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Weighted Ostrowski, Ostrowski-Grüss and Ostrowski–Čebyšev type inequalities on time scales

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Abstract. In this paper we derive some weighted Ostrowski, Ostrowski-Grüss and Ostrowski–Čebyšev type inequalities on time scales. We also give some other interesting inequalities on time scales as special cases.

1. Introduction

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [28], the so-called Ostrowski inequality, which can also be shown by using Montgomery identity [26]. In 1997, by combining Montgomery identity and Grüss's integral inequality, Dragomir and Wang [12] proved the following Ostrowski–Grüss type integral inequality, which is a connection between Ostrowski's inequality and Grüss's inequality.

Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. Then, for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma). \quad (1.1)$$

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In [11], DRAGOMIR and BARNETT pointed out a new estimation on the left hand side of (1.1) as follows.

Theorem B. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b). Then we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \\ \leq \frac{M}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2 \quad (1.2)$$

for all $x \in [a,b]$, where $M = \sup_{a<t<b}|f''(t)| < \infty$.

Recently, AHMAD et. al [2] developed some new Ostrowski and Čebyšev type inequalities involving two functions, by using an identity of DRAGOMIR and BARNETT proved in [11]. In [35], TSENG, HWANG and DRAGOMIR established the following generalizations of weighted Ostrowski type inequalities for mappings of bounded variation.

Theorem C. *Let us have $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ continuous and positive on (a,b) and let $h : [a,b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a,b]$. Let $c = h^{-1}\big(\big(1-\frac{\alpha}{2}\big)h(a) + \frac{\alpha}{2}h(b)\big)$ and $d = h^{-1}\big(\frac{\alpha}{2}h(a) + \big(1-\frac{\alpha}{2}\big)h(b)\big)$. Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation. Then, for all $x \in [c,d]$, we have*

$$\left| \int_a^b f(t)g(t)\mathrm{d}t - \left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\mathrm{d}t \right| \leq K\bigvee_a^b(f), \quad (1.3)$$

where

$$K := \begin{cases} \dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)\mathrm{d}t + \left|h(x) - \dfrac{h(a)+h(b)}{2}\right|, & 0 \leq \alpha \leq \dfrac{1}{2}, \\ \max\left\{\dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)\mathrm{d}t + \left|h(x) - \dfrac{h(a)+h(b)}{2}\right|, \quad \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)\mathrm{d}t\right\}, & \dfrac{1}{2} < \alpha < \dfrac{2}{3}, \\ \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)\mathrm{d}t, & \dfrac{2}{3} \leq \alpha \leq 1 \end{cases} \quad (1.4)$$

and $\bigvee_a^b(f)$ denotes the total varfiation of f on the interval $[a,b]$. In (1.4), the constant $\frac{1-\alpha}{2}$ for $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{\alpha}{2}$for $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

The theory of calculus on time scales was initiated by Hilger [13] in 1988 in order to unify continuous and discrete analysis, and it has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in population dynamics [4], economics [3], physics [34], space weather [17] and so on. Since then, many authors have studied the theory of certain integral inequalities on time scales (see [7], [8], [9], [10], [15], [18], [19], [20], [21], [22], [23], [24], [27], [29], [30], [31], [32], [33], [36], [38]). For example, by using the Montgomery identity on time scales, Bohner and Matthews [8] established the following Ostrowski inequality on time scales which unify discrete, continuous and many other cases.

Theorem D. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s))\Delta s \right| \leq \frac{M}{b-a} [h_2(t, a) + h_2(t, b)], \tag{1.5}$$

where $h_2(\cdot, \cdot)$ is defined by Definition 8 below and $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.5) *cannot be replaced by a smaller one.*

Recently, Karpuz and Özkan [15] generalized Ostrowski's inequality and Montgomery's identity on arbitrary time scales by the means of generalized polynomials of arbitrary order on time scales. By introducing a parameter, Liu, Ngô and Chen [24] also extended a generalization of the above inequality on time scales. In [20], Liu and Ngô derived an inequality of Ostrowski–Grüss type on time scales by using the Grüss inequality on time scales. Then, Ngô and Liu [27] gave a sharp Grüss type inequality on time scales and applied it to the sharp Ostrowski-Grüss inequality on time scales. Motivated by the ideas in [8], [24], [27], [37], Tuna and Daghan [36] studied generalizations of Ostrowski and Ostrowski–Grüss type inequality on time scales. More recently, Liu, Tuna and Jiang [25] derived a weighted Montgomery identity on time scales and then established weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales, respectively.

Motivated by the above research, the purpose of this paper is to obtain weighted Ostrowski, Ostrowski-Grüss and Ostrowski–Čebyšev type inequalities on time scales. We also give some other interesting inequalities on time scales as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. The weighted Ostrowski, Ostrowski-Grüss and Ostrowski–Čebyšev type inequalities on time scales are derived in Section 3.

2. Time scales essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to HILGER's Ph.D. thesis [13], the books [5], [6], [16], and the survey [1].

Definition 1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T} : a \le t \le b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : \; s < t\}$.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 3. If $\sigma(t) > t$, *then we say that* t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.

Definition 4. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the *graininess function.* The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 5. Let $f : \mathbb{T} \to \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon \left| \sigma(t) - s \right|, \quad \forall\, s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

Theorem E. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

Definition 6. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous (denote* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [5, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 7. Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $F : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ-integral of f as

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

Theorem F. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^b [\alpha f(t) + \beta g(t)]\, \Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t,$

(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$

(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$

(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t,$

Theorem G. *If f is Δ-integrable on $[a, b]$, then so is $|f|$, and*

$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b |f(t)|\, \Delta t.$$

Definition 8. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t, s) = 1 \quad \text{for all } s, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau \quad \text{for all } s, t \in \mathbb{T}.$$

3. Main results

Theorem 1. *Let $0 \le k \le 1$, $g : [a, b] \to [0, \infty)$ be rd continuous and positive and $h : [a, b] \to \mathbb{R}$ be differentiable such that $h^{\Delta}(t) = g(t)$ on $[a, b]$.*

Let $a, b, t, x \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be twice differentiable. Then for all $x \in [a, b]$, we have

$$\begin{aligned}
&\Bigg|(1-\alpha)^2 f(x) + \alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \alpha\frac{f^{\Delta}(a)+f^{\Delta}(b)}{2\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right) \\
&\quad - \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)f^{\Delta}(\sigma(t))\Delta t\right) \\
&\quad - \frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b g(t)f(\sigma(t))\Delta t\Bigg| \\
&\quad \le \frac{M}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\Delta s\Delta t, \qquad (3.1)
\end{aligned}$$

where $M = \sup_{a\le t<b}\left|f^{\Delta\Delta}(t)\right| < \infty$ and

$$W(x,t) = \begin{cases} h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a \le t < x, \\ h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & x \le t \le b. \end{cases} \qquad (3.2)$$

Proof. Using item (4) of Theorem F and (3.2), we have (see also [25])

$$\begin{aligned}
(1-\alpha)^2 f(x) = &-\alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b g(t)f(\sigma(t))\Delta t \\
&+ \frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b W(x,t)f^{\Delta}(t)\Delta t, \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
(1-\alpha)f^{\Delta}(t) = &-\alpha\frac{f^{\Delta}(a)+f^{\Delta}(b)}{2} + \frac{1}{\int_a^b g(t)\Delta t}\int_a^b g(s)f^{\Delta}(\sigma(s))\Delta s \\
&+ \frac{1}{\int_a^b g(t)\Delta t}\int_a^b W(t,s)f^{\Delta\Delta}(s)\Delta s. \qquad (3.4)
\end{aligned}$$

Substituting $(1-\alpha)f^{\Delta}(t)$ in (3.4) into the right hand side of (3.3), we obtain

$$(1-\alpha)^2 f(x) = -\alpha(1-\alpha)\frac{f(a)+f(b)}{2} - \alpha\frac{f^{\Delta}(a)+f^{\Delta}(b)}{2\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right)$$

$$+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)f^{\Delta}(\sigma(t))\Delta t\right)$$
$$+\frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b g(t)f(\sigma(t))\Delta t$$
$$+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)f^{\Delta\Delta}(s)\Delta s\Delta t. \tag{3.5}$$

From (3.5) and using the properties of modulus, the inequality (3.1) is proved. □

Corollary 1. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 1, we have*

$$\Bigg|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{f'(a)+f'(b)}{2\int_a^b g(t)\mathrm{d}t}\left(\int_a^b W(x,t)\mathrm{d}t\right)$$
$$-\frac{1}{\left(\int_a^b g(t)\mathrm{d}t\right)^2}\left(\int_a^b W(x,t)\mathrm{d}t\right)\left(\int_a^b g(t)f'(t)\mathrm{d}t\right)-\frac{1-\alpha}{\int_a^b g(t)\mathrm{d}t}\int_a^b g(t)f(t)\mathrm{d}t\Bigg|$$
$$\le\frac{M}{\left(\int_a^b g(t)\mathrm{d}t\right)^2}\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\mathrm{d}s\mathrm{d}t,$$

where $g(t)=h'(t)$ on $[a,b]$, $M=\sup_{a\le t<b}|f''(t)|<\infty$ and

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\le t<x,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\le t\le b.\end{cases}$$

Corollary 2. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 1, we have*

$$\Bigg|(1-\alpha)^2 f(x)+\alpha(1-\alpha)\frac{f(a)+f(b)}{2}+\alpha\frac{\Delta f(a)+\Delta f(b)}{2\sum_{t=a}^{b-1}g(t)}\left(\sum_{t=a}^{b-1}W(x,t)\right)$$
$$-\frac{1}{\left(\sum_{t=a}^{b-1}g(t)\right)^2}\left(\sum_{t=a}^{b-1}W(x,t)\right)\left(\sum_{t=a}^{b-1}g(t)\Delta f(t+1)\right)-\frac{1-\alpha}{\sum_{t=a}^{b-1}g(t)}\sum_{t=a}^{b-1}g(t)f(t+1)\Bigg|$$
$$\le\frac{M}{\left(\sum_{t=a}^{b-1}g(t)\right)^2}\sum_{t=a}^{b-1}\sum_{s=a}^{b-1}|W(x,t)|\,|W(t,s)|\,,$$

where $g(t) = h(t+1) - h(t)$ *on* $[a, b-1]$, $M = \sup_{a \le t < b} \left|\Delta^2 f(t)\right| < \infty$ *and*

$$W(x,t) = \begin{cases} h(t) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)\right), & a \le t < x-1, \\ h(t) - \left(\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b)\right), & x \le t \le b-1. \end{cases}$$

Corollary 3. *In the case of* $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\} (q > 1)$ *in Theorem 1, we have*

$$\Bigg|(1-\alpha)^2 f(x) + \alpha(1-\alpha)\frac{f(a)+f(b)}{2} + \alpha \frac{\mathrm{D}_q f(a) + \mathrm{D}_q f(b)}{2\int_a^b g(t)\mathrm{d}_q t}\left(\int_a^b W(x,t)\mathrm{d}_q t\right)$$
$$- \frac{1}{\left(\int_a^b g(t)\mathrm{d}_q t\right)^2}\left(\int_a^b W(x,t)\mathrm{d}_q t\right)\left(\int_a^b g(t)\mathrm{D}_q f(qt)\mathrm{d}_q t\right)$$
$$- \frac{1-\alpha}{\int_a^b g(t)\mathrm{D}_q t}\int_a^b g(t) f(qt)\mathrm{d}_q t\Bigg| \le \frac{M}{\left(\int_a^b g(t)\mathrm{d}_q t\right)^2}\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\mathrm{d}_q s\,\mathrm{d}_q t,$$

where $g(t) = (\mathrm{D}_q h)(t)$ *on* $[a,b]$ *and* $M = \sup_{a \le t < b} |(\mathrm{D}_q^2 f)(t)| < \infty$. *Here, for* $s, t \in q^{\mathbb{Z}} \cup \{0\}$ *with* $t \ge s$, *we have*

$$(\mathrm{D}_q f)(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{and} \quad \int_s^t v(\eta)\mathrm{d}_q \eta = (q-1)\sum_{\ell = \log_q(s)}^{\log_q(t/q)} v(q^\ell) q^\ell,$$

where we adopt the convention that $\log_q(0) := -\infty$ *and* $\log_q(\infty) := \infty$ *(see* [14]*).*

Remark 1. In the case of $\alpha = 0$ and $h(t) = t$, Corollary 1 reduces to Theorem B.

Theorem 2. *Let* $0 \le k \le 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\Delta}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, t, x \in \mathbb{T}$, $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then for all* $x \in [a,b]$, *we have*

$$\Bigg|\frac{1}{b-a}\left[(1-\alpha)f(x) + \alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\Delta t$$
$$- \frac{f(b)-f(a)}{(b-a)^2}\int_a^b W(x,t)\Delta t - \frac{1}{b-a}\int_a^b g(t) f(\sigma(t))\Delta t\Bigg|$$
$$\le \left[\frac{1}{b-a}\int_a^b W^2(x,t)\Delta t - \left(\frac{1}{b-a}\int_a^b W(x,t)\Delta t\right)^2\right]^{\frac{1}{2}}$$

$$\times \left[\frac{1}{b-a} \int_a^b \left(f^{\Delta}(t) \right)^2 \Delta t - \left(\frac{f(b)-f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \tag{3.6}$$

where

$$W(x,t) = \begin{cases} h(t) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right), & a \le t < x, \\ h(t) - \left(\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b) \right), & x \le t \le b. \end{cases}$$

Proof. We have

$$\frac{1}{b-a} \int_a^b W(x,t) f^{\Delta}(t) \Delta t - \left(\frac{1}{b-a} \int_a^b W(x,t) \Delta t \right) \left(\frac{1}{b-a} \int_a^b f^{\Delta}(t) \Delta t \right)$$
$$= \frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(W(x,t) - W(x,s) \right) \left(f^{\Delta}(t) - f^{\Delta}(s) \right) \Delta t \Delta s. \tag{3.7}$$

We also have

$$\int_a^b W(x,t) f^{\Delta}(t) \Delta t$$
$$= \left[(1-\alpha) f(x) + \alpha \frac{f(a)+f(b)}{2} \right] \int_a^b g(t) \Delta t - \int_a^b g(t) f(\sigma(t)) \Delta t \tag{3.8}$$

and

$$\frac{1}{b-a} \int_a^b f^{\Delta}(t) \Delta t = \frac{f(b)-f(a)}{b-a}. \tag{3.9}$$

Using the Cauchy-Schwartz inequality, we may write

$$\left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(W(x,t) - W(x,s) \right) \left(f^{\Delta}(t) - f^{\Delta}(s) \right) \Delta t \Delta s \right|$$
$$\le \left(\frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(W(x,t) - W(x,s) \right)^2 \Delta t \Delta s \right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(f^{\Delta}(t) - f^{\Delta}(s) \right)^2 \Delta t \Delta s \right)^{\frac{1}{2}}. \tag{3.10}$$

However

$$\frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(W(x,t) - W(x,s) \right)^2 \Delta t \Delta s$$
$$= \frac{1}{b-a} \int_a^b W^2(x,t) \Delta t - \left(\frac{1}{b-a} \int_a^b W(x,t) \Delta t \right)^2,$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^{\Delta}(t)-f^{\Delta}(s)\right)^2 \Delta t\Delta s$$
$$=\frac{1}{b-a}\int_a^b \left(f^{\Delta}(t)\right)^2 \Delta t-\left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)^2.$$

Using (3.7)-(3.12), we obtain the inequality (3.6). □

Corollary 4. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 2, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\mathrm{d}t\right.\\&\qquad\left.-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b W(x,t)\mathrm{d}t-\frac{1}{b-a}\int_a^b g(t)f(t)\mathrm{d}t\right|\\&\le\left[\frac{1}{b-a}\int_a^b W^2(x,t)\mathrm{d}t-\left(\frac{1}{b-a}\int_a^b W(x,t)\mathrm{d}t\right)^2\right]^{\frac{1}{2}}\\&\quad\times\left[\frac{1}{b-a}\int_a^b \left(f'(t)\right)^2\mathrm{d}t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

where $g(t)=h'(t)$ on $[a,b]$ and

$$W(x,t)=\begin{cases}h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\le t<x,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\le t\le b.\end{cases}$$

Corollary 5. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 2, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\sum_{t=a}^{b-1} g(t)\right.\\&\qquad\left.-\frac{f(b)-f(a)}{(b-a)^2}\sum_{t=a}^{b-1} W(x,t)-\frac{1}{b-a}\sum_{t=a}^{b-1} g(t)f(t+1)\right|\\&\le\left[\frac{1}{b-a}\sum_{t=a}^{b-1} W^2(x,t)-\left(\frac{1}{b-a}\sum_{t=a}^{b-1} W(x,t)\right)^2\right]^{\frac{1}{2}}\end{aligned}$$

$$\times\left[\frac{1}{b-a}\sum_{t=a}^{b-1}(\Delta f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=h(t+1)-h(t)$ *on* $[a,b-1]$ *and*

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right), & a\le t<x-1,\\ h(t)-\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right), & x\le t\le b-1.\end{cases}$$

Corollary 6. *In the case of* $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}(q>1)$ *in Theorem 2, we have*

$$\left|\frac{1}{b-a}\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\mathrm{d}_q t\right.$$
$$\left.-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b W(x,t)\mathrm{d}_q t-\frac{1}{b-a}\int_a^b g(t)f(qt)\mathrm{d}_q t\right|$$
$$\le\left[\frac{1}{b-a}\int_a^b W^2(x,t)\mathrm{d}_q t-\left(\frac{1}{b-a}\int_a^b W(x,t)\mathrm{d}_q t\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{b-a}\int_a^b(\mathrm{D}_q f(t))^2\,\mathrm{d}_q t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

where $g(t)=(\mathrm{d}_q h)(t)$ *on* $[a,b]$.

Theorem 3. *Let* $0\le k\le 1$, $g:[a,b]\to[0,\infty)$ *be rd continuous and positive and* $h:[a,b]\to\mathbb{R}$ *be differentiable such that* $h^{\Delta}(t)=g(t)$ *on* $[a,b]$*. Let* $a,b,t,x\in\mathbb{T}$, $a<b$ *and* $p,r:[a,b]\to\mathbb{R}$ *be twice differentiable. Then for all* $x\in[a,b]$*, we have*

$$\left|(1-\alpha)^2p(x)r(x)\right.$$
$$+\frac{\alpha}{4\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right)\left[r(x)\left(p^{\Delta}(a)+p^{\Delta}(b)\right)+p(x)\left(r^{\Delta}(a)+r^{\Delta}(b)\right)\right]$$
$$-\frac{1-\alpha}{2\int_a^b g(t)\Delta t}\left(r(x)\int_a^b g(t)p(\sigma(t))\Delta t+p(x)\int_a^b g(t)r(\sigma(t))\Delta t\right)$$
$$+\frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right)-\frac{1}{2\left(\int_a^b g(t)\Delta t\right)^2}$$

$$\times\left(\int_a^b W(x,t)\Delta t\right)\left(r(x)\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t + p(x)\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t\right)\Bigg|$$
$$\leq \frac{1}{2\left(\int_a^b g(t)\Delta t\right)^2}\left(|r(x)|\,M + |p(x)|\,N\right)\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\Delta s\Delta t, \tag{3.11}$$

where $M = \sup_{a\leq t<b}\left|p^{\Delta\Delta}(t)\right| < \infty$, $N = \sup_{a\leq t<b}\left|r^{\Delta\Delta}(t)\right| < \infty$, *and*

$$W(x,t) = \begin{cases} h(t) - \left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right), & a\leq t<x, \\ h(t) - \left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right), & x\leq t\leq b. \end{cases}$$

Proof. We have

$$\begin{aligned}
&(1-\alpha)^2 p(x) + \alpha\frac{p^{\Delta}(a)+p^{\Delta}(b)}{2\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right) + \alpha(1-\alpha)\frac{p(a)+p(b)}{2} \\
&\quad - \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t\right) \\
&\quad - \frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b g(t)p(\sigma(t))\Delta t \\
&= \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)p^{\Delta\Delta}(s)\Delta s\Delta t
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&(1-\alpha)^2 r(x) + \alpha\frac{r^{\Delta}(a)+r^{\Delta}(b)}{2\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right) + \alpha(1-\alpha)\frac{r(a)+r(b)}{2} \\
&\quad - \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t\right) \\
&\quad - \frac{1-\alpha}{\int_a^b g(t)\Delta t}\int_a^b g(t)r(\sigma(t))\Delta t \\
&= \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b\int_a^b W(x,t)W(t,s)r^{\Delta\Delta}(s)\Delta s\Delta t
\end{aligned} \tag{3.13}$$

Multiplying both sides of (3.12) and (3.13) by $r(x)$ and $p(x)$ respectively, adding the resultant identities and rewriting, we have:

$$
\begin{aligned}
&2(1-\alpha)^2 p(x)r(x) \\
&\quad + \frac{\alpha}{2\int_a^b g(t)\Delta t}\left(\int_a^b W(x,t)\Delta t\right)\left[r(x)\left(p^{\Delta}(a)+p^{\Delta}(b)\right)+p(x)\left(r^{\Delta}(a)+r^{\Delta}(b)\right)\right] \\
&\quad - \frac{1-\alpha}{\int_a^b g(t)\Delta t}\left(r(x)\int_a^b g(t)p(\sigma(t))\Delta t + p(x)\int_a^b g(t)r(\sigma(t))\Delta t\right) \\
&\quad + \alpha(1-\alpha)\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right) - \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2} \\
&\quad \times\left(\int_a^b W(x,t)\Delta t\right)\left(r(x)\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t + p(x)\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t\right) \\
&= \frac{1}{\left(\int_a^b g(t)\Delta t\right)^2}\left(r(x)\int_a^b\int_a^b W(x,t)W(t,s)p^{\Delta\Delta}(s)\Delta s\Delta t\right. \\
&\quad \left. + p(x)\int_a^b\int_a^b W(x,t)W(t,s)r^{\Delta\Delta}(s)\Delta s\Delta t\right).
\end{aligned}
$$

Using properties of the modulus, we get (3.11). □

Corollary 7. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 3, we have*

$$
\begin{aligned}
&\left|(1-\alpha)^2 p(x)r(x)\right. \\
&\quad + \frac{\alpha}{4\int_a^b g(t)\mathrm{d}t}\left(\int_a^b W(x,t)\mathrm{d}t\right)\left[r(x)\left(p'(a)+p'(b)\right)+p(x)\left(r'(a)+r'(b)\right)\right] \\
&\quad - \frac{1-\alpha}{2\int_a^b g(t)\mathrm{d}t}\left(r(x)\int_a^b g(t)p(t)\mathrm{d}t + p(x)\int_a^b g(t)r(t)\mathrm{d}t\right) \\
&\quad + \frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2}+p(x)\frac{r(a)+r(b)}{2}\right) \\
&\quad \left. - \frac{1}{2\left(\int_a^b g(t)\mathrm{d}t\right)^2}\left(\int_a^b W(x,t)\mathrm{d}t\right)\left(r(x)\int_a^b g(t)p'(t)\mathrm{d}t + p(x)\int_a^b g(t)r'(t)\mathrm{d}t\right)\right| \\
&\le \frac{1}{2\left(\int_a^b g(t)\mathrm{d}t\right)^2}\left(|r(x)|\,M + |p(x)|\,N\right)\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

where $g(t) = h'(t)$ *on* $[a, b]$ *and* $M = \sup_{a \le t < b} |p''(t)| < \infty$, $N = \sup_{a \le t < b} |r''(t)| < \infty$, *and*

$$W(x,t) = \begin{cases} h(t) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)\right), & a \le t < x, \\ h(t) - \left(\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b)\right), & x \le t \le b. \end{cases}$$

Corollary 8. *In the case of* $\mathbb{T} = \mathbb{Z}$ *in Theorem 3, we have*

$$\begin{aligned} &\Bigg|(1-\alpha)^2 p(x) r(x) \\ &\quad + \frac{\alpha}{4 \sum_{t=a}^{b-1} g(t)} \left(\sum_{t=a}^{b-1} W(x,t)\right) \left[r(x)\left(\Delta p(a) + \Delta p(b)\right) + p(x)\left(\Delta r(a) + \Delta r(b)\right)\right] \\ &\quad - \frac{1-\alpha}{2 \sum_{t=a}^{b-1} g(t)} \left(r(x) \sum_{t=a}^{b-1} g(t) p(t+1) + p(x) \sum_{t=a}^{b-1} g(t) r(t+1)\right) \\ &\quad + \frac{\alpha(1-\alpha)}{2} \left(r(x) \frac{p(a)+p(b)}{2} + p(x) \frac{r(a)+r(b)}{2}\right) - \frac{1}{2\left(\sum_{t=a}^{b-1} g(t)\right)^2} \\ &\quad \times \left(\sum_{t=a}^{b-1} W(x,t)\right) \left(r(x) \sum_{t=a}^{b-1} g(t) \Delta p(t+1) + p(x) \sum_{t=a}^{b-1} g(t) \Delta r(t+1)\right)\Bigg| \\ &\le \frac{1}{2\left(\sum_{t=a}^{b-1} g(t)\right)^2} \left(|r(x)| M + |p(x)| N\right) \sum_{t=a}^{b-1} \sum_{s=a}^{b-1} |W(x,t)| \, |W(t,s)|, \end{aligned}$$

where $g(t) = h(t+1) - h(t)$ *on* $[a, b-1]$,

$$M = \sup_{a \le t < b} \left|\Delta^2 p(t)\right| < \infty, \quad N = \sup_{a \le t < b} \left|\Delta^2 r(t)\right| < \infty,$$

and

$$W(x,t) = \begin{cases} h(t) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)\right), & a \le t < x-1, \\ h(t) - \left(\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b)\right), & x \le t \le b-1. \end{cases}$$

Corollary 9. *In the case of* $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\} (q > 1)$ *in Theorem 3, we have*

$$\begin{aligned} &\Bigg|(1-\alpha)^2 p(x) r(x) + \frac{\alpha}{4 \int_a^b g(t) \mathrm{d}_q t} \\ &\quad \times \left(\int_a^b W(x,t) \mathrm{d}_q t\right) \left[r(x)\left(\mathrm{D}_q p(a) + \mathrm{D}_q p(b)\right) + p(x)\left(\mathrm{D}_q r(a) + \mathrm{D}_q r(b)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&- \frac{1-\alpha}{2\int_a^b g(t)\mathrm{d}_q t}\left(r(x)\int_a^b g(t)p(qt)\mathrm{d}_q t + p(x)\int_a^b g(t)r(qt)\mathrm{d}_q t\right)\\
&+ \frac{\alpha(1-\alpha)}{2}\left(r(x)\frac{p(a)+p(b)}{2} + p(x)\frac{r(a)+r(b)}{2}\right) - \frac{1}{2\left(\int_a^b g(t)\mathrm{d}_q t\right)^2}\\
&\times\left(\int_a^b W(x,t)\mathrm{d}_q t\right)\left(r(x)\int_a^b g(t)\mathrm{D}_q p(qt)\mathrm{d}_q t + p(x)\int_a^b g(t)\mathrm{D}_q r(qt)\mathrm{d}_q t\right)\Bigg|\\
\leq\ &\frac{1}{2\left(\int_a^b g(t)\mathrm{d}_q t\right)^2}\left(|r(x)|\,M + |p(x)|\,N\right)\int_a^b\int_a^b |W(x,t)|\,|W(t,s)|\,\mathrm{d}_q s\,\mathrm{d}_q t,
\end{aligned}
$$

where $g(t) = (\mathrm{D}_q h)(t)$ *on* $[a,b]$,

$$
M = \sup_{a\leq t<b}\left|(\mathrm{D}_q^2 p)(t)\right| < \infty \quad \textit{and} \quad N = \sup_{a\leq t<b}\left|(\mathrm{D}_q^2 r)(t)\right| < \infty.
$$

Remark 2. In the case of $\alpha = 0$ and $h(t) = t$, Corollary 7 reduces to [2, Theorem 5].

Theorem 4. *Let* $0 \leq k \leq 1$, $g : [a,b] \to [0,\infty)$ *be rd continuous and positive and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\Delta}(t) = g(t)$ *on* $[a,b]$. *Let* $a, b, t, x \in \mathbb{T}$, $a < b$ *and* $p, r : [a,b] \to \mathbb{R}$ *be twice differentiable. Then for all* $x \in [a,b]$, *we have*

$$
\begin{aligned}
&\Bigg|(1-\alpha)^4 p(x)r(x)\\
&+ \frac{\alpha^2}{4\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)^2\left(p^{\Delta}(a)+p^{\Delta}(b)\right)\left(r^{\Delta}(a)+r^{\Delta}(b)\right)\\
&+ \alpha(1-\alpha)^2\left(\int_a^b W(x,t)\Delta t\right)\left(p(x)\frac{r^{\Delta}(a)+r^{\Delta}(b)}{2\int_a^b g(t)\Delta t} + r(x)\frac{p^{\Delta}(a)+p^{\Delta}(b)}{2\int_a^b g(t)\Delta t}\right)\\
&- \frac{(1-\alpha)^2}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\\
&\times\left(p(x)\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t + r(x)\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t\right)\\
&- \frac{(1-\alpha)^3}{\int_a^b g(t)\Delta t}\left(p(x)\int_a^b g(t)r(\sigma(t))\Delta t + r(x)\int_a^b g(t)p(\sigma(t))\Delta t\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)+\frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)\Delta t}\int_a^b W(x,t)\Delta t\\
&\times\left(\left(p^\Delta(a)+p^\Delta(b)\right)(r(a)+r(b))\ +\left(r^\Delta(a)+r^\Delta(b)\right)(p(a)+p(b))\right)\\
&-\frac{\alpha}{2\left(\int_a^b g(t)\Delta t\right)^3}\left(\int_a^b W(x,t)\Delta t\right)^2\left(\left(p^\Delta(a)+p^\Delta(b)\right)\left(\int_a^b g(t)r^\Delta(\sigma(t))\Delta t\right)\right.\\
&\left.+\left(r^\Delta(a)+r^\Delta(b)\right)\left(\int_a^b g(t)p^\Delta(\sigma(t))\Delta t\right)\right)\\
&-\frac{\alpha\,(1-\alpha)}{2\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\left(\left(p^\Delta(a)+p^\Delta(b)\right)\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\left(r^\Delta(a)+r^\Delta(b)\right)\int_a^b g(t)p(\sigma(t))\Delta t\right)-\frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)\\
&\times\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r^\Delta(\sigma(t))\Delta t\ +\frac{r(a)+r(b)}{2}\int_a^b g(t)p^\Delta(\sigma(t))\Delta t\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}\left(p(a)+p(b)\right)\left(r(a)+r(b)\right)\\
&+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b g(t)p(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)\Delta t}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^4}\left(\int_a^b W(x,t)\Delta t\right)^2\int_a^b g(t)p^\Delta(\sigma(t))\Delta t\int_a^b g(t)r^\Delta(\sigma(t))\Delta t\\
&+\frac{1-\alpha}{\left(\int_a^b g(t)\Delta t\right)^3}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)p^\Delta(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.\left.+\int_a^b g(t)r^\Delta(\sigma(t))\Delta t\int_a^b g(t)p(\sigma(t))\Delta t\right)\right|\\
\leq\ &\frac{MN}{\left(\int_a^b g(t)\Delta t\right)^4}\left(\int_a^b\int_a^b|W(x,t)|\,|W(t,s)|\,\Delta s\Delta t\right)^2, \qquad (3.14)
\end{aligned}
$$

where $M = \sup_{a \le t < b} |p^{\Delta\Delta}(t)| < \infty$, $N = \sup_{a \le t < b} |r^{\Delta\Delta}(t)| < \infty$, *and*

$$W(x,t) = \begin{cases} h(t) - \left(\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b)\right), & a \le t < x, \\ h(t) - \left(\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b)\right), & x \le t \le b. \end{cases}$$

Proof. Multiplying the left and right sides of the identities (3.12) and (3.13), we get

$$\begin{aligned}
&(1-\alpha)^4 p(x) r(x) \\
&\quad + \frac{\alpha^2}{4\left(\int_a^b g(t)\Delta t\right)^2} \left(\int_a^b W(x,t)\Delta t\right)^2 \left(p^\Delta(a) + p^\Delta(b)\right)\left(r^\Delta(a) + r^\Delta(b)\right) \\
&\quad + \alpha(1-\alpha)^2 \left(\int_a^b W(x,t)\Delta t\right) \left(p(x)\frac{r^\Delta(a) + r^\Delta(b)}{2\int_a^b g(t)\Delta t} + r(x)\frac{p^\Delta(a) + p^\Delta(b)}{2\int_a^b g(t)\Delta t}\right) \\
&\quad - \frac{(1-\alpha)^2}{\left(\int_a^b g(t)\Delta t\right)^2} \left(\int_a^b W(x,t)\Delta t\right) \\
&\quad \times \left(p(x)\int_a^b g(t) r^\Delta(\sigma(t))\Delta t + r(x)\int_a^b g(t) p^\Delta(\sigma(t))\Delta t\right) \\
&\quad - \frac{(1-\alpha)^3}{\int_a^b g(t)\Delta t} \left(p(x)\int_a^b g(t) r(\sigma(t))\Delta t + r(x)\int_a^b g(t) p(\sigma(t))\Delta t\right) \\
&\quad + \alpha(1-\alpha)^3 \left(p(x)\frac{r(a)+r(b)}{2} + r(x)\frac{p(a)+p(b)}{2}\right) + \frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)\Delta t}\int_a^b W(x,t)\Delta t \\
&\quad \times \left(\left(p^\Delta(a) + p^\Delta(b)\right)(r(a)+r(b)) + \left(r^\Delta(a) + r^\Delta(b)\right)(p(a)+p(b))\right) \\
&\quad - \frac{\alpha}{2\left(\int_a^b g(t)\Delta t\right)^3} \left(\int_a^b W(x,t)\Delta t\right)^2 \left(\left(p^\Delta(a) + p^\Delta(b)\right)\left(\int_a^b g(t) r^\Delta(\sigma(t))\Delta t\right)\right. \\
&\quad \left. + \left(r^\Delta(a) + r^\Delta(b)\right)\left(\int_a^b g(t) p^\Delta(\sigma(t))\Delta t\right)\right) \\
&\quad - \frac{\alpha(1-\alpha)}{2\left(\int_a^b g(t)\Delta t\right)^2} \left(\int_a^b W(x,t)\Delta t\right) \left(\left(p^\Delta(a) + p^\Delta(b)\right)\int_a^b g(t) r(\sigma(t))\Delta t\right. \\
&\quad \left. + \left(r^\Delta(a) + r^\Delta(b)\right)\int_a^b g(t) p(\sigma(t))\Delta t\right) - \frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)\Delta t\right)^2}\left(\int_a^b W(x,t)\Delta t\right)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}\left(p(a)+p(b)\right)\left(r(a)+r(b)\right)\\
&+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\Delta t\right)^2}\int_a^b g(t)p(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)\Delta t}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(\sigma(t))\Delta t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
&+\frac{1}{\left(\int_a^b g(t)\Delta t\right)^4}\left(\int_a^b W(x,t)\Delta t\right)^2\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t\\
&+\frac{1-\alpha}{\left(\int_a^b g(t)\Delta t\right)^3}\left(\int_a^b W(x,t)\Delta t\right)\left(\int_a^b g(t)p^{\Delta}(\sigma(t))\Delta t\int_a^b g(t)r(\sigma(t))\Delta t\right.\\
&\left.+\int_a^b g(t)r^{\Delta}(\sigma(t))\Delta t\int_a^b g(t)p(\sigma(t))\Delta t\right)\\
=&\frac{1}{\left(\int_a^b g(t)\Delta t\right)^4}\left(\int_a^b\int_a^b W(x,t)W(t,s)p^{\Delta\Delta}(s)\Delta s\Delta t\right)\\
&\times\left(\int_a^b\int_a^b W(x,t)W(t,s)r^{\Delta\Delta}(s)\Delta s\Delta t\right).
\end{aligned}
$$

Using properties of the modulus, we get (3.14). □

Corollary 10. *In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 4, we have*

$$
\begin{aligned}
&\Bigg|(1-\alpha)^4p(x)r(x)+\frac{\alpha^2}{4\left(\int_a^b g(t)\mathrm{d}t\right)^2}\left(\int_a^b W(x,t)\mathrm{d}t\right)^2\left(p'(a)+p'(b)\right)\left(r'(a)+r'(b)\right)\\
&+\alpha(1-\alpha)^2\left(\int_a^b W(x,t)\mathrm{d}t\right)\left(p(x)\frac{r'(a)+r'(b)}{2\int_a^b g(t)\mathrm{d}t}+r(x)\frac{p'(a)+p'(b)}{2\int_a^b g(t)\mathrm{d}t}\right)\\
&-\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\mathrm{d}t\right)^2}\left(\int_a^b W(x,t)\mathrm{d}t\right)\left(p(x)\int_a^b g(t)r'(t)\mathrm{d}t+r(x)\int_a^b g(t)p'(t)\mathrm{d}t\right)\\
&-\frac{(1-\alpha)^3}{\int_a^b g(t)\mathrm{d}t}\left(p(x)\int_a^b g(t)r(t)\mathrm{d}t+r(x)\int_a^b g(t)p(t)\mathrm{d}t\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha(1-\alpha)^3\left(p(x)\frac{r(a)+r(b)}{2}+r(x)\frac{p(a)+p(b)}{2}\right)+\frac{\alpha^2(1-\alpha)}{4\int_a^b g(t)\mathrm{d}t}\\
&\times\left(\int_a^b W(x,t)\mathrm{d}t\right)\left(\left(p'(a)+p'(b)\right)\left(r(a)+r(b)\right)\ +\left(r'(a)+r'(b)\right)\left(p(a)+p(b)\right)\right)\\
&-\frac{\alpha}{2\left(\int_a^b g(t)\mathrm{d}t\right)^3}\left(\int_a^b W(x,t)\mathrm{d}t\right)^2\left(\left(p'(a)+p'(b)\right)\left(\int_a^b g(t)r'(t)\mathrm{d}t\right)\right.\\
&\left.+\left(r'(a)+r'(b)\right)\left(\int_a^b g(t)p'(t)\mathrm{d}t\right)-\frac{\alpha\left(1-\alpha\right)}{2\left(\int_a^b g(t)\mathrm{d}t\right)^2}\right.\\
&\times\int_a^b W(x,t)\mathrm{d}t\left(\left(p'(a)+p'(b)\right)\int_a^b g(t)r(t)\mathrm{d}t\ +\left(r'(a)+r'(b)\right)\int_a^b g(t)p(t)\mathrm{d}t\right)\\
&+\frac{\alpha^2(1-\alpha)^2}{4}\left(p(a)+p(b)\right)\left(r(a)+r(b)\right)-\frac{\alpha(1-\alpha)}{\left(\int_a^b g(t)\mathrm{d}t\right)^2}\\
&\times\int_a^b W(x,t)\mathrm{d}t\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r'(t)\mathrm{d}t\ +\frac{r(a)+r(b)}{2}\int_a^b g(t)p'(t)\mathrm{d}t\right)\\
&+\frac{(1-\alpha)^2}{\left(\int_a^b g(t)\mathrm{d}t\right)^2}\int_a^b g(t)p(t)\mathrm{d}t\int_a^b g(t)r(t)\mathrm{d}t\\
&-\frac{\alpha(1-\alpha)^2}{\int_a^b g(t)\mathrm{d}t}\left(\frac{p(a)+p(b)}{2}\int_a^b g(t)r(t)\mathrm{d}t+\frac{r(a)+r(b)}{2}\int_a^b g(t)p(t)\mathrm{d}t\right)\\
&+\frac{1}{\left(\int_a^b g(t)\mathrm{d}t\right)^4}\left(\int_a^b W(x,t)\mathrm{d}t\right)^2\left(\int_a^b g(t)p'(t)\mathrm{d}t\right)\left(\int_a^b g(t)r'(t)\mathrm{d}t\right)\\
&+\frac{1-\alpha}{\left(\int_a^b g(t)\mathrm{d}t\right)^3}\left(\int_a^b W(x,t)\mathrm{d}t\right)\\
&\times\left.\left(\int_a^b g(t)p'(t)\mathrm{d}t\int_a^b g(t)r(t)\mathrm{d}t\ +\int_a^b g(t)r'(t)\mathrm{d}t\int_a^b g(t)p(t)\mathrm{d}t\right)\right|\\
&\leq\frac{MN}{\left(\int_a^b g(t)\mathrm{d}t\right)^4}\left(\int_a^b\int_a^b\left|W(x,t)\right|\left|W(t,s)\right|\mathrm{d}s\mathrm{d}t\right)^2,
\end{aligned}
$$

where $g(t)=h'(t)$ *on* $[a,b]$ *and* $M=\sup\limits_{a\leq t<b}|p''(t)|<\infty$, $N=\sup\limits_{a\leq t<b}|r''(t)|<\infty$, *and*

$$W(x,t)=\begin{cases} h(t)-\left(\left(1-\dfrac{\alpha}{2}\right)h(a)+\dfrac{\alpha}{2}h(b)\right), & a\le t<x,\\ h(t)-\left(\dfrac{\alpha}{2}h(a)+\left(1-\dfrac{\alpha}{2}\right)h(b)\right), & x\le t\le b.\end{cases}$$

Remark 3. In the case of $\alpha=0$ and $h(t)=t$, Corollary 10 reduces to [2, Theorem 4].

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