coupler* of $\mathcal{O}_n$ to a coupler of $(\operatorname{Im}_k(\mathcal{O}_n), \operatorname{Ker}_k(\mathcal{O}_n))$ (notice that $\operatorname{Im}_k(\mathcal{O}_n) = \binom{X_n}{k} = \operatorname{Im}_k(\mathcal{OP}_n)$ and $\operatorname{Ker}_k(\mathcal{O}_n) = \{\pi \in \operatorname{Ker}_k(\mathcal{OP}_n) \mid (1,n) \notin \pi\}$).

Example 1.10. Consider the following transformations of $\mathcal{OP}_5$ of rank 3:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 4 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 5 & 5 \end{pmatrix}, \qquad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 3 & 4 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 5 & 5 \end{pmatrix}, \qquad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 5 & 5 \end{pmatrix},$$

$$\alpha_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 2 & 3 & 4 \end{pmatrix}, \qquad \alpha_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 3 & 5 \end{pmatrix},$$

$$\alpha_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 5 & 5 \end{pmatrix} \quad \text{and} \quad \alpha_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 3 & 4 & 5 \end{pmatrix}.$$

Then, we have $\{\operatorname{im}\alpha_1, \ldots, \operatorname{im}\alpha_{10}\} = \binom{X_5}{3}$, $\{\ker\alpha_1, \ldots, \ker\alpha_{10}\} = \operatorname{Ker}_3(\mathcal{OP}_5)$ and $\{\ker\alpha_1, \ldots, \ker\alpha_6\} = \operatorname{Ker}_3(\mathcal{O}_5)$. Moreover, for instance,

- $(\{\operatorname{im}\alpha_1, \operatorname{im}\alpha_2\}, \{\ker\alpha_1, \ker\alpha_2, \ker\alpha_3, \ker\alpha_4, \ker\alpha_{10}\})$ [Figure 1],
- $(\{\operatorname{im}\alpha_7, \operatorname{im}\alpha_8, \operatorname{im}\alpha_9, \operatorname{im}\alpha_{10}\}, \{\ker\alpha_4, \ker\alpha_5, \ker\alpha_6\})$ [Figure 2],
- $(\{\operatorname{im}\alpha_1, \operatorname{im}\alpha_7, \operatorname{im}\alpha_{10}\}, \{\ker\alpha_2, \ker\alpha_4, \ker\alpha_5\})$ [Figure 3] and
- $(\{\operatorname{im}\alpha_6, \operatorname{im}\alpha_9\}, \{\ker\alpha_3, \ker\alpha_5, \ker\alpha_6, \ker\alpha_9, \ker\alpha_{10}\})$ [Figure 4]

are 3-couplers of $\mathcal{OP}_5$.

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							
ker α_7		*		*					*	*
ker α_8	*	*						*	*	
ker α_9	*						*	*		
ker α_{10}				*			*			*

Figure 1

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							
ker α_7		*		*					*	*
ker α_8	*	*						*	*	
ker α_9	*						*	*		
ker α_{10}				*			*			*

Figure 2

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							
ker α_7		*		*					*	*
ker α_8	*	*						*	*	
ker α_9	*						*	*		
ker α_{10}				*			*			*

Figure 3

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							
ker α_7		*		*					*	*
ker α_8	*	*						*	*	
ker α_9	*						*	*		
ker α_{10}				*			*			*

Figure 4

On the other hand,

- $(\{\operatorname{im}\alpha_1, \operatorname{im}\alpha_7, \operatorname{im}\alpha_{10}\}, \{\ker\alpha_2, \ker\alpha_4, \ker\alpha_5\})$ [Figure 5]
- $(\{\operatorname{im}\alpha_1, \operatorname{im}\alpha_2\}, \{\ker\alpha_1, \ker\alpha_2, \ker\alpha_3, \ker\alpha_4\})$ [Figure 6]

are 3-couplers of $\mathcal{O}_5$.

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							

Figure 5

	im α_1	im α_2	im α_3	im α_4	im α_5	im α_6	im α_7	im α_8	im α_9	im α_{10}
ker α_1						*			*	*
ker α_2					*	*		*	*	
ker α_3				*	*		*	*		
ker α_4			*		*	*				
ker α_5		*	*	*	*					
ker α_6	*	*	*							

Figure 6

(As usual, the symbol $*$ inside a box means that the corresponding $\mathcal{H}$-class contains an idempotent.)

Next, we consider the following subsets of $OP(n,k)$:

(1) $S_A = OP(n,k-1) \cup (J_k \setminus L_\alpha^{\mathcal{OP}_n})$, with $\alpha \in \mathcal{OP}_n$ such that $A = \operatorname{im}\alpha$, for each $A \in \binom{X_n}{k}$;

(2) $S_\pi = OP(n,k-1) \cup (J_k \setminus R_\alpha^{\mathcal{OP}_n})$, with $\alpha \in \mathcal{OP}_n$ such that $\pi = \ker\alpha$, for each $\pi \in \operatorname{Ker}_k(\mathcal{OP}_n)$;

(3) $S_{(\mathcal{I},\mathcal{K})} = OP(n,k-1) \cup \left(\bigcup\{L_\alpha^{\mathcal{OP}_n} \mid \alpha \in \mathcal{OP}_n \text{ and } \operatorname{im}\alpha \in \mathcal{I}\}\right) \cup$
$\cup \left(\bigcup\{R_\alpha^{\mathcal{OP}_n} \mid \alpha \in \mathcal{OP}_n \text{ and } \ker\alpha \in \mathcal{K}\}\right)$,
for each k-coupler $(\mathcal{I},\mathcal{K})$ of $\mathcal{OP}_n$.

It is routine matter to prove that each of these subsets is a (proper) subsemigroup of $OP(n,k)$.

Before we give the description of the maximal subsemigroups of the ideals of the semigroup $\mathcal{OP}_n$, we recall a result presented in [6] by the first and third authors (see also [20]). Let $O(n,k)$ denote the ideal of $\mathcal{O}_n$ of all elements of rank less than or equal to k, i.e. $O(n,k) = OP(n,k) \cap \mathcal{O}_n$. Thus, we have:

Theorem 1.11. *Let $n \geq 3$ and $2 \leq k \leq n-1$. Then a subsemigroup of $O(n,k)$ is maximal if and only if it belongs to one of the following types:*

(1) $S_A \cap \mathcal{O}_n$, *with* $A \in \binom{X_n}{k}$;

(2) $S_\pi \cap \mathcal{O}_n$, *with* $\pi \in \mathrm{Ker}_k(\mathcal{O}_n)$ *such that* π *does not admit any interval of* X_n *as a transversal;*

(3) $S'_{(\mathcal{I},\mathcal{K})} = O(n,k-1) \cup \big(\bigcup\{L^{\mathcal{O}_n}_\alpha \mid \alpha \in \mathcal{O}_n \text{ and } \operatorname{im}\alpha \in \mathcal{I}\}\big) \cup$
$\cup \big(\bigcup\{R^{\mathcal{O}_n}_\alpha \mid \alpha \in \mathcal{O}_n \text{ and } \ker\alpha \in \mathcal{K}\}\big)$,
with $(\mathcal{I},\mathcal{K})$ *a* k*-coupler of* $\mathcal{O}_n$.

Regarding the maximal subsemigroups of $OP(n,k)$, we first prove:

Lemma 1.12. *Let* S *be a maximal subsemigroup of* $OP(n,k)$. *Then* $S = \bigcup\{H^{\mathcal{OP}_n}_\alpha \mid \alpha \in S\}$.

Proof. Let $T = \bigcup\{H^{\mathcal{OP}_n}_\alpha \mid \alpha \in S\}$. Then clearly $S \subseteq T$. By Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S$. Hence $H^{\mathcal{OP}_n}_\varepsilon \cap S = \emptyset$ and so $T \neq OP(n,k)$. The result follows by proving that T is a subsemigroup of $OP(n,k)$. Clearly, by the maximality of S and Lemma 1.7, we have $OP(n,k-1) \subsetneq S$. So, it suffices to show that, for all $\alpha,\beta \in T \cap J_k$ such that $\alpha\beta \in J_k$, we get $\alpha\beta \in T$. Therefore, let $\alpha,\beta \in T \cap J_k$ be such that $\alpha\beta \in J_k$. Take $\alpha',\beta' \in S$ such that $\alpha \in H^{\mathcal{OP}_n}_{\alpha'}$ and $\beta \in H^{\mathcal{OP}_n}_{\beta'}$. Then $\operatorname{im}\alpha' = \operatorname{im}\alpha \,\#\, \ker\beta = \ker\beta'$ and $\alpha\beta \in R^{\mathcal{OP}_n}_\alpha \cap L^{\mathcal{OP}_n}_\beta$, whence $\alpha'\beta' \in R^{\mathcal{OP}_n}_{\alpha'} \cap L^{\mathcal{OP}_n}_{\beta'} = R^{\mathcal{OP}_n}_\alpha \cap L^{\mathcal{OP}_n}_\beta = H^{\mathcal{OP}_n}_{\alpha\beta}$ and so, as $\alpha'\beta' \in S$, we obtain $\alpha\beta \in H^{\mathcal{OP}_n}_{\alpha'\beta'} \subseteq T$, as required. □

Now, we have:

Theorem 1.13. *Let* $n \geq 3$ *and* $2 \leq k \leq n-1$. *Then a subsemigroup of* $OP(n,k)$ *is maximal if and only if it belongs to one of the following types:*

(1) S_A, *with* $A \in \binom{X_n}{k}$;

(2) S_π, *with* $\pi \in \mathrm{Ker}_k(\mathcal{OP}_n)$;

(3) $S_{(\mathcal{I},\mathcal{K})}$, *with* $(\mathcal{I},\mathcal{K})$ *a* k*-coupler of* $\mathcal{OP}_n$.

Proof. We begin by showing that each of these subsemigroups of $OP(n,k)$ is maximal.

First, let $A \in \binom{X_n}{k}$ and let $\alpha \in \mathcal{OP}_n$ be such that $\operatorname{im}\alpha = A$. Take an idempotent $\varepsilon \in (J_k \setminus L^{\mathcal{OP}_n}_\alpha) \cap R^{\mathcal{OP}_n}_\alpha$. As $L^{\mathcal{OP}_n}_\varepsilon \subseteq S_A$ and, by Lemma 1.1, $L^{\mathcal{OP}_n}_\varepsilon \alpha = L^{\mathcal{OP}_n}_\alpha$, we have $\langle S_A, \alpha\rangle = OP(n,k)$. Thus, S_A is maximal.

Similarly, being $\pi \in \mathrm{Ker}_k(\mathcal{OP}_n)$ and being $\alpha \in \mathcal{OP}_n$ such that $\ker\alpha = \pi$, the $\mathcal{L}$-class $L^{\mathcal{OP}_n}_\alpha$ contains at least one idempotent $\varepsilon \in J_k \setminus R^{\mathcal{OP}_n}_\alpha$ and so $R^{\mathcal{OP}_n}_\varepsilon \subseteq S_\pi$ and, by Lemma 1.1, $\alpha R^{\mathcal{OP}_n}_\varepsilon = R^{\mathcal{OP}_n}_\alpha$, whence $\langle S_\pi, \alpha\rangle = OP(n,k)$. Thus, S_π is maximal.

Finally, regarding the subsemigroups of type (3), let $(\mathcal{I},\mathcal{K})$ be a k-coupler of $\mathcal{OP}_n$. As $\mathcal{I}$ and $\mathcal{K}$ are proper subsets of $\binom{X_n}{k}$ and $\mathrm{Ker}_k(\mathcal{OP}_n)$, respectively, we may take $\gamma \in \mathcal{OP}_n$ such that $\operatorname{im}\gamma \in \binom{X_n}{k} \setminus \mathcal{I}$ and $\ker\gamma \in \operatorname{Ker}_k(\mathcal{OP}_n) \setminus \mathcal{K}$. Then, there exist $\alpha, \beta \in \mathcal{OP}_n$ such that $\operatorname{im}\alpha \in \mathcal{I}$, $\ker\beta \in \mathcal{K}$, $\operatorname{im}\gamma \# \ker\beta$ and $\operatorname{im}\alpha \# \ker\gamma$. Now, by Lemma 1.1, we have $\gamma R_\beta^{\mathcal{OP}_n} = R_\gamma^{\mathcal{OP}_n}$. As $R_\beta^{\mathcal{OP}_n} \subseteq S_{(\mathcal{I},\mathcal{K})}$, we obtain $R_\gamma^{\mathcal{OP}_n} \subseteq \langle S_{(\mathcal{I},\mathcal{K})}, \gamma\rangle$. On the other hand, by Lemma 1.1, we also have $L_\alpha^{\mathcal{OP}_n} R_\gamma^{\mathcal{OP}_n} = J_k$. Since $L_\alpha^{\mathcal{OP}_n} \subseteq S_{(\mathcal{I},\mathcal{K})}$, we deduce that $\langle S_{(\mathcal{I},\mathcal{K})}, \gamma\rangle = OP(n,k)$. Thus, $S_{(\mathcal{I},\mathcal{K})}$ is maximal.

For the converse part, let S be a maximal subsemigroup of the ideal $OP(n,k)$.

If $S \cap R_\alpha^{\mathcal{OP}_n} = \emptyset$, for some $\alpha \in J_k$, then $S = S_{\ker\alpha}$, by the maximality of S. Similarly, if $S \cap L_\alpha^{\mathcal{OP}_n} = \emptyset$, for some $\alpha \in J_k$, then $S = S_{\operatorname{im}\alpha}$. Thus, suppose that S has at least one element from each $\mathcal{R}$-class and each $\mathcal{L}$-class of $\mathcal{OP}_n$ contained in J_k. If $S \cap \mathcal{O}_n = O(n,k)$ then $S = OP(n,k)$, by Proposition 1.8. Therefore $S \cap \mathcal{O}_n \subsetneq O(n,k)$. Let $\bar{S}$ be a maximal subsemigroup of $O(n,k)$ such that $S \cap \mathcal{O}_n \subseteq \bar{S}$. By Theorem 1.11, we have three possible cases for $\bar{S}$.

First, suppose that $\bar{S} = S_\pi \cap \mathcal{O}_n$, for some $\pi \in \operatorname{Ker}_k(\mathcal{O}_n)$. As $\operatorname{Ker}_k(\mathcal{O}_n) \subseteq \operatorname{Ker}_k(\mathcal{OP}_n)$, we may take $\alpha \in S$ such that $\ker\alpha = \pi$. Moreover, we have $H_\alpha^{\mathcal{OP}_n} \cap \mathcal{O}_n \neq \emptyset$. Now, as $H_\alpha^{\mathcal{OP}_n} \subseteq S$ (by Lemma 1.12), we have $(S \cap \mathcal{O}_n) \cap R_\alpha^{\mathcal{OP}_n} \neq \emptyset$, whence $\bar{S} \cap R_\alpha^{\mathcal{OP}_n} \neq \emptyset$ and so $S_\pi \cap R_\alpha^{\mathcal{OP}_n} \neq \emptyset$, which is a contradiction. Thus, $\bar{S}$ cannot be of this type.

Secondly, we suppose that $\bar{S} = S_{A_1} \cap \mathcal{O}_n$, for some $A_1 \in \binom{X_n}{k}$. Let $A_2, \ldots, A_r$ $(r \geq 2)$ be distinct elements of $\binom{X_n}{k}$ such that, for all $\alpha \in J_k$, $L_\alpha^{\mathcal{OP}_n} \cap \mathcal{O}_n \cap S = \emptyset$ if and only if $\operatorname{im}\alpha \in \{A_1, \ldots, A_r\}$. For each $i \in \{1, \ldots, r\}$, let $\alpha_i \in \mathcal{OP}_n$ be such that $\operatorname{im}\alpha_i = A_i$. Notice that, for $i \in \{1, \ldots, r\}$, as S has at least one element from each $\mathcal{L}$-class of $\mathcal{OP}_n$ contained in J_k, in particular, we have $L_{\alpha_i}^{\mathcal{OP}_n} \cap S \neq \emptyset$ and, on the other hand, as a consequence of Lemma 1.12, if $\alpha \in L_{\alpha_i}^{\mathcal{OP}_n} \cap S$ then $(1,n) \in \ker\alpha$. Now, let

$$\mathcal{K} = \{\ker\alpha \mid \alpha \in L_{\alpha_i}^{\mathcal{OP}_n} \cap S, \text{ for some } i \in \{1, \ldots, r\}\}.$$

Notice that, clearly, $\mathcal{K} \neq \emptyset$. Also, let

$$\mathcal{I} = \{A \in \tbinom{X_n}{k} \mid A \text{ is not a transversal of } \pi, \text{ for all } \pi \in \mathcal{K}\}.$$

Observe that, as $(1,n) \in \pi$, for all $\pi \in \mathcal{K}$, then $\{A \in \binom{X_n}{k} \mid 1, n \in A\} \subseteq \mathcal{I}$ and so, in particular, $\mathcal{I} \neq \emptyset$. Furthermore, it is a routine matter to check that the pair $(\mathcal{I}, \mathcal{K})$ is a k-coupler of $\mathcal{OP}_n$. Next, we show that $S \cap J_k \subseteq S_{(\mathcal{I},\mathcal{K})}$. Take $\alpha \in S \cap J_k$. If $\operatorname{im}\alpha \in \mathcal{I}$, then $\alpha \in S_{(\mathcal{I},\mathcal{K})}$, by definition. On the other hand, suppose that $\operatorname{im}\alpha \notin \mathcal{I}$. Then, there exists $\pi \in \mathcal{K}$ such that $\operatorname{im}\alpha \# \pi$. As $\pi \in \mathcal{K}$, then $\pi = \ker\beta$, for some $\beta \in L_{\alpha_i}^{\mathcal{OP}_n} \cap S$ and $i \in \{1, \ldots, r\}$. From $\operatorname{im}\alpha \# \ker\beta$ it follows that $\alpha\beta \in R_\alpha^{\mathcal{OP}_n} \cap L_\beta^{\mathcal{OP}_n} = R_\alpha^{\mathcal{OP}_n} \cap L_{\alpha_i}^{\mathcal{OP}_n}$. Moreover, $\alpha\beta \in S$, whence

$\alpha\beta \in L_{\alpha_i}^{\mathcal{OP}_n} \cap S$ and so $\ker\alpha = \ker(\alpha\beta) \in \mathcal{K}$. Then $\alpha \in S_{(\mathcal{I},\mathcal{K})}$, by definition. So, we have proved that $S \cap J_k \subseteq S_{(\mathcal{I},\mathcal{K})}$. Therefore $S \subseteq S_{(\mathcal{I},\mathcal{K})}$ and thus $S = S_{(\mathcal{I},\mathcal{K})}$, by the maximality of S.

Finally, suppose that $\bar{S} = S'_{(\mathcal{I}',\mathcal{K}')}$ (as defined in Theorem 1.11), for some k-coupler $(\mathcal{I}',\mathcal{K}')$ of $\mathcal{O}_n$. Let

$$\mathcal{I} = \mathcal{I}' \cap \{\operatorname{im}\alpha \mid \alpha \in S \text{ and } \ker\alpha \in \operatorname{Ker}_k(\mathcal{O}_n) \setminus \mathcal{K}'\}$$

and

$$\mathcal{K} = \{\pi \in \operatorname{Ker}_k(\mathcal{OP}_n) \mid A \text{ is not a transversal of } \pi, \text{ for all } A \in \mathcal{I}\}.$$

Clearly, $\mathcal{K}' \subseteq \mathcal{K}$, whence $\mathcal{K} \neq \emptyset$. On the other hand, from the definition of $\mathcal{I}$ and from $S \cap \mathcal{O}_n \subseteq S'_{(\mathcal{I}',\mathcal{K}')}$, in view of Lemma 1.12, we may deduce that $R_\beta^{\mathcal{OP}_n} \cap L_\alpha^{\mathcal{OP}_n} \cap S = \emptyset$, for all $\alpha, \beta \in J_k$ such that $\ker\beta \in \operatorname{Ker}_k(\mathcal{O}_n) \setminus \mathcal{K}'$ and $\operatorname{im}\alpha \in \binom{X_n}{k} \setminus \mathcal{I}$. As S has at least one element from each $\mathcal{R}$-class of $\mathcal{OP}_n$ contained in J_k, in particular, it follows that $\mathcal{I} \neq \emptyset$. Furthermore, it is a routine matter to check that the pair $(\mathcal{I},\mathcal{K})$ is a k-coupler of $\mathcal{OP}_n$. Next, we aim to prove that $S = S_{(\mathcal{I},\mathcal{K})}$. Take $\alpha \in J_k \cap S$ and suppose that $\alpha \notin S_{(\mathcal{I},\mathcal{K})}$. Then, $\operatorname{im}\alpha \in \binom{X_n}{k} \setminus \mathcal{I}$ and $\ker\alpha \in \operatorname{Ker}_k(\mathcal{OP}_n) \setminus \mathcal{K}$. Hence, there exists $A \in \mathcal{I}$ such that $A \# \ker\alpha$ and, by the definition of $\mathcal{I}$, we have $A = \operatorname{im}\beta$, for some $\beta \in S$ such that $\ker\beta \in \operatorname{Ker}_k(\mathcal{O}_n) \setminus \mathcal{K}'$. Thus, from $\operatorname{im}\beta = A \# \ker\alpha$, it follows that $\beta\alpha \in R_\beta^{\mathcal{OP}_n} \cap L_\alpha^{\mathcal{OP}_n} \cap S$ and so $R_\beta^{\mathcal{OP}_n} \cap L_\alpha^{\mathcal{OP}_n} \cap S \neq \emptyset$, with $\ker\beta \in \operatorname{Ker}_k(\mathcal{O}_n) \setminus \mathcal{K}'$ and $\operatorname{im}\alpha \in \binom{X_n}{k} \setminus \mathcal{I}$, which contradicts the above deduction. Therefore, $\alpha \in S_{(\mathcal{I},\mathcal{K})}$, whence $S \subseteq S_{(\mathcal{I},\mathcal{K})}$ and then $S = S_{(\mathcal{I},\mathcal{K})}$, by the maximality of S, as required. □

2. Maximal subsemigroups of the ideals of $\mathcal{OR}_n$

As for $\mathcal{OP}_n$, the semigroup $\mathcal{OR}_n$ is the union of its $\mathcal{J}$-classes $\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_n$, where

$$\bar{J}_k = \{\alpha \in \mathcal{OR}_n \mid \operatorname{rank}\alpha = k\}$$

for $k = 1, \ldots, n$. Notice that $\bar{J}_k \cap \mathcal{OP}_n$ is the $\mathcal{J}$-class J_k of $\mathcal{OP}_n$, for $k = 1, \ldots, n$, and $\bar{J}_1 = J_1$ and $\bar{J}_2 = J_2$. Observe also that, for $\alpha \in \mathcal{OP}_n$, we have $L_\alpha^{\mathcal{OP}_n} = L_\alpha^{\mathcal{OR}_n} \cap \mathcal{OP}_n$, $R_\alpha^{\mathcal{OP}_n} = R_\alpha^{\mathcal{OR}_n} \cap \mathcal{OP}_n$ and $H_\alpha^{\mathcal{OP}_n} = H_\alpha^{\mathcal{OR}_n} \cap \mathcal{OP}_n$.

Analogously to $\mathcal{OP}_n$, the ideals of the semigroup $\mathcal{OR}_n$ are unions of the $\mathcal{J}$-classes $\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_k$, i.e. the sets

$$OR(n,k) = \{\alpha \in \mathcal{OR}_n \mid \operatorname{rank}\alpha \leq k\},$$

with $k = 1, \ldots, n$.

For $\alpha \in OR(n,k)$, with $k = 1, \ldots, n$, we also have $L_\alpha^{OR(n,k)} = L_\alpha^{\mathcal{OR}_n}$, $R_\alpha^{OR(n,k)} = R_\alpha^{\mathcal{OR}_n}$ and $H_\alpha^{OR(n,k)} = H_\alpha^{\mathcal{OR}_n}$. Moreover, a result similar to Lemma 1.1 holds for elements of $\mathcal{OR}_n$:

Lemma 2.1. *Let $k \in \{1, 2, \ldots, n\}$ and let $\alpha, \beta \in \bar{J}_k$ be such that $\operatorname{im}\alpha \,\#\, \ker\beta$. Then $\alpha R_\beta^{\mathcal{OR}_n} = R_{\alpha\beta}^{\mathcal{OR}_n} = R_\alpha^{\mathcal{OR}_n}$, $L_\alpha^{\mathcal{OR}_n}\beta = L_{\alpha\beta}^{\mathcal{OR}_n} = L_\beta^{\mathcal{OR}_n}$, $\alpha H_\beta^{\mathcal{OR}_n} = H_\alpha^{\mathcal{OR}_n}\beta = H_{\alpha\beta}^{\mathcal{OR}_n}$ and $L_\alpha^{\mathcal{OR}_n} R_\beta^{\mathcal{OR}_n} = \bar{J}_k$.*

As $\mathcal{OR}_1 = \mathcal{OP}_1$ and $\mathcal{OR}_2 = \mathcal{OP}_2$, in what follows, we consider $n \geq 3$.

Next, recall that a dihedral group $\mathcal{D}_n$ of order $2n$ can abstractly be defined by the group presentation

$$\langle x, y \mid x^n = y^2 = 1, xy = yx^{-1} \rangle.$$

Let

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \in \bar{J}_n.$$

Hence, we have $\bar{J}_n = \langle g, h \rangle$ and, as $g^n = h^2 = (gh)^2 = 1$, it is easy to see that $\bar{J}_n$ is a dihedral group of order $2n$. Furthermore, Catarino and Higgins [5] proved:

Proposition 2.2. *Let $k \in \{3, \ldots, n\}$ and let $\alpha \in \mathcal{OR}_n$ be an element of rank k. Then $|H_\alpha| = 2k$. Moreover, if α is an idempotent, then H_α is a dihedral group of order $2k$.*

Thus, each $\mathcal{H}$-class of rank k of $\mathcal{OR}_n$ has k orientation-preserving transformations and k orientation-reversing transformations, for $k \in \{3, \ldots, n\}$.

Notice that, since $\bar{J}_1 = J_1$ and $\bar{J}_2 = J_2$, for $\alpha \in \bar{J}_k$ with $k = 1, 2$, we have $|H_\alpha^{\mathcal{OR}_n}| = k$.

Let us consider again the dihedral group $\mathcal{D}_n$ of order $2n$. Observe that

$$\mathcal{D}_n = \{1 = x^0, x, x^2, \ldots, x^{n-1}\} \cup \{y, xy, x^2y, \ldots, x^{n-1}y\}.$$

It is easy to show that the subgroups of $\mathcal{D}_n$ are of the form $\langle x^d \rangle$ (a cyclic group of order n/d) and of the form $\langle x^d, x^i y \rangle$ (a dihedral group of order $2n/d$), for each positive divisor d of n and each $0 \leq i < d$. It follows that $\langle x \rangle$ and $\langle x^p, x^i y \rangle$, with p a prime divisor of n and $0 \leq i < p$, are the maximal subsemigroups of $\mathcal{D}_n$.

Now, for a prime divisor p of n and $0 \leq i < p$, consider the dihedral group $V_{p,i} = \langle g^p, g^i h \rangle$ of order $2n/p$. Then, the above observation can be rewrote as:

Lemma 2.3. *The group $J_n = \langle g \rangle$ and the groups $V_{p,i}$, with p a prime divisor of n and $0 \leq i < p$, are the maximal subsemigroups of $\bar{J}_n$.*

Next, we recall the following well known result (see [4], [17]).

Proposition 2.4. $\mathcal{OR}_n = \langle u_1, g, h \rangle$.

In fact, more generally, we have:

Proposition 2.5. *Let* $\alpha \in \bar{J}_{n-1}$, γ *an element of* J_n *of order* n *and* $\beta \in \bar{J}_n \setminus J_n$. *Then* $\mathcal{OR}_n = \langle \alpha, \gamma, \beta \rangle$.

Proof. If $\alpha \in \bar{J}_{n-1} \cap \mathcal{OP}_n$ then, by Proposition 1.4, we have $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$. If $\alpha \in \bar{J}_{n-1} \setminus \mathcal{OP}_n$ then $\alpha\beta \in \bar{J}_{n-1} \cap \mathcal{OP}_n$ and, again by Proposition 1.4, we obtain $\mathcal{OP}_n = \langle \alpha\beta, \gamma \rangle$. Therefore, $u_1, g \in \langle \alpha, \gamma, \beta \rangle$. As $\beta \in \bar{J}_n \setminus \mathcal{OP}_n$, there exists $i \in \{1, \dots, n\}$ such that $\beta = \left(\begin{smallmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 & n \\ i-1 & i-2 & \cdots & 1 & n & \cdots & i+1 & i \end{smallmatrix}\right)$. On the other hand, the transformation

$$\delta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 & n \\ n-i+2 & n-i+3 & \cdots & n & 1 & \cdots & n-i & n-i+1 \end{pmatrix}$$

is an element of $\mathcal{OP}_n$ and $h = \beta\delta \in \langle \alpha, \gamma, \beta \rangle$. Therefore, by Proposition 2.4, we deduce that $\mathcal{OR}_n = \langle \alpha, \gamma, \beta \rangle$. □

We have now all the ingredients to describe the maximal subsemigroups of $\mathcal{OR}_n$.

Theorem 2.6. *A subsemigroup* S *of the semigroup* $\mathcal{OR}_n$ *is maximal if and only if* $S = OR(n, n-2) \cup \bar{J}_n$ *or* $S = OR(n, n-1) \cup J_n$ *or* $S = OR(n, n-1) \cup V_{p,i}$, *for some prime divisor* p *of* n *and* $0 \le i < p$.

Proof. Let S be a maximal subsemigroup of $\mathcal{OR}_n$. Then, by Proposition 2.5, we have $S = OR(n, n-2) \cup T$, for some $T \subset (\bar{J}_{n-1} \cup \bar{J}_n)$ such that $T \cap \bar{J}_{n-1} = \emptyset$ or T does not contain any element of J_n of order n or $T \cap (\bar{J}_n \setminus J_n) = \emptyset$. In the latter two cases, we must have $\bar{J}_{n-1} \subseteq T$, by the maximality of S. Thus, $S = OR(n, n-1) \cup T'$, for some $T' \subset \bar{J}_n$. Clearly, T' must be a maximal subsemigroup of $\bar{J}_n$, whence $S = OR(n, n-1) \cup J_n$ or $S = OR(n, n-1) \cup V_{p,i}$, for some prime divisor p of n and $0 \le i < p$, accordingly with Lemma 2.3. On the other hand, if $T \cap \bar{J}_{n-1} = \emptyset$ then $S \subseteq OR(n, n-2) \cup \bar{J}_n$ and so $S = OR(n, n-2) \cup \bar{J}_n$, by the maximality of S.

The converse part follows immediately from Proposition 2.5 and Lemma 2.3. □

From now on we consider the ideals $OR(n, k)$ of $\mathcal{OR}_n$, for $k \in \{1, \dots, n-1\}$. Since $OR(n, 1) = OP(n, 1)$ and $OR(n, 2) = OP(n, 2)$, in what follows, we take $k \ge 3$.

Notice that, as $\alpha h \in OP(n,k)$, for all $\alpha \in OR(n,k) \setminus OP(n,k)$, by using Lemma 1.7, it is easy to conclude:

Lemma 2.7. $OR(n,k) = \langle \bar{J}_k \rangle$.

In fact, moreover, we have:

Proposition 2.8. *$OR(n,k) = \langle J_k, \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.*

Proof. Let $\alpha \in \bar{J}_k \setminus J_k$ and take an idempotent $\varepsilon \in L_\alpha^{\mathcal{OR}_n}$. Since $\operatorname{im}\alpha = \operatorname{im}\varepsilon \,\#\, \ker\varepsilon$, we have $\alpha R_\varepsilon^{\mathcal{OR}_n} = R_\alpha^{\mathcal{OR}_n}$, by Lemma 2.1. Hence, $\alpha(R_\varepsilon^{\mathcal{OR}_n} \cap J_k) = R_\alpha^{\mathcal{OR}_n} \setminus J_k$ and so

$$R_\alpha^{\mathcal{OR}_n} = (R_\alpha^{\mathcal{OR}_n} \cap J_k) \cup (R_\alpha^{\mathcal{OR}_n} \setminus J_k) = (R_\alpha^{\mathcal{OR}_n} \cap J_k) \cup \alpha(R_\varepsilon^{\mathcal{OR}_n} \cap J_k) \subseteq \langle J_k, \alpha \rangle.$$

Now, let ε' be an idempotent of $R_\alpha^{\mathcal{OR}_n}$ and take $\alpha' \in H_{\varepsilon'}^{\mathcal{OR}_n} \setminus J_k$. Notice that $\alpha' \in R_\alpha^{\mathcal{OR}_n} \subseteq \langle J_k, \alpha \rangle$. As $\operatorname{im}\varepsilon' \,\#\, \ker\varepsilon' = \ker\alpha'$, we have $L_{\varepsilon'}^{\mathcal{OR}_n}\alpha' = L_{\alpha'}^{\mathcal{OR}_n} = L_{\varepsilon'}^{\mathcal{OR}_n}$, by Lemma 2.1. Thus $(L_{\varepsilon'}^{\mathcal{OR}_n} \cap J_k)\alpha' = L_{\varepsilon'}^{\mathcal{OR}_n} \setminus J_k$, whence

$$L_{\varepsilon'}^{\mathcal{OR}_n} = (L_{\varepsilon'}^{\mathcal{OR}_n} \cap J_k) \cup (L_{\varepsilon'}^{\mathcal{OR}_n} \setminus J_k) = (L_{\varepsilon'}^{\mathcal{OR}_n} \cap J_k) \cup (L_{\varepsilon'}^{\mathcal{OR}_n} \cap J_k)\alpha' \subseteq \langle J_k, \alpha \rangle.$$

Finally, as $\operatorname{im}\varepsilon' \,\#\, \ker\varepsilon' = \ker\alpha$, we have $L_{\varepsilon'}^{\mathcal{OR}_n} R_\alpha^{\mathcal{OR}_n} = \bar{J}_k$, again by Lemma 2.1. Therefore, $\bar{J}_k \subseteq \langle J_k, \alpha \rangle$ and so, by Lemma 2.7, $OR(n,k) = \langle J_k, \alpha \rangle$, as required. □

As an immediate consequence of Proposition 2.8, we have:

Corollary 2.9. *$OR(n,k-1) \cup J_k$ is a maximal subsemigroup of $OR(n,k)$.*

Also, combining Proposition 2.8 with Corollary 1.9, we have:

Corollary 2.10. *$OR(n,k) = \langle E(J_k), \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.*

Before we present our description of the maximal subsemigroups of the ideals of $\mathcal{OR}_n$, we prove the following result:

Lemma 2.11. *Let S be a maximal subsemigroup of $OR(n,k)$ containing at least one orientation-reversing transformation of rank k. Then $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S \cap \mathcal{OP}_n\}$.*

Proof. Let $\alpha \in S$. As clearly $OR(n,k-1) \subseteq S$, it suffices to consider $\alpha \in \bar{J}_k$. Take $\beta \in H_\alpha^{\mathcal{OR}_n}$ and suppose that $\beta \notin S$. Hence, by the maximality of S, we have $OR(n,k) = \langle S, \beta \rangle$. Let $\varepsilon \in E(J_k)$. Then, there exist $t \geq 0$, $r_0, r_1, \ldots, r_t \geq 0$ and $\alpha_1, \ldots, \alpha_t \in S$ such that $\varepsilon = \beta^{r_0}\alpha_1\beta^{r_1}\alpha_2 \cdots \beta^{r_{t-1}}\alpha_t\beta^{r_t}$. As $\alpha\mathcal{H}\beta$, it follows that $\tau = \alpha^{r_0}\alpha_1\alpha^{r_1}\alpha_2 \cdots \alpha^{r_{t-1}}\alpha_t\alpha^{r_t}\mathcal{H}\varepsilon$. Furthermore, $\tau \in S$. Now, since ε is a power of τ, it follows that also $\varepsilon \in S$. Thus $E(J_k) \subseteq S$. Since S also contains an orientation-reversing transformation of rank k, by Corollary 2.10, we have $S = OR(n,k)$, a contradiction. Therefore $H_\alpha^{\mathcal{OR}_n} \subseteq S$. This shows that $H_\alpha^{\mathcal{OR}_n} \subseteq S$ for all $\alpha \in S$, i.e. $\bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S\} \subseteq S$ and thus $\bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S\} = S$. Since each $\mathcal{H}$-class of $\mathcal{OR}_n$ contains an orientation-preserving transformation, we obtain $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S \cap \mathcal{OP}_n\}$, as required. □

In general, if S' is a subsemigroup of $OP(n,k)$ containing $OP(n,k-1)$, then (using an argument similar to that considered in the proof of Lemma 1.12) it is easy to show that $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S'\}$ is a subsemigroup of $OR(n,k)$. Furthermore, if $S' \subsetneq OP(n,k)$ then also $S \subsetneq OR(n,k)$. In fact, in this case, by Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S'$. It follows that $H_\varepsilon^{\mathcal{OR}_n} \cap S' = \emptyset$ and so also $\varepsilon \notin S$.

Finally, we have:

Theorem 2.12. *Let $n \geq 4$ and $3 \leq k \leq n-1$. Let S be a subsemigroup of $OR(n,k)$. Then, S is maximal if and only if $S = OR(n,k-1) \cup J_k$ or $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S'\}$, for some maximal subsemigroup S' of $OP(n,k)$.*

Proof. First, let S be a maximal subsemigroup of $OR(n,k)$ and suppose that $S \neq OR(n,k-1) \cup J_k$. Then S must contain an orientation-reversing transformation of rank k and so $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S \cap \mathcal{OP}_n\}$, by Lemma 2.11. Clearly, $S \cap \mathcal{OP}_n$ is a proper subsemigroup of $OP(n,k)$, whence there exists a maximal subsemigroup S' of $OP(n,k)$ such that $S \cap \mathcal{OP}_n \subseteq S'$. Then, as in the proof of Lemma 2.11, $\bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S'\}$ is a proper subsemigroup of $OR(n,k)$ and, as it contains S, it follows that $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S'\}$, by the maximality of S.

Conversely, if $S = OR(n,k-1) \cup J_k$, then S is a maximal subsemigroup of $OR(n,k)$, by Corollary 2.9. Hence, let us suppose that $S = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in S'\}$, for some maximal subsemigroup S' of $OP(n,k)$. Then, by the above observation, S is a proper subsemigroup of $OR(n,k)$. Moreover, S must contain an orientation-reversing transformation of rank k. Let $\hat{S}$ be a maximal subsemigroup of $OR(n,k)$ such that $S \subseteq \hat{S}$. Then $\hat{S}$ also contains an orientation-reversing transformation of rank k and so, by Lemma 2.11, $\hat{S} = \bigcup\{H_\alpha^{\mathcal{OR}_n} \mid \alpha \in \hat{S} \cap \mathcal{OP}_n\}$. On the other hand, $S' \subseteq S \cap \mathcal{OP}_n \subseteq \hat{S} \cap \mathcal{OP}_n \subsetneq OP(n,k)$, whence $S' = S \cap \mathcal{OP}_n = \hat{S} \cap \mathcal{OP}_n$, by the maximality of S'. It follows that $S = \hat{S}$ and thus S is a maximal subsemigroup of $OR(n,k)$, as required. □

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References

[1] A. Ja. Aĭzenštat, Defining relations of the semigroup of endomorphisms of a finite linearly ordered set, *Sibirsk. Matem. Žurn.* **3** (1962), 161–169.

[2] R. E. Arthur and N. Ruškuc, Presentations for two extensions of the monoid of order-preserving mappings on a finite chain, *Southeast Asian Bull. Math.* **24** (2000), 1–7.

[3] P. A. Bayramov, On the problem of completeness in a symmetric semigroup of finite degree, *Diskret. Analiz* **8** (1966), 3–27 (in *Russian*).

[4] P. M. Catarino, Monoids of orientation-preserving transformations of a finite chain and their presentations, Semigroups and Applications, (J. M. Howie and N. Ruškuc, eds.), *World Scientific*, 1998, 39–46.

[5] P. M. Catarino and P. M. Higgins, The monoid of orientation-preserving mappings on a chain, *Semigroup Forum* **58** (1999), 190–206.

[6] I. Dimitrova and J. Koppitz, On the maximal subsemigroups of some transformation semigroups, *Asian-Eur. J. Math.* **1** (2008), 189–202.

[7] I. Dimitrova and J. Koppitz, The maximal subsemigroups of the ideals of some semigroups of partial injections, *Discuss. Math. Gen. Algebra Appl.* **29** (2009), 153–167.

[8] M. S. Dummit and R. M. Foote, Abstract Algebra, *Prentice-Hall, Englewood Clifs*, 1991.

[9] V. H. Fernandes, The monoid of all injective order-preserving partial transformations on a finite chain, *Semigroup Forum* **62** (2001), 178–204.

[10] V. H. Fernandes, G. M. S. Gomes and M. M. Jesus, Congruences on monoids of transformations preserving the orientation on a finite chain, *J. Algebra* **321** (2009), 743–757.

[11] O. Ganyushkin and V. Mazorchuk, On the structure of IO_n, *Semigroup Forum* **66** (2003), 455–483.

[12] G. U. Garba, On the idempotent ranks of certain semigroups of order-preserving transformations, *Portugal. Math.* **51** (1994), 185–204.

[13] G. M. S. Gomes and J. M. Howie, On the rank of certain semigroups of order-preserving transformations, *Semigroup Forum* **51** (1992), 275–282.

[14] I. Gyudzhenov and I. Dimitrova, On the maximal subsemigroups of the semigroup of all monotone transformations, *Discuss. Math. Gen. Algebra Appl.* **26** (2006), 199–217.

[15] J. M. Howie, Products of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* **17** (1971), 223–236.

[16] J. M. Howie, Fundamentals of Semigroup Theory, *Oxford University Press*, 1995.

[17] D. McAlister, Semigroups generated by a group and an idempotent, *Comm. Algebra* **26** (1998), 515–547.

[18] K. Todorov and L. Kračolova, On the maximal subsemigroups of the ideals of finite symmetric semigroup, *Simon Stevin* **59** (1985), 129–140.

[19] H. Yang and X. Yang, Maximal subsemigroups of finite transformation semigroups $K(n, r)$, *Acta Math. Sin. (Engl. Ser.)* **20** (2004), 475–482.

[20] X. Yang, A classification of maximal subsemigroups of finite order-preserving transformation semigroups, *Comm. Algebra* **28** (2000), 1503–1513.

[21] X. Yang, Maximal subsemigroups of the finite singular transformation semigroup, *Comm. Algebra* **29** (2001), 1175–1182.

[22] P. Zhao, X. Bo and Y. Mei, Locally maximal idempotent-generated subsemigroups of singular orientation-preserving transformation semigroups, *Semigroup Forum* **77** (2008), 187–195.

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