Publ. Math. Debrecen
81/1-2 (2012), 65–79
DOI: 10.5486/PMD.2012.5041

A characterization of Beckenbach families admitting discontinuous Jensen affine functions

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Abstract. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that $\mathcal{F} := \{\mathbb{R} \ni x \mapsto F(x, a, b) \in \mathbb{R} : a, b \in \mathbb{R}\}$ is a Beckenbach family. Additionally, we assume that for each $a, b \in \mathbb{R}$ the functions $\mathbb{R} \ni x \mapsto F(x, a, b) \in \mathbb{R}$ are monotonic. We show that if there exists a function which is discontinuous at some point and Jensen affine with respect to the family $\mathcal{F}$, then there exists a strictly increasing and continuous function $h : \mathbb{R} \to \mathbb{R}$ and continuous $G, H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$F(u, a, b) = h(G(a, b)u + H(a, b)), \tag{$*$}$$

for all $u, a, b \in \mathbb{R}$. As a consequence we get an independent proof of theorem of J. Matkowski. Finally, we characterize Beckenbach families of the form $(*)$.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an open interval. A family $\mathcal{F} := \{\psi : I \to \mathbb{R} : \psi - function\}$ is called a continuous Beckenbach family on I (see [1] for more details) if it consists of continuous functions and for every pair of points $(x, y), (\tilde{x}, \tilde{y}) \in I \times \mathbb{R}$ such that $x \neq \tilde{x}$ there exists exactly one function $\psi \in \mathcal{F}$ such that $\psi(x) = y$ and $\psi(\tilde{x}) = \tilde{y}$. As a special case, it is often considered a Beckenbach family determined by a function. Precisely, we call a function $F : I \times \mathbb{R}^2 \to \mathbb{R}$ a two-parameter, continuous Beckenbach family on I if F is continuous and for every

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Mathematics Subject Classification: 26B25.
Key words and phrases: Beckenbach convexity, convex function with respect to a given function, affine (Jensen affine) function, Cauchy equation.
This research has been supported by the University of Silesia Mathematics Department, Iterative Functional Equations and Real Analysis program.

$(x,y),(\tilde{x},\tilde{y}) \in I \times \mathbb{R}$ there exists exactly one $(a,b) \in \mathbb{R}^2$ such t hat

$$F(x,a,b) = y \quad \text{and} \quad F(\tilde{x},a,b) = \tilde{y}.$$

Furthermore, if for fixed $(a,b) \in \mathbb{R}^2$ the function $I \ni x \mapsto F(x,a,b) \in \mathbb{R}$ is monotonic, then the Beckenbach family will be called monotonic. It is easy to see that each function in a monotonic Beckenbach family is either strictly monotonic or constant.

Obviously, a family of affine functions, i.e.

$$\mathcal{F} := \{\mathbb{R} \ni x \mapsto ax + b \in \mathbb{R} : a, b \in \mathbb{R}\},$$

is a continuous, monotonic, two-parameter Beckenbach family.

An important generalization is a Beckenbach family of the form

$$\mathcal{F}_\alpha := \{a\alpha(\cdot) + b : a, b \in \mathbb{R}\},$$

where $\alpha : I \to \mathbb{R}$ is a strictly monotonic and continuous function. Obviously, $\mathcal{F}_\alpha$ is a continuous, monotonic, two-parameter Beckenbach family.

For a given Beckenbach family $\mathcal{F}$, we will call a function $\varphi : I \longrightarrow \mathbb{R}$ $\mathcal{F}$-Jensen affine or Jensen affine with respect to the family $\mathcal{F}$, when for fixed $x, y \in I$ we have: if

$$\begin{cases} \varphi(x) = \psi(x), \\ \varphi(y) = \psi(y), \end{cases} \tag{1}$$

for some $\psi \in \mathcal{F}$, then

$$\varphi\Big(\frac{x+y}{2}\Big) = \psi\Big(\frac{x+y}{2}\Big). \tag{2}$$

Additionally, if $F : I \times \mathbb{R}^2 \to \mathbb{R}$ is a two-parameter, continuous Beckenbach family then a function $\varphi : I \to \mathbb{R}$ will be called F-Jensen affine if it is $\mathcal{F}$-Jensen affine, where $\mathcal{F} := \{F(\cdot,a,b) : a,b \in \mathbb{R}\}$.

Now, in the paper [4] Janusz Matkowski has shown that a discontinuous at least at one point, Jensen affine function with respect to the family $\mathcal{F}_\alpha$ exists if and only if α is a homographic function. For completness, we give here a part of this Theorem.

Theorem 1.1 (see [4, Thm. 8, p. 443])**.** *Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha : I \to \mathbb{R}$ be continuous and strictly monotonic. There exists a discontinuous at least at one point, $\mathcal{F}_\alpha$-Jensen affine function if and only if, there are $p, q, r, s \in \mathbb{R}$, $ps \neq rq$, such that*

$$\alpha(x) = \frac{px + q}{rx + s}, \quad x \in I,$$

i.e., α is a homographic function.

Obviously, in the case when $I = \mathbb{R}$ the function α in the above theorem is an affine function.

Now, in this spirit prof. Janusz Matkowski asked (oral communication) about characterization of Beckenbach families having discontinuous (at least at one point) Jensen affine function. In this paper we give a partial answer to this question. Namely, we give a description of two-parameter, continuous monotonic Beckenbach families on the whole (!) real line $\mathbb{R}$, having discontinuous, Jensen affine functions with respect to this family (see Theorem 2.5). We prove that such families are of the form (3). As a consequence, we get (see Theorem 2.4) an independent proof of a part of the thesis of Theorem 1.1 (concerning the case $I = \mathbb{R}$).

Finally, in Theorem 3.1, we present a characterization of two-parameter, continuous Beckenbach families of the form (3).

2. Main results

Let $F : I \times \mathbb{R}^2 \to \mathbb{R}$ be a two-parameter Beckenbach family on I given by the formula

$$F(u, a, b) = h(G(a, b)u + H(a, b)), \tag{3}$$

for $u \in I$ and $a, b \in \mathbb{R}$, where $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and the functions $G, H : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

We call a function $\varphi : I \to \mathbb{R}$ Jensen affine if it satisfies Jensen functional equation (see [3]), i.e.

$$\varphi\Big(\frac{x+y}{2}\Big) = \frac{\varphi(x)+\varphi(y)}{2}, \quad x, y \in I. \tag{4}$$

We start with a description of F-Jensen affine function, where F is given by (3).

Theorem 2.1. *Let $F : I \times \mathbb{R}^2 \to \mathbb{R}$ be a two-parameter Beckenbach family on I of the form* (3). *A function $\tilde{\varphi} : I \to \mathbb{R}$ is F-Jensen affine if and only if there exists a function $\varphi : I \to \mathbb{R}$ satisfying equation* (4) *and such that $\tilde{\varphi} = h \circ \varphi$.*

Proof. Let $\tilde{\varphi} : I \to \mathbb{R}$ be F-Jensen affine. Fix $x_i \in I$ for $i \in \{1, 2\}$ such that

$$\tilde{\varphi}(x_i) = h(G(a, b)x_i + H(a, b)), \tag{5}$$

holds for some $a, b \in \mathbb{R}$. By F-Jensen affinity of $\tilde{\varphi}$ we have

$$\tilde{\varphi}\Big(\frac{x_1+x_2}{2}\Big) = h\Big(G(a, b)\frac{x_1+x_2}{2} + H(a, b)\Big).$$

Since h is invertible we get

$$(h^{-1}\circ\tilde{\varphi})\Big(\frac{x_1+x_2}{2}\Big)=G(a,b)\frac{x_1+x_2}{2}+H(a,b). \tag{6}$$

By (5) we have

$$\frac{(h^{-1}\circ\tilde{\varphi})(x_1)+(h^{-1}\circ\tilde{\varphi})(x_2)}{2}=G(a,b)\frac{x_1+x_2}{2}+H(a,b).$$

Combining (6) and the above we obtain

$$(h^{-1}\circ\tilde{\varphi})\Big(\frac{x_1+x_2}{2}\Big)=\frac{(h^{-1}\circ\tilde{\varphi})(x_1)+(h^{-1}\circ\tilde{\varphi})(x_2)}{2},$$

which completes the proof of the "If" part.

To deal with the "only if" again fix $x_i\in I$ for $i\in\{1,2\}$. Consider $(a,b)\in\mathbb{R}^2$ such that

$$(h\circ\varphi)(x_i)=h(G(a,b)x_i+H(a,b)),$$

for $i\in\{1,2\}$. By the Jensen affinity of φ and the above we get

$$\begin{aligned}(h\circ\varphi)\Big(\frac{x_1+x_2}{2}\Big)&=h\Big(\frac{\varphi(x_1)+\varphi(x_2)}{2}\Big)\\&=h\left(\frac{G(a,b)x_1+H(a,b)+G(a,b)x_2+H(a,b)}{2}\right)\\&=h\Big(G(a,b)\frac{x_1+x_2}{2}+H(a,b)\Big)=F\left(\frac{x_1+x_2}{2}\right).\end{aligned}$$

which is the desired conclusion. □

Take $\varphi: I\to\mathbb{R}$ Jensen affine and discontinuous (for existance of such functions see [3]). By the above theorem the function $h\circ\varphi$ is a discontinuous and F-Jensen affine, where F is of the form (3). Now, it turns out that, in the case when $I=\mathbb{R}$, families of the form (3) are the only continuous, monotonic, two-parameter Beckenbach families having discontinuous F-Jensen affine functions. The rest of this section is devoted to prove this assertation. Thus, we assume that $\mathcal{F}$ is a Beckenbach family on $\mathbb{R}$. We start with some lemmas

Lemma 2.1 (compare [1, Thm. 1])**.** *Let $\psi,\tilde{\psi}\in\mathcal{F}$. If $\psi(x_0)=\tilde{\psi}(x_0)$ and $(\psi-\tilde{\psi})|_{(-\infty,x_0)}<0$, then $(\psi-\tilde{\psi})|_{(x_0,+\infty)}>0$. Furthermore, if $(\psi-\tilde{\psi})|_{(-\infty,x_0)}>0$, then $(\psi-\tilde{\psi})|_{(x_0,+\infty)}<0$.*

A subset A of a linear space X is called J-convex if $\frac{A+A}{2}\subseteq A$ and it is called $\mathbb{Q}$-convex if $rA+(1-r)A\subseteq A$ for all $r\in(0,1)\cap\mathbb{Q}$.

Lemma 2.2 ([7, Lemma]). *If A is J-convex subset of the linear space X such that $X \setminus A$ is also J-convex then it is necessarily $\mathbb{Q}$-convex.*

Lemma 2.3. *Let $\mathcal{F}$ be a continuous Beckenbach family on $\mathbb{R}$. Assume that $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathcal{F}$-Jensen affine and $\psi \in \mathcal{F}$. Then the set*

$$A_\psi := \{u \in \mathbb{R} : \varphi(u) = \psi(u)\},$$

is $\mathbb{Q}$-convex. Furthermore, if $\operatorname{card} A_\psi > 1$, then A_ψ is dense in $\mathbb{R}$.

Proof. Put

$$A_< := \{u \in \mathbb{R} : \varphi(u) < \psi(u)\}, \qquad B_\geq := \{u \in \mathbb{R} : \varphi(u) \geq \psi(u)\}.$$

Obviously, we have $A_< \cap B_\geq = \emptyset$ and $A_< \cup B_\geq = \mathbb{R}$. Additionally, we may assume that each of sets $A_<$ and $B_\geq$ is nonempty.

We show that the sets $A_<$ and $B_\geq$ are J-convex. To this end, consider $x, y \in A_<$ and, on the contrary, suppose that $\frac{x+y}{2} \in B_\geq$, i.e.

$$\varphi\Big(\frac{x+y}{2}\Big) \geq \psi\Big(\frac{x+y}{2}\Big).$$

Take $\eta \in \mathcal{F}$ such that $\eta(x) = \varphi(x)$ and $\eta(y) = \varphi(y)$. Since φ is $\mathcal{F}$-Jensen affine, we have $\eta\big(\frac{x+y}{2}\big) = \varphi\big(\frac{x+y}{2}\big)$. Thus,

$$\eta\Big(\frac{x+y}{2}\Big) \geq \psi\Big(\frac{x+y}{2}\Big).$$

By continuity of functions η and ψ there exists $\xi \in (x, \frac{x+y}{2}]$ such that $\eta(\xi) = \psi(\xi)$. Since $\eta(x) = \varphi(x) < \psi(x)$, by Lemma 2.1 we get $\varphi(y) = \eta(y) > \psi(y)$, i.e. $y \in B_\geq$. Obtained contradiction ends the proof of J-convexity of the set $A_<$.

It remains to prove that the set $B_\geq$ is J-convex. To this end fix $x, y \in B_\geq$ and suppose that $\frac{x+y}{2} \notin B_\geq$. Take $\tilde{\eta} \in \mathcal{F}$ such that $\tilde{\eta}(x) = \varphi(x)$ and $\tilde{\eta}(y) = \varphi(y)$. Since φ is $\mathcal{F}$-Jensen affine we have $\tilde{\eta}(\frac{x+y}{2}) = \varphi(\frac{x+y}{2})$. Thus,

$$\tilde{\eta}\Big(\frac{x+y}{2}\Big) < \psi\Big(\frac{x+y}{2}\Big). \tag{7}$$

By continuity of functions $\tilde{\eta}$ and ψ there exist $\xi \in [x, \frac{x+y}{2})$ and $\tilde{\xi} \in (\frac{x+y}{2}, y]$ such that $\eta(\xi) = \tilde{\psi}(\xi)$ and $\eta(\tilde{\xi}) = \tilde{\psi}(\tilde{\xi})$. Consequently, $\eta = \tilde{\psi}$, which contradics (7) and ends the proof of J-convexity of the set $B_\geq$.

Now, by Lemma 2.2, $A_<$ and $B_\geq$ are $\mathbb{Q}$-convex.

In the same manner we can see that the sets

$$A_{>} := \{u \in \mathbb{R} : \varphi(u) > \psi(u)\}, \qquad B_{\leq} := \{u \in \mathbb{R} : \varphi(u) \leq \psi(u)\},$$

are $\mathbb{Q}$-convex.

Finally, the set $A_\psi = B_{\geq} \cap B_{\leq}$ is $\mathbb{Q}$-convex.

The proof of density of A_ψ is a typical calculation, and we leave it to the reader. □

Directly from the above lemma we get

Lemma 2.4. *Let $\mathcal{F}$ be a continuous Beckenbach family on $\mathbb{R}$. Jf $\varphi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{F}$-Jensen affine and discontinuous at a point $x_0 \in \mathbb{R}$, then the graph of φ, i.e. $W_\varphi := \{(x, \varphi(x)) : x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$.*

Proof. We begin by proving that

$$\operatorname{int} \operatorname{cl} W_\varphi \neq \emptyset. \tag{8}$$

Since φ is discontinuous at x_0 its graph is not included in a graph of any function of the family $\mathcal{F}$. Thus there exist $\phi, \tilde{\phi} \in \mathcal{F}$, $\phi \neq \tilde{\phi}$ and such that $\operatorname{card} A_\phi > 1$ and $\operatorname{card} A_{\tilde{\phi}} > 1$. Let $z \in \mathbb{R}$ be such that $\phi(z) = \tilde{\phi}(z)$ (there exists at most one such a point). By virtue of Lemma 2.3, the sets A_ϕ and $A_{\tilde{\phi}}$ are $\mathbb{Q}$-convex in $\mathbb{R}$. Consider the set $W := \{(x, r\phi(x) + (1-r)\tilde{\phi}(x)) : x \in \mathbb{R}, r \in (0,1)\} \setminus \{(z, \phi(z))\}$. Obviously W is open. We show that the graph W_φ is dense in the set W. Let $P := (m, M) \times (z, Z) \subseteq W$. Without loss of generality we may assume that

$$\phi|_{(m,M)} < \tilde{\phi}|_{(m,M)}.$$

Due to the Lemma 2.3 the sets A_ϕ and $A_{\tilde{\phi}}$ are dense in $\mathbb{R}$. Thus, there exist $m', M' \in \mathbb{R}$ such that $m < m' < M' < M$ and $\varphi(m') = \phi(m')$, $\varphi(M') = \tilde{\phi}(M')$. Take $\beta \in \mathcal{F}$ such that $\beta(m') = \phi(m')$ and $\beta(M') = \tilde{\phi}(M')$.

Since A_β is $\mathbb{Q}$-convex, by virtue of Lemma 2.1 we have $\{(x, \varphi(x)) : x \in A_\beta\} \cap P \neq \emptyset$. That completes the proof of (8).

Now, fix $(u, \varphi(u)) \in \operatorname{int} \operatorname{cl} W_\varphi$ and let $U \subseteq \mathbb{R}$ be nonepmty open set. Let $(v, w), (v, \tilde{w}) \in U$ be such that $w < \tilde{w}$. Without loss of generality we may assume that $u < v$.

Take functions $\psi, \tilde{\psi} \in \mathcal{F}$ such that $\psi(u) = \tilde{\psi}(u) = \varphi(u)$ and $\psi(v) = w$, a $\tilde{\psi}(v) = \tilde{w}$. By Lemma 2.1 we have $\psi|_{(u,+\infty)} < \tilde{\psi}|_{(u,+\infty)}$. Thus the set

$$O := \{(x, r\psi(x) + (1-r)\tilde{\psi}(x)) : r \in (0,1),\ x \in (u, +\infty)\},$$

is open. Now, by (8), there exists $r > 0$ such that $K((u, \varphi(u)), r) \subseteq \operatorname{cl} W_\varphi$. Hence, we get $O \cap K((u, \varphi(u)), r) \subseteq \operatorname{cl} W_\varphi$. Thus, there exists $\tilde{x} > u$ such that $(\tilde{x}, \varphi(\tilde{x})) \in O$.

Take a function $\beta \in \mathcal{F}$ satisfying $\beta(u) = \varphi(u)$ and $\beta(\tilde{x}) = \varphi(\tilde{x})$. Since $\varphi(\tilde{x}) < \beta(\tilde{x}) < \tilde{\varphi}(\tilde{x})$ we have $\varphi|_{(u,+\infty)} < \beta|_{(u,+\infty)} < \tilde{\varphi}|_{(u,+\infty)}$. Particularly $\varphi(v) < \beta(v) < \tilde{\varphi}(v)$.

Let $s > 0$ be such that $K((v, \beta(v)), s) \subseteq U$. Since the set A_β is $\mathbb{Q}$-convex and dense in $\mathbb{R}$, there exists a point $\tilde{v} \in \mathbb{R}$ such that $(\tilde{v}, \varphi(\tilde{v})) \in K((v, \beta(v)), s) \subseteq U$. This completes the proof. □

As an immidiate consequence we get

Corollary 2.1. *Let $\mathcal{F}$ be a continuous Beckenbach family on $\mathbb{R}$. Jf $\varphi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{F}$-Jensen affine and discontinuous at $x_0 \in \mathbb{R}$, then φ is discontinuous at each point.*

Now, the existence of discontinuous $\mathcal{F}$-Jensen affine function allows us to prove the crucial result for the proof Theorem 2.3. We state it as

Corollary 2.2. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous, two-parameter Beckenbach family. Furthermore, let a function $\varphi : \mathbb{R} \to \mathbb{R}$ be F-Jensen affine and discontinuous at least at one point. Then the set*

$$\mathcal{D}_\varphi := \{(a, b) \in \mathbb{R}^2 : \operatorname{card} A_{F(\cdot,a,b)} > 1\},$$

is dense in $\mathbb{R}^2$.

Proof. Fix $u, v \in \mathbb{R}$ such that $u < v$. Let $\pi_{u,v} : \mathbb{R}^2 \to \mathbb{R}^2$ be function defined by the formula

$$\pi_{u,v}(a, b) = (F(u, a, b), F(v, a, b)), \quad (a, b) \in \mathbb{R}^2.$$

Notice, that by assumptions on F, the function $\pi_{u,v}$ is a continuous bijection. According to Invariance of Domain Theorem (see [2]) $\pi_{u,v}$ is a homeomorphism.

Now, consider the rectangle $(c, d) \times (\tilde{c}, \tilde{d}) \subseteq \mathbb{R}^2$. Choose points $c_1, d_1, \tilde{c}_1, \tilde{d}_1$ such that $c < c_1 < d_1 < d$ and $\tilde{c} < \tilde{c}_1 < \tilde{d}_1 < \tilde{d}$. Let the functions $\psi, \tilde{\psi} \in \mathcal{F}$ satisfy $\psi(u) = c_1$, $\psi(v) = \tilde{c}_1$ and $\tilde{\psi}(u) = d_1$, $\tilde{\psi}(v) = \tilde{d}_1$. Obviously, by virtue of Lemma 2.1, we get $\psi|_{[u,v]} < \tilde{\psi}|_{[u,v]}$. Hence, by continuity of ψ and $\tilde{\psi}$ we have $\psi|_{[\tilde{u},\tilde{v}]} < \tilde{\psi}|_{[\tilde{u},\tilde{v}]}$ for some $\tilde{u} < u$ and $v < \tilde{v}$. Since, by Lemma 2.4 the graph of φ is dense, there exist points $(p, \varphi(p))$, $(\tilde{p}, \varphi(\tilde{p})) \in \mathbb{R}^2$ such that $(p, \varphi(p)) \in (\tilde{u}, u) \times (\psi(p), \tilde{\psi}(p))$ and $(\tilde{p}, \varphi(\tilde{p})) \in (v, \tilde{v}) \times (\psi(\tilde{p}), \tilde{\psi}(\tilde{p}))$.

Now, let $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ be such that $F(\cdot, \tilde{a}, \tilde{b})$ satisfies $F(p, \tilde{a}, \tilde{b}) = \varphi(p)$ and $F(\tilde{p}, \tilde{a}, \tilde{b}) = \varphi(\tilde{p})$. By virtue of Lemma 2.1 we have $\psi|_{(p,\tilde{p})} < F(\cdot, \tilde{a}, \tilde{b})|_{(p,\tilde{p})} <$

$\tilde{\psi}|_{(p,\tilde{p})}$. Thus, in particular, there exists $(s,t)\in(c,d)\times(\tilde{c},\tilde{d})$ such that $F(u,\tilde{a},\tilde{b})=s$ and $F(v,\tilde{a},\tilde{b})=t$. Consequently, the set

$$\{(F(u,a,b),F(v,a,b)):(a,b)\in D_\varphi\}=\{\pi_{u,v}(a,b):(a,b)\in D_\varphi\}=\pi_{u,v}(D_\varphi).$$

is dense in $\mathbb{R}^2$. Since the function $\pi_{u,v}^{-1}$ is continuous, we get a density of the set $\mathcal{D}_\varphi$. □

Now, let X be a linear space and D be a convex subset of X. After [5], a function $f:D\to\mathbb{R}$ satisfying

$$\min\{f(x),f(y)\}\le f\Big(\frac{x+y}{2}\Big)\le\max\{f(x),f(y)\},\quad x,y\in D,$$

will be called midpoint-quasiaffine. Furthermore, a midpoint-quasiaffine function that satisfies

$$\min\{f(x),f(y)\}<f\Big(\frac{x+y}{2}\Big)<\max\{f(x),f(y)\}\quad\text{if}\quad f(x)\neq f(y),$$

will be called strictly midpoint-quasiaffine.

If a function f satisfies a condition

$$\lim_{\substack{r\to 0+\\ r\in\mathbb{Q}}} f((1-r)x+ry)=f(x), \tag{9}$$

for all $x,y\in D$, then we say that f is $\mathbb{Q}$-radially continuous on D.

In the proof of Theorem 2.3 we will use the following

Theorem 2.2 (see [5, Thm. 4])**.** *Let $f:X\to\mathbb{R}$ be a nonconstant function. Then f is a strictly midpoint-quasiaffine and $\mathbb{Q}$-radially continuous function if and only if it can be represented in the form $f=g\circ\alpha$, where $\alpha:X\to\mathbb{R}$ is an additive function and $g:\mathbb{R}\to\mathbb{R}$ is an upper semicontinuous strictly increasing function which is continuous on the range of the additive function α. Furthermore, the representation $f=g\circ\alpha$ is unique in the following sense: If $f=g'\circ\alpha'$ with an additive α' and upper semicontinuous strictly increasing g', then there exists a positive constant $q>0$ such that*

$$\alpha'(x)=q\alpha(x),\quad x\in X\quad\textit{and}\quad g'(t)=g(t/q),\quad t\in\mathbb{R}. \tag{10}$$

Obviously, we will use above theorem in the case $X=\mathbb{R}$.

Theorem 2.3. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous, monotonic, two-parameter Beckenbach family. If there exists a discontinuous at least at one point, F-Jensen affine function, then there exists continuous and strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, such that*

$$F(u,a,b) = g((g^{-1}(F(1,a,b)) - g^{-1}(F(0,a,b)))u + g^{-1}(F(0,a,b))), \tag{11}$$

for all $u, a, b \in \mathbb{R}$.

Proof. Put $\mathcal{F} := \{F(\cdot,a,b) : a,b \in \mathbb{R}\}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a discontinuous, $\mathcal{F}$-Jensen affine function. Obviously, φ is nonconstant.

We show that φ is strictly midpoint-quasiaffine.

To this end, consider $x, y \in \mathbb{R}$. Since the family $\mathcal{F}$ consists of monotonic functions, we have

$$\min\{\psi(x),\psi(y)\} \le \psi\Big(\frac{x+y}{2}\Big) \le \max\{\psi(x),\psi(y)\}, \tag{12}$$

for $\psi \in \mathcal{F}$. Now, let $\psi \in \mathcal{F}$ satisfies $\psi(x) = \varphi(x)$ and $\psi(y) = \varphi(y)$. By $\mathcal{F}$-Jensen affinity of φ we have $\psi(\frac{x+y}{2}) = \varphi(\frac{x+y}{2})$. Thus,

$$\min\{\varphi(x),\varphi(y)\} \le \varphi\Big(\frac{x+y}{2}\Big) \le \max\{\varphi(x),\varphi(y)\}. \tag{13}$$

Moreover, if $\varphi(x) \ne \varphi(y)$, then ψ is strictly monotonic. Consequently, each of inequalities (12) and (13) are strict.

Now, we will show that φ is $\mathbb{Q}$-radially continuous.
Again, fix $x, y \in \mathbb{R}$ and take $\psi \in \mathcal{F}$ such that $\psi(x) = \varphi(x)$ and $\psi(y) = \varphi(y)$. By virtue of Lemma 2.3 the set A_ψ is $\mathbb{Q}$-convex. Hence

$$\lim_{\substack{r\to 0+\\ r\in\mathbb{Q}}} \varphi((1-r)x + ry) = \lim_{\substack{r\to 0+\\ r\in\mathbb{Q}}} \psi((1-r)x + ry) = \psi(x) = \varphi(x).$$

Since the function φ satisfies the assumptions of Theorem 2.2, there exists a strictly increasing, upper semicontinuous function $g : \mathbb{R} \to \mathbb{R}$ and an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\varphi = g \circ \alpha$. By virtue of Lemma 2.4 the graph of φ is dense in $\mathbb{R}^2$. Consequently, g (as increasing) is a continuous surjection. By (1), (2) and the fact that g is invertible we have

$$\begin{cases} \alpha(x) = (g^{-1} \circ F)(x, a(x,y), b(x,y)), \\ \alpha(y) = (g^{-1} \circ F)(y, a(x,y), b(x,y)), \end{cases} \tag{14}$$

and

$$\alpha\Big(\frac{x+y}{2}\Big) = (g^{-1} \circ F)\Big(\frac{x+y}{2}, a(x,y), b(x,y)\Big),$$

for all $x, y \in \mathbb{R}$, where $a, b : \mathbb{R}^2 \to \mathbb{R}$ are functions. Hence, by additivity of α we get

$$\frac{\alpha(x) + \alpha(y)}{2} = (g^{-1} \circ F)\Big(\frac{x+y}{2}, a(x,y), b(x,y)\Big).$$

Combining the above and (14) we obtain

$$\frac{1}{2}\big((g^{-1} \circ F)(x, a(x,y), b(x,y)) + (g^{-1} \circ F)(y, a(x,y), b(x,y))\big) \\ = (g^{-1} \circ F)\Big(\frac{x+y}{2}, a(x,y), b(x,y)\Big),$$

for all $x, y \in \mathbb{R}$.

Now, fix $(a, b) \in \mathcal{D}_\varphi$. Notice, that for all $x, y \in A_{F(\cdot,a,b)}$ we have $a(x, y) = a$ and $b(x, y) = b$. Thus, we get

$$\frac{1}{2}((g^{-1} \circ F)(x, a, b) + (g^{-1} \circ F)(y, a, b)) = (g^{-1} \circ F)\left(\frac{x+y}{2}, a, b\right), \qquad (15)$$

for all $x, y \in A_{F(\cdot,a,b)}$. By virtue of Collorary 2.2 the set $A_{F(\cdot,a,b)}$ is dense in $\mathbb{R}$. By (15) and a continuity of the function $g^{-1} \circ F$ the equality

$$\frac{1}{2}((g^{-1} \circ F)(x, a, b) + (g^{-1} \circ F)(y, a, b)) = (g^{-1} \circ F)\left(\frac{x+y}{2}, a, b\right),$$

holds for all $x, y \in \mathbb{R}$. Now, take a notice that the above formula states a Jensen affinity of a continuous function $g^{-1} \circ F(\cdot, a, b)$. Hence, there exist constants $\tilde{c}(a, b), \tilde{d}(a, b) \in \mathbb{R}$ such that

$$g^{-1}(F(u, a, b)) = \tilde{c}(a, b)u + \tilde{d}(a, b), \quad u \in \mathbb{R}, (a, b) \in \mathcal{D}_\varphi. \qquad (16)$$

Putting $u = 0$ we get

$$\tilde{d}(a, b) = g^{-1}(F(0, a, b)), \quad (a, b) \in \mathcal{D}_\varphi,$$

and putting $u = 1$ in (16) we get

$$\tilde{c}(a, b) = g^{-1}(F(1, a, b)) - g^{-1}(F(0, a, b)), \quad (a, b) \in \mathcal{D}_\varphi.$$

Thus

$$g^{-1}(F(u,a,b)) = (g^{-1}(F(1,a,b)) - g^{-1}(F(0,a,b)))u + g^{-1}(F(0,a,b)),$$

for all $u \in \mathbb{R}$ and $(a,b) \in \mathcal{D}_\varphi$. Since the functions at each sides of the above equality are continuous and the set $\mathcal{D}_\varphi$ is dense, we obtain

$$g^{-1}(F(u,a,b)) = (g^{-1}(F(1,a,b)) - g^{-1}(F(0,a,b)))u + g^{-1}(F(0,a,b)),$$

for all $u, a, b \in \mathbb{R}$, which is equivalent to (11) and completes the proof. □

Now, let us mention an important consequence of the above theorem.

Theorem 2.4. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a strictly increasing and continuous function. If for a two-parameter Beckenbach family $F : \mathbb{R}^3 \to \mathbb{R}$ of the form*

$$F(u,a,b) = a\alpha(u) + b, \quad u, a, b \in \mathbb{R},$$

exists a discontinuous at least at one point, F-Jensen affine function, then

$$\alpha(u) = cu + d, \quad u \in \mathbb{R},$$

for some $c, d \in \mathbb{R}$.

Proof. Due to [6]. Since F is a monotonic Beckenbach family, by virtue of Theorem 2.3, there exists a strictly increasing and continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g^{-1}(a\alpha(u) + b) = \big(g^{-1}(a\alpha(1) + b) - g^{-1}(a\alpha(0) + b)\big)\, u + g^{-1}(a\alpha(0) + b), \quad (17)$$

for $a, b, u \in \mathbb{R}$.

Let $\beta : \mathbb{R} \to \mathbb{R}$ be defined by the formula $\beta(x) = \frac{\alpha(x)-\alpha(0)}{\alpha(1)-\alpha(0)}$, for $x \in \mathbb{R}$. By (17) we get

$$\begin{aligned} g^{-1}(a(\alpha(1) - \alpha(0))\beta(u) + a\alpha(0) + b) &= (g^{-1}(a(\alpha(1) - \alpha(0))\beta(1) + a\alpha(0) + b) \\ &= -g^{-1}(a(\alpha(1) - \alpha(0))\beta(0) + a\alpha(0) + b))u+ \\ &= g^{-1}(a(\alpha(1) - \alpha(0))\beta(0) + a\alpha(0) + b), \end{aligned}$$

for $a, b, u \in \mathbb{R}$. Since $\beta(0) = 0, \beta(1) = 1$, we have

$$\begin{aligned} &g^{-1}(a(\alpha(1) - \alpha(0))\beta(u) + a\alpha(0) + b) \\ &\qquad = \big(g^{-1}(a\alpha(1) + b) - g^{-1}(a\alpha(0) + b)\big)\, u + g^{-1}(a\alpha(0) + b), \end{aligned}$$

for $a, b, u \in \mathbb{R}$. Let $A = a(\alpha(1) - \alpha(0))$ and $B = a\alpha(0) + b$. We have

$$g^{-1}(A\beta(u) + B) = \big(g^{-1}(A + B) - g^{-1}(B)\big)\, u + g^{-1}(B), \tag{18}$$

for $A, B, u \in \mathbb{R}$.

Now, let $h : \mathbb{R} \to \mathbb{R}$ be defined by the formula $h(x) = \frac{g(x)-g(0)}{g(1)-g(0)}$. Putting $u = 1$ in (17) we get $a\alpha(u) + b = g(g^{-1}(a\alpha(u) + b))$. Hence the function g is a surjection. Thus, by (18) and formula $g^{-1}(u) = h^{-1}\big(\frac{u-g(0)}{g(1)-g(0)}\big)$ for $u \in \mathbb{R}$, we have

$$\begin{aligned} h^{-1}&\Big(\frac{A\beta(u) + B - g(0)}{g(1) - g(0)}\Big) \\ &= \Big(h^{-1}\Big(\frac{A + B - g(0)}{g(1) - g(0)}\Big) - h^{-1}\Big(\frac{B - g(0)}{g(1) - g(0)}\Big)\Big)\, u + h^{-1}\Big(\frac{B - g(0)}{g(1) - g(0)}\Big), \end{aligned}$$

for $A, B, u \in \mathbb{R}$. Now, putting $A = \tilde{a}(g(1) - g(0)),\ B = \tilde{b}(g(1) - g(0)) + g(0)$ we obtain

$$h^{-1}(\tilde{a}\beta(u) + \tilde{b}) = (h^{-1}(\tilde{a} + \tilde{b}) - h^{-1}(\tilde{b}))u + h^{-1}(\tilde{b}), \tag{19}$$

for $\tilde{a}, \tilde{b}, u \in \mathbb{R}$. Take $\tilde{a} = 1, \tilde{b} = 0$ in (19). We have

$$h^{-1}(\beta(u)) = (h^{-1}(1) - h^{-1}(0))u + h^{-1}(0), \quad u \in \mathbb{R}.$$

Since $h(0) = 0, h(1) = 1$ we get $h^{-1}(\beta(u)) = u$ for $u \in \mathbb{R}$. Thus, $h = \beta$.
Using this equality and putting $\tilde{b} = 0$ in (19), we get

$$h^{-1}(ah(u)) = h^{-1}(a)u, \qquad a, u \in \mathbb{R}.$$

Since h is a bijection for every $a \in \mathbb{R}$ there exists $w \in \mathbb{R}$ such that $a = h(w)$. Thus

$$h^{-1}(h(w)h(u)) = wu, \quad w, u \in \mathbb{R}.$$

Or equivalently,

$$h(uw) = h(u)h(w), \quad u, w \in \mathbb{R}.$$

By virtue of Theorem 13.1.3 (see [3]), there exists a constant $c > 0$ such that $h(u) = |u|^c sgn(u)$ for $u \in \mathbb{R}$.
Now, putting $\tilde{a} = \tilde{b} = 1, u > 0$ in (19) we get

$$(u^c + 1)^{\frac{1}{c}} = (2^{\frac{1}{c}} - 1)u + 1, \qquad u > 0,$$

Let $C = 2^{\frac{1}{c}} - 1$. We have

$$(u^c + 1) = (Cu + 1)^c, \qquad u > 0.$$

Differentiating both sides of equality we get

$$cu^{c-1} = Cc(Cu + 1)^{c-1}, \qquad u > 0.$$

Thus,

$$\frac{1}{C} = \left(C + \frac{1}{u}\right)^{c-1}, \qquad u > 0.$$

Since the left handside of the above equality is constant we get $c = 1$. That means $h(u) = |u| sgn(u) = u$, $u \in \mathbb{R}$. Finally, the equality $h = \beta$ and the definition of β complete the proof. □

We end this section with

Theorem 2.5. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous, monotonic two-parameter Beckenbach family on $\mathbb{R}$. If there exists a discontinuous at least at one point, F-Jensen affine function, then there exist a strictly increasing, continuous $h : \mathbb{R} \to \mathbb{R}$ and continuous functions $G, H : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$F(u, a, b) = h(G(a, b)u + H(a, b)), \quad u, a, b \in \mathbb{R}.$$

Proof. Taking $h := g$, $G(\cdot,\cdot) := g^{-1}(F(1,\cdot,\cdot)) - g^{-1}(F(0,\cdot,\cdot))$ and $H(\cdot,\cdot) := g^{-1}(F(0,\cdot,\cdot))$ in formula (11), we get the thesis. □

3. Characterization of Beckenbach families of the form (3)

Now, we give a characterization of Beckenbach families of the form (3). We start with an easy to prove

Remark 3.1. A family of functions $\mathcal{F}$ is a (continuous) Beckenbach family iff the family $h(\mathcal{F})$ is a (continuous) Beckenbach family for some homeomorphism $h : \mathbb{R} \to \mathbb{R}$.

Lemma 3.1. *Let $G, H : \mathbb{R}^2 \to \mathbb{R}$ be functions and $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function given by*

$$F(u, a, b) = G(a, b)u + H(a, b), \quad u, a, b \in \mathbb{R}. \tag{20}$$

The function F is a two-parameter, continuous Beckenbach family on $\mathbb{R}$ iff the function $\hat{I} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by formula

$$\hat{I}(a,b) = (G(a,b), H(a,b)), \quad (a,b) \in \mathbb{R}^2,$$

is a homeomorphism.

Proof. Assume, that $\mathcal{F}$ is a Beckenbach family. Putting $u = 0$ in (20) we get continuity of the function H. Now, since the function $(a,b) \mapsto F(1,a,b) - H(a,b) = G(a,b)$ is continuous, the function $\hat{I}$ is also continuous.

We show that $\hat{I}$ is a bijection. To this aim fix $(c,d) \in \mathbb{R}^2$. Consider the points $(0,d), (1,c+d) \in \mathbb{R}^2$. There exists exactly one point $(a,b) \in \mathbb{R}^2$ such that $F(0,a,b) = d$ and $F(1,a,b) = c+d$. Equivalently $(G(a,b), H(a,b)) = (c,d)$. Hence $\hat{I}$ is bijection. The Invariance of Domain Theorem completes the "If" part of the proof.

To the proof of reverse implication fix (x_i, y_i) for $i \in \{1,2\}$. There exists exactly one element $(c,d) \in \mathbb{R}^2$ such that $y_i = cx_i + d$ for $i \in \{1,2\}$. Since $\hat{I}$ is a bijection there exists exactly one $(a,b) \in \mathbb{R}^2$ such that $\hat{I}(a,b) = (c,d)$. Finally, there exists exactly one solution of the system of equalities $G(a,b)x_i + H(a,b) = y_i$ for $i \in \{1,2\}$, which completes the proof. □

The last theorem provides a criterion for a function of a form (3) to be a Beckenbach family.

Theorem 3.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism and $G, H : \mathbb{R}^2 \to \mathbb{R}$. Furthermore, let $F : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function given by*

$$F(u,a,b) = h(G(a,b)u + H(a,b)), \quad u,a,b \in \mathbb{R}.$$

The function F is a two-parameter, continuous Beckenbach family on $\mathbb{R}$ iff the function $\hat{I} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by formula

$$\hat{I}(a,b) = (G(a,b), H(a,b)), \quad (a,b) \in \mathbb{R}^2,$$

is a homeomorphism.

Proof. An immediate consequence of the Lemma 3.1 and Remark 3.1. □

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(Received November 8, 2010; revised August 25, 2011)