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Common dynamics of two Pisot substitutions with the same incidence matrix

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Abstract. The matrix of a substitution is not sufficient to completely determine the dynamics associated with it, even in the simplest cases since there are many words with the same abelianization. In this paper we study the common points of the canonical broken lines associated with two different irreducible Pisot unimodular substitutions σ_1 and σ_2 having the same incidence matrix. We prove that if 0 is an inner point to the Rauzy fractal associated with the substitution σ_1 and σ_1 verifies the Pisot conjecture then these common points can be generated with a substitution on an alphabet of so-called balanced pairs, and we obtain in this way the intersection of the interior of two Rauzy fractals.

1. Introduction

Let σ_1 and σ_2 be two different irreducible Pisot substitutions having the same incidence matrix. Although the fixed points of the substitutions have the same letter frequencies, they usually have different properties, e.g., their Rauzy fractals can be very different (The Rauzy fractal is defined by projection of the canonical stepped line, associated with a fixed or periodic point of substitution σ_i on the contracting space of the incidence matrix M_{σ_i}, see Section 2 for definition).

A classic example is given by the Tribonacci substitution and the flipped

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Tribonacci substitution, i.e.,

$$\sigma_1 : \begin{cases} a \to ab \\ b \to ac \\ c \to a \end{cases} \qquad \text{and} \quad \sigma_2 : \begin{cases} a \to ab \\ b \to ca \\ c \to a. \end{cases}$$

The incidence matrix of σ_1 and σ_2 is $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. The dominant eigenvalue satisfies the relation $X^3 - X^2 - X - 1 = 0$.

The standard example of a Rauzy fractal is given by the first substitution σ_1, it was first studied by Rauzy [17]. It is a topological disc [16], simply connected. It is a well-known fact that the second fractal is not simply connected [21], compare Figure 1.

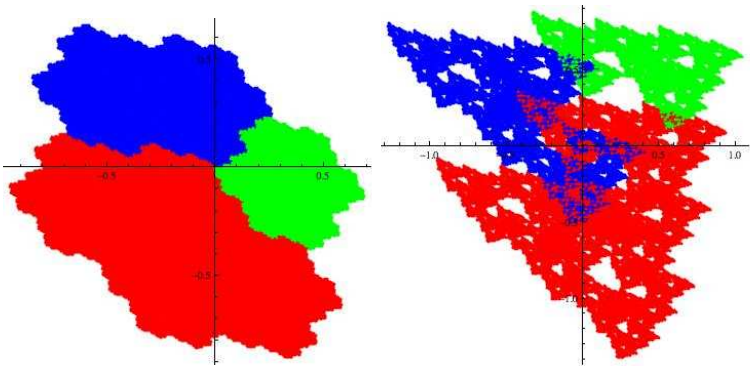

Figure 1. The Rauzy fractals of σ_1 and σ_2

In [23] Victor Sirvent studies the common point of the broken lines of these two Tribonacci substitutions. He defines a dynamical system associated with the product of the prefix automata associated with each one of these substitutions. He shows its self-similar structure and uses it to study the topological and dynamical properties of its geometric realization in the plane and on the two-dimensional torus. In this case the topological structure of the realized symbolic space is a dendrite. The realized dynamic is a domain exchange map on the dendrite (see Figure 2). In [24], Bernd Sing and Victor Sirvent extend this result and study a sequence of dynamical systems defined on sets $\mathcal{F}_k$, a part from the common dynamics of flipped irreducible Pisot substitutions with the same incidence matrix. This common dynamics $\mathcal{F}_k$, is given through the family of the product automata $(\mathcal{U}_1 \times \mathcal{U}_2)(k)$ of the prefix automata associated with the substitution $(\sigma_1)^k$ and $(\sigma_2)^k$ for $k \geq 1$, with the property $\mathcal{F}_k \subset \mathcal{F}_p$ if and only if k divides p, see [24]. They study the adic system associated with the substitutions σ_1 and σ_2. Since the adic systems considered here have geometric realizations given by solutions to graph-directed iterated functions systems, Sing and Sirvent study the topological and measure-theoretic properties of the solution of those iterated functions systems which describe the common dynamics. They show that these sets $\mathcal{F}_k$ have zero Lebesque measure. They also prove that the closure of the union of all the sets $\mathcal{F}_k$ for all $k \geq 1$ is contained in the intersection of the two Rauzy fractals $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ associated with σ_1 and σ_2. See [24, Theorem 1].

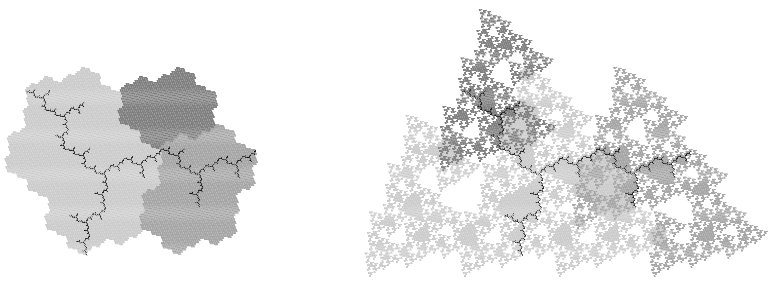

Figure 2. Geometric realization of the first common dynamics $\mathcal{F}_1$.

In this paper we study the common dynamics of two unimodular irreducible Pisot substitutions σ_1 and σ_2 having the same incidence matrix M.

We first prove that if the origin is an inner point of the Rauzy fractal associated with the substitution σ_1, then the intersection has nonempty interior and has a positive measure.

Proposition 1. *Let σ_1 and σ_2 be two unimodular irreducible Pisot substitution with the same incidence matrix. We denote by $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ their two associated Rauzy fractals. Suppose that* 0 *is an inner point to $\mathcal{R}_{\sigma_1}$, then the intersection of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ has nonempty interior and has a positive measure.*

We study the closure of the intersection of the interior of two Rauzy fractals. This set is obtained by taking the closure of the projection of the common points from the two stepped lines associated with the fixed points of σ_1 and σ_2 respectively. We then prove, under the additional assumption that σ_1 verifies the Pisot conjecture, that these common points from the two stepped lines can be obtained as a fixed point of a new substitution Σ on a different alphabet.

By projecting these common points onto the stable space of the matrix M and taking the closure of this set, we obtain a new Rauzy fractal associated with the substitution Σ. This means that the closure of the intersection of the interior of two Rauzy fractals is again a Rauzy fractal.

The Pisot conjecture states that all substitutive systems associated with unimodular irreducible Pisot substitution have pure discrete spectra. An equivalent definition is that the Rauzy fractal of an unimodular irreducible Pisot substitution generates a lattice tiling of the contractive plane of M_σ. It is equivalent to the fact that the associated substitutive dynamical system is measure-theoretically isomorphic to a toral translation. There exists a variety of conditions which imply the Pisot conjecture, we state for unimodular irreducible Pisot substitution a finiteness property analogues to the well-known (F) property in beta-numeration, which is a sufficient condition to get a tiling. In particular, the finiteness property (F) is equivalent to the fact that 0 is an exclusive inner point of the Rauzy fractal. Notice that an exclusive inner point to a Rauzy fractal is a point from one subtile which does not belong to any other tile from the multiple tiling (for more details see [21, chapter 4]).

Definition 1.1. We say that a set is substitutive if it is the closure of the projection of the vertices of a stepped line associated with a substitution Σ on a contracting space of the incidence matrix of Σ.

Remark 1.1. A substitutive set can be expressed as the attractor of some directed iterated functional system (GIFS).

The main result of this paper is the following:

Theorem 1.1. *Let σ_1 and σ_2 be two unimodular irreducible Pisot substitutions with the same incidence matrix. Let $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ be the two associated Rauzy fractals; suppose that 0 is an inner point of $\mathcal{R}_{\sigma_1}$. Then the closure of the intersection of the interiors of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ has non-empty interior, and it is a substitutive set. There is a terminating algorithm to obtain the substitution for the intersection.*

Remark 1.2. The fact that 0 is an inner point to the Rauzy fractal implies the-at the substitution satisfies the Pisot conjecture, see [21].

In Section 2, we introduce the notations that we will use in the sequel. We explain the projection method to obtain the Rauzy fractal associated with an unimodular Pisot substitution. In section 3 we recall some important properties of Rauzy fractals. A Rauzy fractal is the closure of its interior in all cases. Using a theorem of Lagarias and Wang, Sirvent and Wang prove in [25] that the Rauzy fractal is the closure of its interior in the irreducible case. Ei, Ito and Rao extend this result to the reducible case [11]. In Section 4, we prove at first that if the origin is an inner point to the Rauzy fractal of σ_1, then the intersection of the interior of the two associated Rauzy fractals is nonempty and has positive measure. We then generalize the result of Sirvent and Sing and we characterize the common dynamics of two Rauzy fractals with the same incidence matrix. We obtain a method to characterize all the common points of two stepped lines associated with σ_1 and σ_2 respectively. We prove that if σ_1 satisfies the Pisot conjecture, then the closure of the intersection of the interior of the two Rauzy fractals associated with σ_1 and σ_2 is a Rauzy fractal. We define an algorithm to obtain the substitution associated with the common points. In Section 5 we give a few examples and make some remarks.

2. Substitutions and Rauzy fractals

2.1. General setting. Let $\mathcal{A} := \{a_1, \dots, a_d\}$ be a finite set of cardinality d called alphabet. The free monoid $\mathcal{A}^*$ on the alphabet $\mathcal{A}$ with empty word ε is defined as the set of finite words on the alphabet $\mathcal{A}$, that is $\mathcal{A}^* := \bigcup_{k\in\mathbb{N}} \mathcal{A}^k$, endowed with the concatenation as binary operation. We denote by $\mathcal{A}^{\mathbb{N}}$ the set of infinite sequence on $\mathcal{A}$. The topology of $\mathcal{A}^{\mathbb{N}}$ is the product topology of discrete topology on each copy of $\mathcal{A}$.

The length of a word $w \in \mathcal{A}^n$ with $n \in \mathbb{N}$ is defined as $|w| = n$. For any letter $a \in \mathcal{A}$, we define the number of occurrences of a in $w = w_1 w_2 \dots w_{n-1} w_n$ by $|w|_a = \sharp\{i | w_i = a\}$.

Let $l : \mathcal{A}^* \to \mathbb{Z}^d : w \mapsto (|w|_a)_{a\in\mathcal{A}} \in \mathbb{Z}^d$ be the natural homomorphism obtained by abelianization of the free monoid, called the abelianization map.

A substitution over the alphabet $\mathcal{A}$ is an endomorphism of the free monoid $\mathcal{A}^*$ such that the image of each letter of $\mathcal{A}$ is a nonempty word.
A substitution σ is primitive if there exists an integer k such that, for each pair $(a, b) \in \mathcal{A}^2$, $|\sigma^k(a)|_b > 0$. We will always suppose that the substitution is primitive, this implies that for all letter $j \in \mathcal{A}$ the length of the successive iterations $\sigma^k(j)$ tends to infinity.

A substitution σ naturally extends to the set $\mathcal{A}^{\mathbb{N}}$ of infinite sequences by setting

$$\sigma(u_0 u_1 u_2 \dots) = \sigma(u_0)\sigma(u_1)\sigma(u_2)\dots$$

We associate to every substitution σ its incidence matrix M which is the $d\times d$ matrix obtained by abelianization, i.e. $M_{i,j} = |\sigma(j)|_i$. We have that $l(\sigma(w)) = Ml(w)$ for all $w \in \mathcal{A}^*$.

Remark 2.1. The incidence matrix of a primitive substitution is a primitive matrix, so by the Perron-Frobenius theorem, it has a simple real positive dominant eigenvalue β.

2.2. Rauzy fractals.

Definition 2.1. A Pisot number is an algebraic integer $\beta > 1$ such that each Galois conjugate $\beta^{(i)}$ of β satisfies $|\beta^{(i)}| < 1$.

We say that σ is an irreducible Pisot substitution if there exists one eigenvalue of M which is strictly greater than 1 and all other eigenvalues are strictly less than 1 in modulus. An equivalent definition is that the largest eigenvalue is a Pisot number, and the characteristic polynomial is irreducible. A substitution is unimodular if the determinant of its incidence matrix equals ± 1.

One can prove that any irreducible Pisot substitution is a primitive substitution (see [14]).

Remark 2.2. Note that there exist substitutions whose largest eigenvalue is Pisot but whose incidence matrix has eigenvalues that are not conjugate to the dominant eigenvalue. An example is $1 \to 12$, $2 \to 3$, $3 \to 4$, $4 \to 5$, $5 \to 1$. The characteristic polynomial is reducible. Such substitutions are called Pisot reducible.

Definition 2.2. Let σ be a substitution and $u \in \mathcal{A}^{\mathbb{N}}$, u is a fixed point of σ if $\sigma(u) = u$. The infinite word u is a periodic point of σ if there exists $k \in \mathbb{N}$ such that $\sigma^k(u) = u$.

Let σ be a primitive substitution, then there exists a finite number of periodic points (see [9]). We associate to any periodic point u of the substitution a symbolic dynamical system (Ω_u, S) where S is the shift map on $\mathcal{A}^{\mathbb{N}}$ given by $S(a_0a_1\ldots) = a_1a_2\ldots$ and Ω_u is the closure of $\{S^m(u) : m \geq 0\}$ in $\mathcal{A}^{\mathbb{N}}$.

Remark 2.3. If σ is a primitive substitution then the symbolic dynamical system (Ω_u, S) does not depend on u; we denote it by (Ω_σ, S).

In the word of substitutions, geometrical objects appeared in 1982 in the work of Rauzy [17]. The motivation of Rauzy was to build a domain exchange in $\mathbb{R}^2$ that generalizes the theory of interval exchange transformations. Thurston [22] introduced the same geometrical object in the context of numeration systems in non-integer bases.

To build a Rauzy fractal we restrict to the case of a unit Pisot substitution. We will use the projection method to obtain the Rauzy fractal.

Definition 2.3. A stepped line $L = (x_n)$ in $\mathbb{R}^d$ is a sequence (it could be finite or infinite) of points in $\mathbb{R}^d$ such that $x_{n+1} - x_n$ belongs to a finite set.
A canonical stepped line is a stepped line such that $x_0 = 0$ and for all $n \geq 0$, $x_{n+1} - x_n$ belongs to the canonical basis of $\mathbb{R}^d$.

Using the abelianization map, to any finite or infinite word W, we can associate a canonical stepped line in $\mathbb{R}^d$ as the sequence $(l(V_n))$, where V_n are the prefixes of length n of W.

We introduce a suitable decomposition of the space. We denote m the algebraic degree of the Pisot number β; one has $m \leq d$, since the characteristic polynomial of M may be reducible. We denote E_s, the beta-contracting space of the matrix M generated by the eigenspaces associated to the beta-conjugates. Let E_u be the beta-expanding line of M, i.e., the real line generated by the beta-eigenvector u_β. Let E_n be the invariant space of M that satisfies $\mathbb{R}^d = E_s \oplus E_u \oplus E_n$. It is trivial if and only if the substitution is irreducible.

Let $\pi_s : \mathbb{R}^d \to E_s$ be the linear projection on the contracting space, along $E_u \oplus E_n$, according to the natural decomposition $\mathbb{R}^d = E_s \oplus E_u \oplus E_n$.

2.3. Definition of the Rauzy fractal. An interesting property of the canonical stepped line associated with a periodic point of irreducible Pisot substitution is that it remains within bounded distance from the expanding direction given by the right Perron-Frobenius eigenvector of M. In the reducible case, the discrete line may have other expanding directions, but the projection of the discrete line by π_s still provides a bounded set; for more details we refer to [11].

Definition 2.4. Let σ be a primitive unimodular Pisot substitution with dominant eigenvalue β. The Rauzy fractal of σ is the closure of the projection of the vertices of the canonical stepped line associated with any periodic point $u = (u_k)_{k\in\mathbb{N}}$ of σ on the beta-contracting space E_s, i.e.,

$$\mathcal{R}_\sigma := \overline{\{\pi_s(l(u_0 \dots u_{k-1})),\ k \in \mathbb{N}\}}.$$

For each $i \in \mathcal{A}$ the subtiles of the central tile $\mathcal{R}_\sigma$ are naturally defined, depending on the letter associated with the vertex of the stepped line that is projected. On these sets, for $i \in \mathcal{A}$:

$$\mathcal{R}_\sigma(i) := \overline{\{\pi_s(l(u_0 \dots u_{k-1}),\ k \in \mathbb{N},\ u_k = i\}}.$$

Remark 2.4. It follows from the primitivity of the substitution σ that the definition of $\mathcal{R}_\sigma$ and $\mathcal{R}_\sigma(i)$ $(i \in \mathcal{A})$ does not depend on the choice of the periodic point $u \in \mathcal{A}^{\mathbb{N}}$ see [2].

We define the subgroup L of $\mathbb{Z}^d$ as:

$$L = \left\{ \sum_{i=1}^{d} n_i e_i : \sum_{i=1}^{d} n_i = 0,\ n_i \in \mathbb{Z} \right\}$$

Let Γ be the projection of L on the stable space, i.e., $\Gamma = \pi_s(L)$. In the irreducible case, the translation by Γ of the Rauzy fractal covers the stable space. Hence the Rauzy fractal has positive measure. The projection from the orbit of the periodic point to the Rauzy fractal extends by continuity to all of Ω_u; if we take the quotient by Γ, this projection gives a semi conjugacy between (Ω_u, S) and the translation by $\pi_s(e_i)$ on $\mathcal{R}_\sigma/\Gamma$, where e_i is any vector in the canonical base.

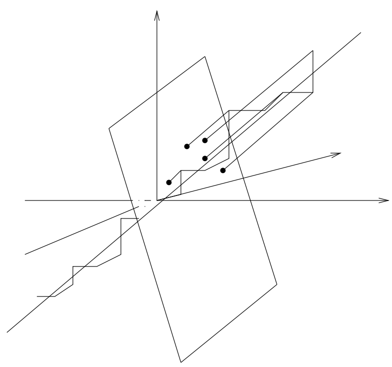

Figure 3. The projection method to get the Rauzy fractal.

3. Properties of Rauzy fractals

The objective of this section is to recall some results about Rauzy fractals associated with Pisot substitutions. An important property of a Rauzy fractal is that it is the closure of its interior. In [25] V. Sirvent and Y. Wang prove in the irreducible case that the Rauzy fractal has positive Lebesque measure [25, Proposition 2.8] and it is the closure of its interior. In the reducible case

H. Ei, S. Ito and H. Rao, prove in [11] that the Rauzy fractal associated to a reducible Pisot substitution is non empty, it is the closure of its interior and $\partial\mathcal{R}_\sigma(i)$ has Lebesgue measure 0, where $\partial\mathcal{R}_\sigma(i)$ denotes the boundary of $\mathcal{R}_\sigma(i)$ see [11, Theorem 1.6].

In both cases, the authors use a theorem of Jeffrey C. Lagarias and Yang Wang, proved in their paper Substitution Delone Sets [13]. For the sake of completeness, we briefly recall the setting of their paper.

Substitution Delone set families are families of Delone sets $\chi = (X_1, \ldots, X_d)$ which satisfy the inflation functional equation.

Inflation functional equation. The family $(X_1, \ldots, X_d)$, $X_i \subset \mathbb{R}^d$, satisfy the system of equations

$$X_i = \bigvee_{j=1}^{d} (A(X_j) + \mathcal{D}_{ij}), \quad 1 \le i \le d,$$

Where A is an expanding matrix, i.e. all of the eigenvalues of A fall outside the unit circle. The $\mathcal{D}_{ij}$ are finite sets of vectors in $\mathbb{R}^d$ and $\bigvee$ denotes union that counts multiplicity. In [13], Lagarias and Wang characterizes families $\chi = (X_1, \ldots, X_d)$ that satisfy an inflation functional equation, in which X_i is a multiset whose underlying set is discrete. Then they study the inflation functional equation for such solution χ to exist.

The subdivision matrix associated is $B = [\sharp\mathcal{D}_{ji}]_{1\le i,j\le d}$, where $\sharp\mathcal{D}_{ji}$ denotes the cardinality. Let us denote by $\rho(B)$ the modulus of the maximal eigenvalue of B. A multiset X is weakly uniformly discrete if there is a positive radius r and a finite constant $m \ge 1$ such that each ball of radius r contains at most m point of X, counting multiplicities.

A necessary condition for the inflation functional equation with primitive subdivision matrix to have a solution $\chi = (X_1, \ldots, X_n)$ with some X_i weakly uniformly discrete is that

$$\rho(B) \le |\det A|.$$

Multi-tile functional equation. The family of compact set $(\mathcal{R}_1, \ldots, \mathcal{R}_d)$ satisfy the system of equations

$$A(\mathcal{R}_i) = \bigcup_{j=1}^{d} (\mathcal{R}_j + \mathcal{D}_{ji})$$

A necessary condition for the multi-tile functional equation with primitive subdivision matrix to have $\tau = (\mathcal{R}_1, \ldots, \mathcal{R}_d)$ with some T_i with positive Lebesgue measure is that

$$\rho(B) \ge |\det A|.$$

The following theorem is a partial result of Theorem 5.1 in [13]

Theorem 3.1. *Consider an inflation functional equation that has a primitive subdivision matrix* B^T *such that*

$$\rho(B) = |\det A|.$$

If there exist a family of Delone sets $(X_1, \dots, X_d)$ *which is solution of the inflation function equation, then*

(i) *the unique compact solution* $(\mathcal{R}_1, \dots, \mathcal{R}_d)$ *of the multi-tile functional equation consists of sets* $\mathcal{R}_i$ *that have positive Lebesque measure,* $1 \le i \le d$.

(ii) $\mathcal{R}_i = \overline{\mathcal{R}_i^\circ}$ *and* $\partial \mathcal{R}_i$ *has Lebesque measure 0.*

Sirvent and Wang in [25], prove that the Rauzy fractal $\mathcal{R}$ associated to a fixed point of primitive, unimodular and irreducible Pisot substitution has non-empty interior and it is the closure of its interior. They use in their proof a construction of Rauzy fractal using valuation see [25]. They write the decomposition of the Rauzy fractal with its subtiles $(\mathcal{R}_1, \dots, \mathcal{R}_d)$, which is the attractor of the strongly connected graph directed IFS. They apply the theorem of Lagarias and Wang, they verify that a Rauzy fractal verifies a multi-tile functional equation, with

$$\mathcal{D}_{ij} := \{B^{-1}(\Delta([\pi(j)]_{k-1})) : (i,j) \in F_i\}.$$

Where $\Delta(U) = [E_1(U), \dots, E_r(U)]$ with E_i is the valuation map $E_i : \mathcal{A}^* \to \mathbb{C}$ having the property $E(UV) = E(U) + E(V)$ and $E(\sigma(U)) = wE(U)$, with w is a constant. It is shown in Holton and Zamboni [12] that in the valuation the constant w must be an eigenvalue of the incidence matrix M, and there exists an w-eigenvector $v = [v_1, \dots, v_n]^T$ such that $E(U) = \sum_{i=1}^n |U|_i v_i$ for all $U \in \mathcal{A}^*$ (for more detail see [25]). They prove that all $\mathcal{D}_{ij}^m$ are ϵ_0-separated.

In the reducible case, Ei, Ito and Rao apply the theorem of Lagarias and Wang to subtiles of Rauzy fractals, using a construction with stepped surfaces and dual substitutions. A natural decomposition of the image of letter can be seen as prefix-suffix decomposition, this means $\sigma(i)$ can be written as $\sigma(i) = P_k^{(i)} W_k^{(i)} S_k^{(i)}$, where $P_k^{(i)}$ and $S_k^{(i)}$ are prefix and suffix of $\sigma(i)$. Then we obtain the following theorem:

Theorem 3.2. *Let* σ *be an unimodular Pisot substitution and* M *its incidence matrix. Then the subtiles* $\{\mathcal{R}_i\}_{i=1}^d$ *are compact and satisfy the following set equations*

$$M^{-1}\mathcal{R}_i = \left(\bigcup_{j=1}^{d} \bigcup_{W_k^{(j)}=i} \mathcal{R}_j + M^{-1}\pi_s(f(P_k^{(j)})) \right).$$

The action of M^{-1} on the stable space E_s is a linear map, and hence it is equivalent to a $(m-1)\times(m-1)$ real matrix, with m is the degree of the Pisot number (we are in the reducible case). We denote this matrix by A. Since M is an unimodular Pisot matrix, we have that A is expanding and $|\det A| = \lambda$ is the Perron–Frobenius eigenvalue of M. Ei, Ito and Rao define

$$\mathcal{D}_{ji} := \{M^{-1}\pi_s(f(P_k^{(j)}))\}, \quad 1 \leq i,\ j \leq d.$$

where π_s is the linear projection in the contracting plane E_s, along $E_u \oplus E_n$. The set $\mathcal{D}_{ji}$ are completely determined by the substitution σ. Then $\mathcal{R}_1, \dots, \mathcal{R}_d$ are the unique invariant set of the multi-tile functional equation. With stepped surfaces, authors define in [11] a family $\{X_1, \dots, X_d\}$ as solution of the inflation functional equation, for more details see [11]. By the theorem of Lagarias and Wang we obtain immediately the following theorem in the case of unimodular Pisot substitution:

Theorem 3.3. *Let σ be an unimodular Pisot substitution (reducible or irreducible), then*

(i) *The interiors of the subtiles $\mathcal{R}_i$ are not empty.*

(ii) $\mathcal{R}_i = \overline{\mathcal{R}_i^\circ}$ *and ∂R_i has Lebesgue measure* 0.

4. Intersection of Rauzy fractals

Let σ_1 and σ_2 be two unimodular irreducible Pisot substitutions with the same incidence matrix. We consider their respective Rauzy fractals $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$. The intersection of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ is non-empty since it contains 0, and it is a compact set as intersection of two compacts sets.

Proposition 2. *Let σ_1 and σ_2 be two substitutions with the same incidence matrix. We consider $\mathcal{R}_{\sigma_1}$ (resp. $\mathcal{R}_{\sigma_2}$) the Rauzy fractal associated with σ_1 (resp. σ_2). Then the boundary of the intersection of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ is included in the union of the boundaries of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ and has zero measure.*

Proof. Trivial. □

4.1. Intersection and positive measure. We suppose that 0 is an inner point to $\mathcal{R}_{\sigma_1}$. We note $\mathcal{E}$ the closure of the intersection of the interior of $\mathcal{R}_{\sigma_1}$ and the interior of $\mathcal{R}_{\sigma_2}$.

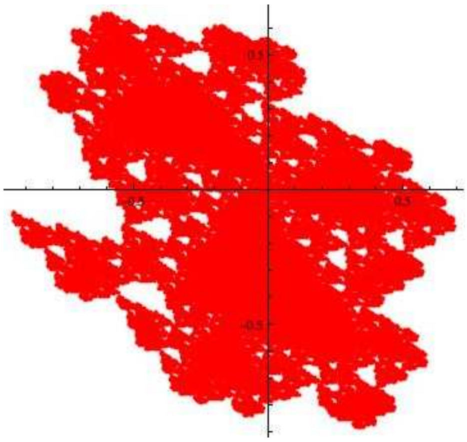

Figure 4. Sets of common points of the Rauzy fractals of Tribonacci and the flipped Tribonacci substitutions.

Proposition 3. *Let σ_1 and σ_2 be two unimodular irreducible Pisot substitution with the same incidence matrix. We consider $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ their associated Rauzy fractal. We suppose that 0 is an inner point to $\mathcal{R}_{\sigma_1}$. Then the set $\mathcal{E}$ has non-empty interior and strictly positive Lebesgue measure.*

Proof. Because 0 is an inner point of $\mathcal{R}_{\sigma_1}$. Then there exists an open set U such that $0 \in U \subset \mathcal{R}_1$. The Rauzy fractal is the closure of its interior [21] and 0 is a point of $\mathcal{R}_{\sigma_2}$, hence there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ from the interior of $\mathcal{R}_{\sigma_2}$ that converges to 0. Thus there exist open sets V_n such that $x_n \in V_n \subset \mathcal{R}_{\sigma_2}$. Since (x_n) converges to 0, there exists $N \in \mathbb{N}$ such that $x_N \in U$. The open set $U \cap V_N$ is non-empty and $U \cap V_N \subset \mathcal{R}_{\sigma_1} \cap \mathcal{R}_{\sigma_2}$. This implies that $\mathcal{E}$ contains a non-empty open set, hence it has strictly positive Lebesgue measure. □

4.2. The main result: Morphism generating the common points of two Pisot substitutions with the same incidence matrix. Let σ_1 and σ_2 be two unimodular irreducible Pisot substitution with the same incidence matrix. We consider $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ their two associated Rauzy fractal. We suppose that σ_1 satisfies the Pisot conjecture, and 0 is an inner point to $\mathcal{R}_{\sigma_1}$. We denote by $\mathcal{E}$ the closure of the intersection of the interior of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$.

Let (Ω_{σ_1}, S) (resp.(Ω_{σ_2}, S)) be the symbolic dynamical systems associated with σ_1 (resp. σ_2). The projection of the stepped line associated with a fixed point of an unimodular irreducible substitution, can be seen as a map from the orbit $\{S^n(u)\}$ of the fixed point to the Rauzy fractal. This map can be extended by continuity to a map $\pi : \Omega_u \to \mathcal{R}$. We consider π_1 (resp. π_2) the resulting map from the symbolic dynamical system (Ω_{σ_1}, S) (resp. (Ω_{σ_2}, S)) onto the Rauzy fractal $\mathcal{R}_{\sigma_1}$ (resp. $\mathcal{R}_{\sigma_2}$).

In this section we will prove that $\mathcal{E}$ can be generated by a substitution which is obtained via an algorithm that generates the common point of the interior of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$.

Definition 4.1. For a dynamical system (X, T) if A is a subset of X, and $x \in A$, we define the first return time of x as $n_x = \inf\{n \in \mathbb{N}^* | T^n(x) \in A\}$ (it is infinite if the orbit of x does not come back to A). We define the induced map of T on A (or first return map) as the map $x \mapsto T^{n_x}(x)$ if the first return time is finite, otherwise it is not defined. We denote this map by T_A.

Definition 4.2. A sequence $u = (u_n)$ is minimal (or uniformly recurrent) if every word occurring in u occurs in an infinite number of positions with bounded gaps, that is, if for every factor W, there exists s such that for every n, W is a factor of $u_n \ldots u_{n+s-1}$.

Lemma 4.1. *Let σ be an irreducible Pisot substitution, and $\mathcal{R}_\sigma$ its associated Rauzy fractal. If σ satisfies the Pisot conjecture, then $\mathcal{R}_\sigma$ is a fundamental domain of E_s for the projection of Γ on the stable space up to a set of measure zero.*

Proof. The proof is given in [21, chapter 4]. □

Lemma 4.2. *Suppose that σ_1 verifies the Pisot conjecture and 0 is an inner point to its associated Rauzy fractal. Let W be a non-empty open set in $\mathcal{E}$, define $V_1 := \pi_1^{-1}(W) \subset \Omega_{\sigma_1}$ and $V_2 := \pi_2^{-1}(W) \subset \Omega_{\sigma_2}$. For any $y \in W$, such that $y = \pi_1(v_1) = \pi_2(v_2)$, any return time of v_2 to V_2 is a return time of v_1 to V_1.*

Proof. We consider $v_1 \in V_1$ and $v_2 \in V_2$ such that $\pi_1(v_1) = \pi_2(v_2)$. Let n be a return time of v_2. By definition, $\pi_1(S^n v_1) = \pi_1(v_1) - \pi_s(l(P_n))$ where P_n is the prefix of length n of v_1. Similarly, we have $\pi_2(S^n v_2) = \pi_2(v_2) - \pi_s(l(P'_n))$, where P'_n is the prefix of length n of v_2. Hence $\pi_1(S^n v_1) = \pi_2(S^n v_2) + \pi_s[l(P'_n) - l(P_n)]$. This means that there exists $\gamma \in \Gamma$ such that $\pi_1(S^n v_1) = \pi_2(S^n v_2) + \pi_s(\gamma)$.

By hypothesis $S^n v_2 \in V_2$ and $\pi_2(S^n v_2)$ is an inner point of $\mathcal{E}$. This implies that $\pi_2(S^n v_2)$ is an inner point of $\mathcal{R}_{\sigma_1}$. Since $\pi_1(S^n v_1) \in \mathcal{R}_{\sigma_1}$, and by hypothesis $\mathcal{R}_{\sigma_1}$ is a fundamental domain, this means that the interior of $\mathcal{R}_{\sigma_1}$ cannot meet $\mathcal{R}_{\sigma_1} + \pi_s(\gamma)$ where $\gamma \in \Gamma$, unless $\pi_s(\gamma) = 0$. So we have $\pi_1(S^n v_1) = \pi_2(S^n v_2)$. Hence if n is a first return time to V_2, it is a return time to V_1. □

Definition 4.3. Let U and V be two finite words, we say that $\binom{U}{V}$ is balanced pair if $l(U) = l(V)$, where l is the abelianization map from $\mathcal{A}^*$ in $\mathbb{Z}^d$.

Definition 4.4. A minimal balanced pair is a balanced pair, such that for every strict prefix U_k, V_k of U and V, respectively, of length k, $l(U_k) \neq l(V_k)$.

Lemma 4.3. *Let σ_1 and σ_2 be two unimodular irreducible Pisot substitutions with the same incidence matrix. Let $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ be their two associated Rauzy fractals; suppose that σ_1 satisfies the Pisot conjecture and 0 is an inner point of $\mathcal{R}_{\sigma_1}$. Let u and v be two fixed points of σ_1 and σ_2 respectively. There exists a finite set E of minimal balanced pairs, $E = \left\{ \binom{U_1}{V_1}, \ldots, \binom{U_p}{V_p} \right\}$, such that the double sequence $\binom{u}{v}$ can be decomposed with elements from E.*

Proof. By definition, $\pi_1(u) = \pi_2(v) = 0$. We consider the open set $\pi_2^{-1}(\mathcal{E}) \in \Omega_{\sigma_2}$. Since (Ω_{σ_2}, S) is a minimal system any point has a return time to $\pi_2^{-1}(\mathcal{E})$. There is a constant which bounds the return time of any point to $\pi_2^{-1}(\mathcal{E})$.

Let n_1 be the first return time, From Lemma 4.2, n_1 is a return time of u to $\pi_1^{-1}(\mathcal{E})$, which implies that $S^{n_1}(u) \in \pi_1^{-1}(\mathcal{E})$. We obtain two prefixes U and V of u and v respectively such that $l(U) = l(V)$. We obtain a first balanced pair $\binom{U}{V}$.

The same argument show that there is an infinite sequence of common return time to the intersection and the difference between two consecutive return time to the intersection is bounded. Hence the double sequence $\binom{u}{v}$ can be decomposed in minimal balanced pairs of bounded length.

We can decompose the first balanced pair $\binom{U}{V}$ into minimal balanced pairs. U and V have bounded length, then the minimal balanced pairs obtained after decomposition are finite and have bounded length.
Consider the image of each of these minimal pairs by σ_1 and σ_2. Each minimal balanced pair will appear, we consider the image of each new minimal balanced pair by σ_1 and σ_2 and iterate. All the minimal balanced pairs will appear after bounded finite time. □

Theorem 4.4. *Let σ_1 and σ_2 be two unimodular irreducible Pisot substitutions with the same incidence matrix. Let $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ be their two associated Rauzy fractals; suppose that 0 is an inner point of $\mathcal{R}_{\sigma_1}$. We denote by $\mathcal{E}$ the closure of the intersection of the interiors of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$. Then $\mathcal{E}$ has non-empty interior, and it is a substitutive set associated to a Pisot substitution Σ on the alphabet of minimal balanced pairs.*

Proof. Let us prove now that these common points can be obtained as the projection of a fixed point of a new substitution defined on the set of the minimal balanced pairs. We give an algorithm to obtain this morphism. From Proposition 4.3, there exist two finite words W_1 and W_2 being prefixes of u and v respectively, such that $l(W_1) = l(W_2)$. We can decompose the balanced pair $\binom{W_1}{W_2}$ into minimal balanced pairs. Let $\binom{v_1}{v_2}$ be the first minimal balanced pair. Then, $\begin{pmatrix}\sigma_1(v_1)\\ \sigma_2(v_2)\end{pmatrix}$ is a balanced pair because σ_1 and σ_2 have the same matrix. We consider the decomposition of this balanced pair $\begin{pmatrix}\sigma_1(v_1)\\ \sigma_2(v_2)\end{pmatrix}$ into minimal balanced pairs. This means that we can write $\begin{pmatrix}\sigma_1(v_1)\\ \sigma_2(v_2)\end{pmatrix} = \binom{u_1 \ldots u_k}{w_1 \ldots w_k}$ where $l(u_1) = l(w_1), \ldots, l(u_k) = l(w_k)$.

Since the set of common return times is bounded, by iteration with σ_1 and σ_2 we obtain in bounded finite time the set of all minimal balanced pairs. We can define the substitution Σ over the finite set of minimal balanced pairs:

$$\Sigma : \begin{pmatrix}U\\ V\end{pmatrix} \longmapsto \begin{pmatrix}\sigma_1(U)\\ \sigma_2(V)\end{pmatrix}.$$

The set $\mathcal{E}$ is obtained as the closure of the projection of the stepped line associated with the fixed point of Σ on the stable space associated with the initial substitution σ_1. The interior is clearly substitutive with respect to the substitution Σ. $\square$

Lemma 4.5. *The substitution Σ is a Pisot substitution.*

Proof. Let λ be the Perron Frobenius eigenvalue of the incidence matrix of σ_1. Let u and v be the two fixed points of σ_1 and σ_2 respectively. We consider the double sequence $\binom{u}{v}$, the fixed point of Σ which begin with A.

We can define two lengths for a prefix of the double sequence $\binom{u}{v}$. One associated to the alphabet E of minimal balanced pair, we denote it $|.|_E$. The second one, we can see the double sequence $\binom{u}{v}$ as a sequence on the initial alphabet of the substitution σ_1 and σ_2, $\mathcal{A} \times \mathcal{A}$ we denote it $|.|_{\mathcal{A}}$.

Let N be the maximal length of balanced pairs, and $c = \frac{1}{N}$, we obtain

$$c|\Sigma^n(A)|_{\mathcal{A}} \leq |\Sigma^n(A)|_E \leq |\Sigma^n(A)|_{\mathcal{A}}$$

The length of $|\Sigma^n(A)|_{\mathcal{A}}$ has an exponential growth, taking the logarithm we obtain

$$\frac{\ln C}{n} + \frac{\ln|\Sigma^n(A)|_{\mathcal{A}}}{n} \leq \frac{\ln|\Sigma^n(A)|_E}{n} \leq \frac{|\Sigma^n(A)|_{\mathcal{A}}}{n}$$

We consider the projection of the minimal balanced pair A on its first coordinate, we obtain $A = \binom{a_1}{a_2}$. We know that $|\Sigma^n(A)|_{\mathcal{A}} = |\sigma_1^n(a_1)| = |\sigma_2^n(a_2)|$ and $|\sigma_1^n(a_1)|$ grows with λ^n. We take the limit when $n \to +\infty$, we deduce

$$\lim_{n\to\infty} \frac{\ln|\Sigma^n(a)|_E}{n} = \lambda$$

where λ is the Perron Frobenius eigenvalue of the incidence matrix of σ_1. We conclude that Σ is a Pisot substitution. We remark that Σ is reducible in almost cases. $\square$

5. Examples

5.1. Algorithm defining the intersection of two Rauzy fractals. We can deduce an effective algorithm to obtain the substitution Σ of the common points of the two substitutions σ_1 and σ_2 with the same incidence matrix. The substitution Σ is defined over the alphabet of minimal balanced pairs. At first we take the fixed points u and v of each substitutions σ_1 and σ_2. There is always a power σ_i^k and at least one symbol $a \in \mathcal{A}$ such that $\sigma_i^k(a)$ starts with a. This means there is a power σ_i^k with at least one fixed point.

Let $\binom{U}{V}$ be the first minimal balanced pair. In the case where the fixed points u and v begin with the same letter a, we obtain the first minimal pair $\binom{a}{a}$. We iterate this first minimal balanced pair with the two substitutions σ_1 and σ_2, this means that $\binom{U}{V} \to \binom{\sigma_1(U)}{\sigma_2(V)}$. The substitutions σ_1 and σ_2 have the same incidence matrix, then the pair $\binom{\sigma_1(a)}{\sigma_2(a)}$ is a balanced pair. We can decompose this new balanced pair with minimal balanced pairs. If we obtain a new minimal balanced pair we iterate it with the same procedure. And we continue this until obtaining all minimal balanced pairs. The set of minimal balanced pairs is finite and bounded, and we obtain all minimal balanced pairs after a finite time which depend on the return times of the minimal balanced pairs.

We associate to each minimal balanced pair a capital letter. We take the image of each element of the finite set of minimal balanced pairs. The substitution Σ is defined as $\Sigma : \binom{U}{V} \longmapsto \binom{\sigma_1(U)}{\sigma_2(V)}$. The balanced pair $\binom{\sigma_1(U)}{\sigma_2(V)}$ can be decomposed with minimal balanced pairs, and we can write the image of each minimal balanced pair with concatenated minimal balanced pairs.

5.1.1. *Example 1.* We consider the two substitutions τ_1 and τ_2 defined as:

$$\tau_1 : \begin{cases} a \to aba \\ b \to ab \end{cases} \quad \text{and} \quad \tau_2 : \begin{cases} a \to aab \\ b \to ba. \end{cases}$$

The Rauzy fractal of τ_2 is the closure of a countable union of disjoint intervals and the Rauzy fractal of τ_1 is an interval, see [15]. We will try to describe an algorithm to obtain the morphism of the common points of these two Rauzy fractals. In this example, the first minimal balanced pair that we can consider is the beginning of the two fixed points associated with τ_1 and τ_2 it will be $\binom{a}{a}$.
We represent the image of the first element of this pair by τ_1 and the second one by τ_2. We obtain : $\binom{a}{a} \xrightarrow{\tau_1, \tau_2} \binom{aba}{aab}$.
We denote by A the minimal balanced pair $\binom{a}{a}$ and by B the minimal balanced pair $\binom{ba}{ab}$.

Hence we obtain $A \to AB$.
The second step is to consider the same process with the new balanced pair $\binom{ba}{ab}$. We consider the image of this balanced pair with the two substitution τ_1 and τ_2, and we obtain:

$$\begin{pmatrix} ba \\ ab \end{pmatrix} \xrightarrow{\tau_1,\tau_2} \begin{pmatrix} ababa \\ aabba \end{pmatrix}.$$

We obtain an other balanced pair $\binom{b}{b}$ and we denote by C the projection over this new balanced pair. We get the image of B which is $ABCA$. We continue with this algorithm and we obtain the image of the balanced pair $\binom{b}{b}$ is the new balanced pair $\binom{ab}{ba}$. We therefore obtain that the image of the letter C is a new letter D. Finally the image of the letter D is $DAAC$. On total, we obtain an alphabet $\mathcal{B}$ on 4 letters and we can define the morphism ϕ as :

$$\Sigma : \begin{cases} A \to AB \\ B \to ABCA \\ C \to D \\ D \to DAAC. \end{cases}$$

We now consider the projection θ of the letters A, B, C, D in the sets of balanced pairs $E = \left\{ \binom{a}{a}, \binom{ba}{ab}, \binom{b}{b}, \binom{ab}{ba} \right\}$
We then get: $\begin{pmatrix} \tau_1^n(a) \\ \tau_2^n(a) \end{pmatrix} = \theta(\Sigma^n(A))$.
The morphism Σ generates all the common points of the two Rauzy fractals associated with τ_1 and τ_2.

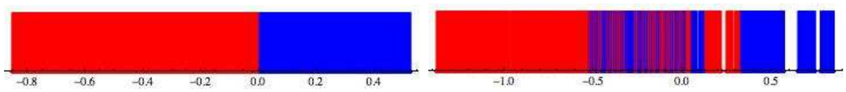

Figure 5. The Rauzy fractals of τ_1 and τ_2.

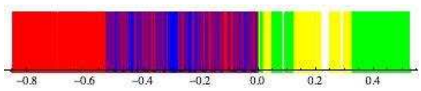

Figure 6. Common points of τ_1 and τ_2 with distinction of blocks defined with Σ

The characteristic polynomial of τ_1 is $x^2 - 3x + 1$. The characteristic polynomial associated with the substitution Σ is $(x-1)(x+1)(x^2-3x+1)$.

5.1.2. *Example 2.* For the two substitutions Tribonacci and the flipped Tribonacci, the whole procedure is more complicated (see Figure[7]). We can define the morphism Σ_1 which generates all the common points as follows:

$$\Sigma_1 : \begin{cases} A \to AB \\ B \to C \\ C \to AD \\ D \to AE \\ E \to F \\ F \to ADDGA \\ G \to AH \\ H \to ID \\ I \to ADJ \\ J \to AHK \\ K \to IDGA. \end{cases}$$

The set E of minimal balanced pairs is:

$$E = \left\{ \begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} ac \\ ca \end{pmatrix}, \begin{pmatrix} ba \\ ab \end{pmatrix}, \begin{pmatrix} cab \\ bca \end{pmatrix}, \begin{pmatrix} aabac \\ caaab \end{pmatrix}, \begin{pmatrix} cab \\ abc \end{pmatrix}, \begin{pmatrix} abac \\ bcaa \end{pmatrix}, \begin{pmatrix} abaca \\ caaab \end{pmatrix}, \begin{pmatrix} cabaab \\ ababca \end{pmatrix}, \begin{pmatrix} ababac \\ bcaaab. \end{pmatrix} \right\}$$

The characteristic polynomial of σ_1 is $x^3 - x^2 - x - 1$. The characteristic polynomial associated with the substitution Σ_1 is $(x^3 - x^2 - x - 1)(x^3 + x^2 + x - 1)$ $(x^5 - x^4 + x^3 - 2x^2 + x - 1)$.

Remark 5.1. Let σ_1 and σ_2 be the two Tribonacci substitutions see page 2. We consider U and V their two fixed points. Then the letter c do not occur in the same position in U and V.

Proof. The minimal balanced pairs represent a decomposition of the two fixed points $U$and V. We remark that in these finite minimal pairs there is no c which appears in the same position. One can then deduce that the letter c does not appear in the same position in two fixed points U and V. □

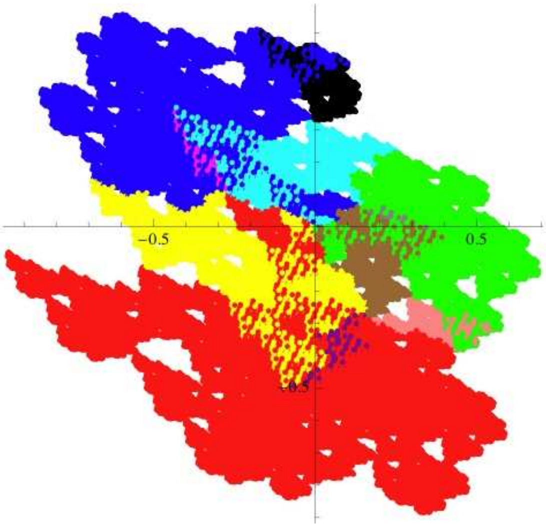

Figure 7. The sets of common points of the Tribonacci substitution and the flipped substitution. Each color stands for a different letter of $\mathcal{B}$ and shows the dynamics of the morphism Σ_1.

5.1.3. *Example 3.* Now we will consider a more general example defined as follows :

$$\delta_i^1 : \begin{cases} a \to a^i b \\ b \to a^{i-1} c \\ c \to a \end{cases} \quad \text{and} \quad \delta_i^2 : \begin{cases} a \to aba^{i-1} \\ b \to aca^{i-2} \\ c \to a \end{cases}$$

δ_i^1 and δ_i^2 have the same incidence matrix. We can define the morphism of their common points for all $i \geq 3$ as :

$$\Sigma_i : \begin{cases} A \to AB \\ B \to AC \\ C \to (AAD)^{i-1}[AAE(AAD)^i]^{i-2}AAE(AAD)^{i-1}A \\ D \to AF \\ E \to (AAD)^{i-3}A \\ F \to (AAD)^{i-1}[AAE(AAD)^i]^{i-3}AAE(AAD)^{i-1}A. \end{cases}$$

The characteristic polynomial of δ_i^1 is $x^3 - ix^2 - (i-1)x - 1$. The characteristic polynomial associated with the substitution Σ_i is $(x^3 - ix^2 - (i-1)x - 1)(x^3 + (i-1)x^2 + ix - 1)$.

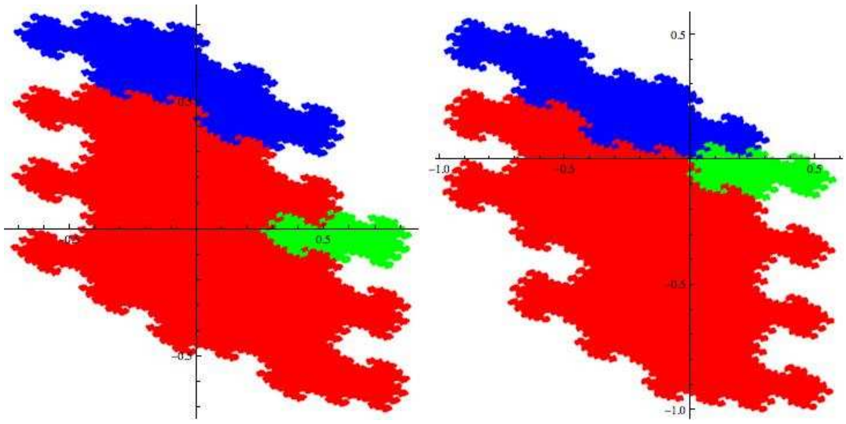

Figure 8. The Rauzy fractals of δ_3^1 and δ_3^2.

The set E_i of minimal balanced pairs is defined as:

$$E_i = \left\{ \begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} a^{i-1}b \\ ba^{i-1} \end{pmatrix}, \begin{pmatrix} a^{i-1}b(a^ib)^{i-2}a^{i-1}c \\ ca^{i-1}(ba^i)^{i-2}ba^{i-1} \end{pmatrix}, \begin{pmatrix} a^{i-2}b \\ ba^{i-2} \end{pmatrix}, \begin{pmatrix} a^{i-3}c \\ ca^{i-3} \end{pmatrix}, \begin{pmatrix} a^{i-1}b(a^ib)^{i-3}a^{i-1}c \\ ca^{i-1}(ba^i)^{i-3}ba^{i-1} \end{pmatrix} \right\}$$

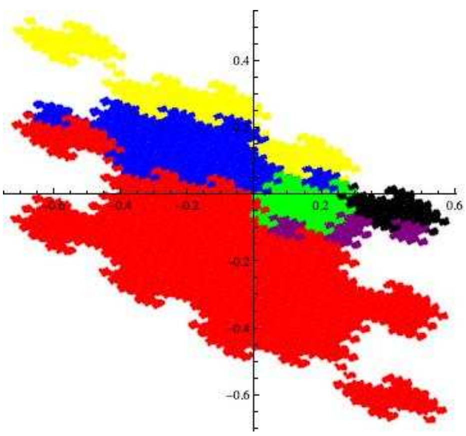

Figure 9. Sets of common points of δ_3^1 and δ_3^2.

Remark 5.2. The property 0 being an inner point cannot be removed. We take this example of substitutions with the same incidence matrix but having the intersection reduced to the origin.

We can give an example where the intersection is reduced to the origin. We consider the two substitutions χ_1 and χ_2 defined as follows:

$$\chi_1 : \begin{cases} a \to aab \\ b \to ab \end{cases} \quad \text{and} \quad \chi_2 : \begin{cases} a \to baa \\ b \to ba. \end{cases}$$

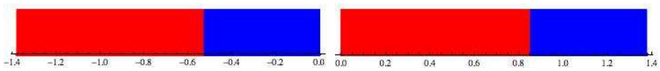

Figure 10. The Rauzy fractals of χ_1 and χ_2.

We consider u_1 and u_2 the two fixed points associated with χ_1 and χ_2 respectively. If $a.x$ is a prefix of u_1 then $b.x$ is a prefix of u_2. We use an inductive argument. For $x = a$ it is verified for $n = 1$. We suppose now that $a.x$ is a prefix of u_1 and that $b.x$ is a prefix of u_2 with $|x| = n$. Then $\chi_1(a.x) = aab\chi_1(x)$ is prefix of u_1, $\chi_2(b.x).b = ba\chi_2(x)b$ is a prefix of u_2 if and only if $\chi_1(x) = \chi_2(x)b$. Furthermore, the two letters, a and b we have:

- $x = a : b.\chi_1(a) = baab = \chi_2(a)b.$
- $x = b : b.\chi_1(b) = bab = \chi_2(b)b.$

We consider now $x = x_1 x_2 \dots x_n$ with $x_i \in \{a, b\}$. We have the following expansion:

$$\begin{aligned} b.\chi_1(x) = b.\chi_1(x_1 x_2 \dots x_n) &= b.\chi_1(x_1) \dots \chi_1(x_n) \\ &= \chi_2(x_1).b.\chi_1(x_2) \dots \chi_1(x_n) \\ &\vdots \\ &= \chi_2(x_1)\chi_2(x_2) \dots \chi_2(x_n).b \end{aligned}$$

We have therefore show that there exists an infinite word u such that $u_1 = a.u$ and $u_2 = b.u$.

Remark 5.3. We are interested to study the closure of the intersection of the interior of two Rauzy fractals associated to two unimodular irreducible Pisot substitutions with the same incidence matrix. We prove that the closure of the intersection of their interiors is a substitutive set. An important question is if we can obtain the intersection of two Rauzy fractals. We do not know if the intersection of two Rauzy fractals is equal to the closure of the intersection of their interior. It obviously contains this closure, but it might also contain an additional part with empty interior.

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