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On the rigidity of spacelike hypersurfaces immersed in the steady state space $\mathcal{H}^{n+1}$

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Abstract. In this paper, as a suitable application of the well known generalized Maximum Principle of Omori–Yau, we obtain rigidity results concerning to complete spacelike hypersurfaces immersed in the half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models the so-called *steady state space*. Moreover, by using an isometrically equivalent model for $\mathcal{H}^{n+1}$, we extend our results to a wider family of spacetimes. Finally, we also study the uniqueness of entire vertical graphs in such ambient spacetimes.

1. Introduction

In the last years, the study of spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ has been of substantial interest from both the physical and mathematical aspects. From a physical point of view, that interest is motivated by their role in the study of different problems in general relativity. From a mathematical point of view, that interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. For example, R. Aiyama in [1] and Y. L. Xin in [27] simultaneous and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $\mathbb{L}^{n+1}$ having the image of its Gauss map contained in a geodesic ball of the hyperbolic space (see also [24] for a weaker first version of this result

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given by B. Palmer). In [6], among other interesting results, J. A. Aledo and L. J. Alías characterized the spacelike hyperplanes in $\mathbb{L}^{n+1}$ as the only complete spacelike hypersurfaces with constant mean curvature which are bounded between two parallel spacelike hyperplanes.

As for the case of de Sitter space $\mathbb{S}_1^{n+1}$, Goddard in [14] conjectured that every complete spacelike hypersurface with constant mean curvature should be totally umbilical. Although the conjecture turned out to be false in its original form, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, J. A. Aledo and L. J. Alías [7] showed that a complete spacelike hypersurface in $\mathbb{S}_1^{n+1}$ such that its image under the Gauss map is contained in a hyperbolic geodesic ball is necessarily compact. As an application of their result, they also conclude that Goddard's conjecture is true under the assumption that the hyperbolic image of the hypersurface is bounded.

In this paper we are concerning to complete spacelike hypersurfaces immersed with bounded mean curvature in the half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models the so-called *steady state space* (cf. Section 2). The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that, in Cosmology, $\mathcal{H}^4$ is the steady state model of the universe proposed by H. Bondi and T. Gold [10], and F. Hoyle [16], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. [26], Section 14.8, and [15], Section 5.2).

Related to our work, A. L. Albujer and L. J. Alías [3] have proved that if a complete spacelike hypersurface with constant mean curvature is bounded away from the infinity of the steady state space $\mathcal{H}^{n+1}$, then its mean curvature must be identically 1. As a consequence of this result, they concluded that the only complete spacelike surfaces with constant mean curvature in $\mathcal{H}^3$ which are bounded away from the infinity are the totally umbilical flat surfaces.

In [11], the second author jointly with A. Caminha have studied complete vertical graphs of constant mean curvature in the hyperbolic and steady state spaces. They first derived suitable formulas for the Laplacians of the height function h and of a support-like function naturally attached to the graph; then, under appropriate restrictions on the values of the mean curvature and the growth of the height function, they obtained necessary conditions for the existence of such a graph. Further, in the 3-dimensional case, they proved Bernstein-type results in each of these ambient spaces.

More recently, the authors have obtained in [12] height estimates concerning to a compact spacelike hypersurface Σ^n immersed with constant mean curvature

H in $\mathcal{H}^{n+1}$, when its boundary is contained into some hyperplane of this spacetime. Furthermore, they apply these estimates to describe the end of a complete spacelike hypersurface and to get theorems of characterization concerning to spacelike hyperplanes in $\mathcal{H}^{n+1}$.

Motivated by the works described above, as a suitable application of the well known generalized Maximum Principle of Omori–Yau [22], [28], we obtain rigidity results in the steady state space $\mathcal{H}^{n+1}$. At first, by imposing a restriction on the normal hyperbolic angle of the hypersurface (that is, the hyperbolic angle between the Gauss map of the hypersurface and the unitary timelike vector field which determines on $\mathcal{H}^{n+1}$ a codimension one spacelike foliation by hyperplanes; see Sections 4 and 2), we prove the following (cf. Theorem 3.2; see also Corollary 3.5).

Let $\psi : \Sigma^n \to \mathcal{H}^{n+1} \subset \mathbb{S}_1^{n+1}$ be a complete spacelike hypersurface bounded away from the future infinite of $\mathcal{H}^{n+1}$, with bounded mean curvature $H \geq 1$. If the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \leq \inf_\Sigma H$, then Σ^n is a hyperplane and its hyperbolic image is exactly a horosphere.

At this point notice that, since there is not exist complete noncompact spacelike hypersurfaces with constant mean curvature $0 \leq H < 1$ in the steady state space which are umbilical, it is natural to consider the restriction $H \geq 1$ (see Remark 3.3). Moreover, we also observe that when Σ^n is a compact spacelike hypersurface immersed with constant mean curvature $H > 1$ in $\mathcal{H}^{n+1}$, and with its boundary $\partial\Sigma$ contained into a spacelike hyperplane of $\mathcal{H}^{n+1}$, a gradient estimate due to S. Montiel (cf. [21], Theorem 7) guarantees that the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \leq H$ (see Remark 3.4).

In Section 5, we reobtain that previous result in the context of the *steady state type* spacetimes (see Theorems 5.1). Moreover, by applying a classical result due to A. Huber [17] concerned with parabolic surfaces, we also obtain another rigidity result concerning to the 3-dimensional case (cf. Theorem 5.3). Finally, in Section 6, we study the rigidity of entire vertical graphs in such ambient spacetimes (cf. Corollaries 6.1 and 6.2).

2. The steady state space $\mathcal{H}^{n+1}$

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz–Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{p \in L^{n+2};\ \langle p, p\rangle = 1\right\}.$$

The induced metric from $\langle\, ,\rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Moreover, if $p \in \mathbb{S}_1^{n+1}$, we can put

$$T_p\left(\mathbb{S}_1^{n+1}\right) = \left\{v \in \mathbb{L}^{n+2};\ \langle v, p\rangle = 0\right\}.$$

Let $a \in \mathbb{L}^{n+2}$ be a past-pointing null vector, that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \dots, 0, 1)$. Then the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{x \in \mathbb{S}_1^{n+1};\ \langle x, a\rangle > 0\right\}$$

is the so-called *steady state space*. Observe that $\mathcal{H}^{n+1}$ is a non-complete manifold, being only half of the de Sitter space. Its boundary, as a subset of $\mathbb{S}_1^{n+1}$, is the null hypersurface

$$\left\{x \in \mathbb{S}_1^{n+1};\ \langle x, a\rangle = 0\right\},$$

whose topology is that of $\mathbb{R} \times \mathbb{S}^{n-1}$ (cf. [21]).

A smooth immersion $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if ψ induces a Riemannian metric on Σ^n, which as usual is also denoted by $\langle\, ,\rangle$. In that case, there exists a unique unitary timelike normal field N globally defined on Σ^n which is future-directed (that is, $\langle N, e_{n+2}\rangle < 0$). Throughout this paper we will refer to N as the future-pointing Gauss map of Σ^n. The mean curvature function of a spacelike hypersurface Σ^n is defined as

$$H = -\frac{1}{n}\operatorname{tr}(A),$$

where A stands for the shape operator (or second fundamental form) of Σ^n with respect to its future-pointing Gauss map N.

Now, we shall consider in $\mathcal{H}^{n+1}$ the timelike field

$$\mathcal{K} = -\langle x, a\rangle\, x + a.$$

We easily see that

$$\overline{\nabla}_V \mathcal{K} = -\langle x, a\rangle\, V, \quad \forall\, V \in \mathfrak{X}(\mathcal{H}^{n+1}),$$

that is, $\mathcal{K}$ is closed and conformal field on $\mathcal{H}^{n+1}$ (cf. [18], Section 5). Then, from Proposition 1 of [20], we have that the n-dimensional distribution $\mathcal{D}$ defined on $\mathcal{H}^{n+1}$ by

$$p \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(p) = \left\{v \in T_p\mathcal{H}^{n+1};\ \langle \mathcal{K}(p), v\rangle = 0\right\}$$

determines a codimension one spacelike foliation $\mathcal{F}(\mathcal{K})$ which is oriented by $\mathcal{K}$. Moreover, from Example 1 of [19], we conclude that the leaves of $\mathcal{F}(\mathcal{K})$ are given by

$$\mathcal{L}_\tau = \left\{x \in \mathbb{S}_1^{n+1};\ \langle x, a\rangle = \tau\right\}, \quad \tau > 0,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$ isometric to the Euclidean space $\mathbb{R}^n$, and having constant mean curvature one with respect to the unit normal fields

$$N_\tau = x - \frac{1}{\tau}a, \quad x \in \mathcal{L}_\tau. \tag{2.1}$$

Remark 2.1. An explicit isometry between the leaves $\mathcal{L}_\tau$ and $\mathbb{R}^n$ can be found at Section 2 of [3].

According to [3], we will say that a spacelike hypersurface Σ^n in $\mathcal{H}^{n+1}$ is *bounded away from the future infinity* if there exists $\overline{\tau} > 0$ such that

$$\psi(\Sigma) \subset \{x \in \mathcal{H}^{n+1};\ \langle x, a\rangle \le \overline{\tau}\},$$

and we will say that it is *bounded away from the past infinity* if there exists $\underline{\tau} > 0$ such that

$$\psi(\Sigma) \subset \{x \in \mathcal{H}^{n+1};\ \langle x, a\rangle \ge \underline{\tau}\}.$$

We will say that Σ^n is *bounded away from the infinity* if it is both bounded away from the past and the future infinity. In other words, Σ^n is bounded away from the infinity if there exist $0 < \underline{\tau} < \overline{\tau}$ such that $\psi(\Sigma)$ is contained in the slab bounded by $\mathcal{L}_{\underline{\tau}}$ and $\mathcal{L}_{\overline{\tau}}$.

3. Rigidity theorems in $\mathcal{H}^{n+1}$

We observe that the Gauss map N of a spacelike hypersurface Σ^n immersed in the steady state space $\mathcal{H}^{n+1}$ can be thought of as a map

$$N : \Sigma^n \to \mathbb{H}^{n+1}$$

taking values in the hyperbolic space

$$\mathbb{H}^{n+1} = \left\{x \in \mathbb{L}^{n+2};\ \langle x, x\rangle = -1,\ \langle x, a\rangle < 0\right\},$$

where a is a non-zero null vector in $\mathbb{L}^{n+2}$, which will be chosen as in the previous section. In this setting, the image $N(\Sigma)$ is called the *hyperbolic image* of Σ^n.

Furthermore, we note that all the horospheres of $\mathbb{H}^{n+1}$ can be realized in the Minkowski model in the following way

$$L_\rho = \left\{x \in \mathbb{H}^{n+1};\ \langle x, a\rangle = \rho\right\}, \quad \rho > 0.$$

In this setting, we refer to the *normal hyperbolic angle* θ of Σ^n as being the hyperbolic angle between the future-pointing Gauss map N of Σ^n and the unitary timelike vector field $\nu = \frac{\mathcal{K}}{|\mathcal{K}|}$, where $|\mathcal{K}| = \sqrt{-\langle \mathcal{K}, \mathcal{K}\rangle}$. In other words, $\cosh\theta = -\langle N, \nu\rangle$.

In order to establish our results, we also will need the well known generalized maximum principle due to H. Omori and S. T. Yau [22], [28].

Lemma 3.1. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from above on Σ^n. Then there is a sequence of points $\{p_k\}$ in Σ^n such that*

$$\lim_{k\to\infty} u(p_k) = \sup_{\Sigma} u, \quad \lim_{k\to\infty} |\nabla u(p_k)| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \Delta u(p_k) \le 0.$$

Now, we can state and prove our first result

Theorem 3.2. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface bounded away from the future infinite of $\mathcal{H}^{n+1}$, with bounded mean curvature $1 \le H \le \alpha$, for some constant α. If the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \le \inf_\Sigma H$, then Σ^n is a hyperplane and its hyperbolic image is exactly a horosphere.*

Proof. Let us recall that the Gauss equation of Σ^n in $\mathcal{H}^{n+1}$ describes the curvature of Σ^n, denoted by R, in terms of its shape operator A, and it is given by

$$\langle R(X,Y)X, Y\rangle = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2 - \langle AX, X\rangle\langle AY, Y\rangle + \langle AX, Y\rangle^2,$$

being $X, Y \in \mathfrak{X}(\Sigma)$. Thus, for all $X \in \mathfrak{X}(\Sigma)$ with $|X| = 1$, we have

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) &= n - 1 + nH\langle AX, X\rangle + \langle AX, AX\rangle \\ &= n - 1 + \left|AX + \frac{nH}{2}X\right|^2 - \frac{n^2H^2}{4}|X|^2,\end{aligned}$$

where Ric_Σ stands for the Ricci curvature of Σ^n. Hence,

$$\mathrm{Ric}_\Sigma \ge n - 1 - \frac{n^2H^2}{4}. \tag{3.1}$$

Thus, since we are supposing that $1 \leq H \leq \alpha$ for some constant α, from (3.1) we get

$$\mathrm{Ric}_\Sigma \geq n - 1 - \frac{n^2\alpha^2}{4},$$

that is, Ric_Σ is bounded from below on Σ^n.

Now, let $u : \Sigma^n \to \mathbb{R}$ be a smooth function given by

$$u(p) = \langle \psi(p), a \rangle,$$

for all $p \in \Sigma^n$. We easily see that the gradient of u in is

$$\nabla u = a^\top,$$

where $a^\top$ denotes the tangential component of a along Σ^n, that is,

$$a = a^\top - \langle N, a \rangle N + \langle \psi, a \rangle \psi = \nabla u - \langle N, a \rangle N + \langle \psi, a \rangle \psi. \tag{3.2}$$

Using Gauss and Weingarten formulas, we obtain

$$\nabla_X \nabla u = -\langle N, a \rangle AX - uX$$

for every $X \in \mathfrak{X}(\Sigma)$. Therefore, the Laplacian of the function u on Σ^n is given by

$$\Delta u = nH\langle N, a \rangle - nu. \tag{3.3}$$

From equation (3.2), it is also easy to see that

$$|\nabla u|^2 = \langle N, a \rangle^2 - u^2. \tag{3.4}$$

Recall here that a is a past-pointing null vector and N is future-pointing, so that $\langle N, a \rangle > 0$.

On the other hand, since Σ^n is supposed to be bounded away from the future infinite of $\mathcal{H}^{n+1}$ and Ric_Σ is bounded from below, we are in position to apply Lemma 3.1 to the function u, obtaining a sequence $\{p_k\}$ in Σ^n such that

$$\lim_{k\to\infty} u(p_k) = \sup_\Sigma u, \quad \lim_{k\to\infty} |\nabla u(p_k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta u(p_k) \leq 0.$$

Consequently, since the mean curvature H is bounded on Σ^n, from equations (3.3) and (3.4), we get a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that

$$0 \geq \lim_{j\to\infty} \Delta u(p_{k_j}) = n \sup_\Sigma u \Big(\lim_{j\to\infty} H(p_{k_j}) - 1 \Big) \geq 0.$$

Then, since $\sup_\Sigma u > 0$, we get $\lim_{j\to\infty} H(p_{k_j}) = 1$ and, hence, $\inf_\Sigma H = 1$. Thus, from our hypothesis on the normal hyperbolic angle θ of Σ^n, $\cosh\theta = 1$ on Σ^n. Therefore, we see that Σ^n is a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$. Moreover, from equation (2.1), we get

$$\langle N, a\rangle = \langle N_\tau, a\rangle = \langle \psi, a\rangle = \tau$$

and, hence, we conclude that the hyperbolic image of Σ^n is exactly the horosphere L_τ. □

Remark 3.3. As a consequence of Bonnet-Myers theorem, a complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ having (not necessarily constant) mean curvature H, such that $|H| \leq c < \frac{2\sqrt{n-1}}{n}$ (c a positive real constant), has to be compact. In fact, for such a bound on H, from (3.1) we get

$$\mathrm{Ric}_\Sigma \geq (n-1) - \frac{n^2c^2}{4} > 0.$$

However, if Σ^n is bounded away from the future infinity of $\mathcal{H}^{n+1}$, then Σ^n is diffeomorphic to $\mathbb{R}^n$; in particular, $\mathcal{H}^{n+1}$ does not possess any compact (without boundary) spacelike hypersurface (cf. [3], Lemma 1). Furthermore, we observe that $\frac{2\sqrt{n-1}}{n} \leq 1$ for $n \geq 2$. On the other hand, taking into account the classification of totally umbilical spacelike hypersurfaces of the de Sitter space (cf. [19], Example 1), it follows from the main theorem of [2] that there exists no totally umbilical complete immersed spacelike hypersurfaces with mean curvature $0 \leq H < 1$ in the steady state space. Therefore, motivated by these reasons, it is natural to restrict our attention to spacelike hypersurfaces immersed with mean curvature $H \geq 1$ in $\mathcal{H}^{n+1}$.

Remark 3.4. Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a spacelike immersion from a compact manifold Σ^n with mean convex boundary $\partial\Sigma$ contained into a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$. Suppose that ψ has constant mean curvature $H > 1$. From Theorem 7 of [21] and taking into account our choice of orientation of Σ^n, we get

$$0 < \langle N, a\rangle \leq H\tau. \tag{3.5}$$

Consequently, from (2.1) and (3.5), we conclude that the normal hyperbolic angle θ of Σ^n satisfies

$$\cosh\theta = -\langle N, \nu\rangle \leq -\langle N, N_\tau\rangle = \frac{1}{\tau}\langle N, a\rangle \leq H.$$

From Theorem 3.2 we obtain the following

Corollary 3.5. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface bounded away from the infinite of $\mathcal{H}^{n+1}$, with bounded mean curvature $1 \leq H \leq \alpha$, for some constant α. Suppose that the hyperbolic image $N(\Sigma)$ is contained in the closure of the interior domain enclosed by a horosphere L_ρ. If $\frac{\rho}{\tau} \leq \inf_\Sigma H$, where $\tau > 0$ is such that $\langle \psi(p), a\rangle \geq \tau$ for all $p \in \Sigma^n$, then Σ^n is a hyperplane and its hyperbolic image is exactly a horosphere.*

Proof. Initially, we observe that the normal hyperbolic angle θ of Σ^n is such that

$$\cosh\theta = -\langle N, \nu\rangle = -\langle N, \psi - \frac{1}{\langle \psi, a\rangle} a\rangle = \frac{1}{\langle \psi, a\rangle}\langle N, a\rangle.$$

Consequently, since we are supposing that Σ^n is over the hyperplane $\mathcal{L}_\tau$,

$$\cosh\theta \leq \frac{1}{\tau}\langle N, a\rangle.$$

Therefore, our hypothesis on the hyperbolic image of Σ^n amounts to

$$\cosh\theta \leq \frac{\rho}{\tau} \leq \inf_\Sigma H$$

and, hence, we finish the proof by applying Theorem 3.2. □

Remark 3.6. Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a spacelike immersion from a compact manifold Σ^n with mean convex boundary $\partial\Sigma$ contained into a hyperplane $\mathcal{L}_\tau$, for some $\tau > 0$. One can reason as in Remark 3.4 to conclude that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere L_ρ, with $\frac{\rho}{\tau} \leq H$.

4. Generalized Robertson–Walker spacetimes

Let M^n be a connected, n-dimensional ($n \geq 2$) Riemannian manifold, $I \subset \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the factors I and M, respectively.

A particular class of Lorentzian manifolds is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$\langle v, w\rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w\rangle + (f \circ \pi_I)(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$. Such a space is called (following the terminology introduced in [9]) a *generalized Robertson-Walker* (GRW) spacetime, and in what follows we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. In particular, when M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a *Robertson-Walker* (RW) spacetime (cf. [23]).

A smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted by $\langle\, ,\rangle$. In this setting, $H = -\frac{1}{n}\operatorname{tr}(A)$ is the mean curvature of Σ^n.

For each $t \in I$, we orient the (spacelike) *slice* $M^n_t = \{t\} \times M^n$ by using its unit normal vector field ∂_t. According to [9], M^n_t has constant mean curvature $H = \frac{f'}{f}(t)$ with respect to ∂_t.

We observe that, since ∂_t is a unitary timelike vector field globally defined on the ambient spacetime, there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation as ∂_t. By using the Cauchy–Schwarz inequality, we get

$$\langle N, \partial_t\rangle \le -1 < 0 \quad \text{on } \Sigma^n.$$

We will refer to that normal vector field N as the future-pointing Gauss map of the spacelike hypersurface Σ^n. In the context of the RW spacetimes, the normal hyperbolic angle θ of Σ^n is the smooth function $\theta : \Sigma^n \to [0, +\infty)$ given by

$$\cosh\theta = -\langle N, \partial_t\rangle. \tag{4.1}$$

As in the previous section, we say that a spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ is *bounded away from the future infinity* of $-I \times_f M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_f M^n;\ t \le \bar{t}\,\}.$$

Analogously, we say that Σ^n is *bounded away from the past infinity* of $-I \times_f M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -I \times_f M^n; t \ge \underline{t}\}.$$

Finally, Σ^n is said to be *bounded away from the infinity* of $-I \times_f M^n$ if it is both bounded away from the past and future infinity of $-I \times_f M^n$. In other words, Σ^n is bounded away from the infinity if there exist $\underline{t} < \bar{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $M^n_{\underline{t}}$ and $M^n_{\bar{t}}$.

Now, let h denote the (vertical) height function naturally attached to Σ^n, namely, $h = (\pi_I)|_\Sigma$. Let $\overline{\nabla}$ and ∇ denote gradients with respect to the metrics of $I \times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $I \times M^n$ is given by

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t \rangle = -\partial_t, \tag{4.2}$$

so, from (4.2), we have that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \langle N, \partial_t \rangle N. \tag{4.3}$$

In particular, from (4.3) and (4.1), we get

$$|\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1 = \cosh^2\theta - 1, \tag{4.4}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n.

The formula collected in the following lemma is a particular case of one obtained by L. J. Alías jointly with the first author (cf. [8], Lemma 4.1).

Lemma 4.1. *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N. Then, by denoting $h = \pi_I \circ \psi$ the height function of Σ, we have*

$$\Delta h = -(\ln f)'(h)(n + |\nabla h|^2) - nH\langle N, \partial_t \rangle.$$

5. Steady state type spacetimes

We observe that the steady state space $\mathcal{H}^{n+1}$ can also be expressed in an isometrically equivalent way as the RW spacetime

$$-\mathbb{R} \times_{e^t} \mathbb{R}^n.$$

To see it, take $b \in \mathbb{L}^{n+2}$ another null vector such that $\langle a, b \rangle = 1$ and let $\Phi : \mathcal{H}^{n+1} \to -\mathbb{R} \times_{e^t} \mathbb{R}^n$ be the map given by

$$\Phi(x) = \left(\ln(\langle x, a \rangle), \frac{x - \langle x, a \rangle b - \langle x, b \rangle a}{\langle x, a \rangle} \right).$$

Then it can easily be checked that Φ is an isometry between both spaces which conserves time orientation (see [3], Section 4). In particular, for all $\tau > 0$, we have that

$$\Phi(\mathcal{L}_\tau) = \{\ln \tau\} \times \mathbb{R}^n$$

and

$$\Phi_*(N_\tau) = \partial_t.$$

Following the ideas of A. L. Albujer and L. J. Alías [3], we now consider a natural extension of the steady state space $\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n$. Let M^n be a connected n-dimensional Riemannian manifold and consider the GRW spacetime

$$-\mathbb{R} \times_{e^t} M^n.$$

We will refer to such wider family of GRW spacetimes as *steady state type* spacetimes. For instance, when M^n is the flat n-torus we get the de Sitter cusp as defined in [13].

In this setting, by applying a similar procedure as in the proof of Theorem 3.2, we obtain the following

Theorem 5.1. *Let M^n be a complete Riemannian manifold with nonnegative sectional curvature and let $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete spacelike hypersurface bounded away from the future infinite of $-\mathbb{R} \times_{e^t} M^n$, with bounded mean curvature $1 \leq H \leq \alpha$, for some constant α. If the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \leq \inf_\Sigma H$, then Σ^n is a slice M^n_t, for some $t \in \mathbb{R}$.*

Proof. From Lemma 4.1 and (4.1), we have that

$$\Delta h = n\left(-H\langle N, \partial_t\rangle - 1\right) - |\nabla h|^2 = n\left(H\cosh\theta\rangle - 1\right) - |\nabla h|^2. \qquad (5.1)$$

On the other hand, since we are supposing that H is bounded, from inequality (16) of [3] we have that the Ricci curvature of Σ^n is bounded from below. Thus, since Σ^n is supposed to be bounded away from the future infinite of $\mathcal{H}^{n+1}$, we are in position to apply Lemma 3.1 to the function h, obtaining a sequence $\{p_k\}$ in Σ^n such that

$$\lim_{k\to\infty} h(p_k) = \sup_\Sigma h, \quad \lim_{k\to\infty} |\nabla h(p_k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta h(p_k) \leq 0.$$

Consequently, since $\cosh\theta$ is also bounded on Σ^n, from (5.1) we have that

$$0 \geq \lim_{j\to\infty} \Delta h(p_{k_j}) \geq n\Big(\lim_{j\to\infty} H(p_{k_j}) - 1\Big) \geq 0,$$

for some subsequence $\{p_{k_j}\}$ of $\{p_k\}$. Then, $\lim_{j\to\infty} H(p_{k_j}) = 1$ and, hence, $\inf_\Sigma H = 1$. Thus, from our hypothesis on the normal hyperbolic angle θ of Σ^n, we conclude that $\cosh\theta = 1$ on Σ^n. Therefore, Σ^n is a slice M^n_t, for some $t \in \mathbb{R}$. □

Remark 5.2. From Lemma 7 of [3], we see that if a steady state type spacetime $-\mathbb{R} \times_{e^t} M^n$ admits a complete spacelike hypersurface Σ^n which is bounded away from the future infinity, then the Riemannian fiber M^n is necessarily complete. On the other hand, by supposing that M^n has nonnegative sectional curvature and that the spacelike hypersurface Σ^n has bounded mean curvature, from the inequality (16) of [3] we get that the Ricci curvature of Σ^n is bounded from below. Consequently, as in Remark 3.3, when the mean curvature H of Σ^n satisfies $|H| \le c < \frac{2\sqrt{n-1}}{n}$ ($c > 0$ constant) we conclude that Σ^n must be compact. Hence, in this case, the ambient steady state type spacetime is necessarily *spatially closed* (that is, its Riemannian fiber is compact; see [9], Proposition 3.2).

In the 3-dimensional case, we obtain the following rigidity result concerning to complete spacelike surfaces of nonnegative Gaussian curvature

Theorem 5.3. *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature and let $\psi : \Sigma^2 \to -\mathbb{R} \times_{e^t} M^2$ be a complete spacelike surface of nonnegative Gaussian curvature, with mean curvature $H \ge 1$. If the normal hyperbolic angle θ of Σ^2 satisfies $\cosh\theta \le H$, then Σ^2 is a slice M_t^2, for some $t \in \mathbb{R}$.*

Proof. By applying Lemma 4.1, we get

$$\Delta e^{-h} = e^{-h}\left(|\nabla h|^2 - \Delta h\right) = 2e^{-h}\left(|\nabla h|^2 + 1 + H\langle N, \partial_t\rangle\right). \tag{5.2}$$

Thus, from (4.1), (4.4) and (5.2), we obtain that

$$\Delta e^{-h} = 2e^{-h}\cosh\theta\,(\cosh\theta - H)$$

and, hence, our hypothesis on the normal hyperbolic angle θ of Σ^2 guarantees that the function e^{-h} is a superharmonic positive function on Σ. However, a classical result due to A. Huber [17] assures that complete surfaces of nonnegative Gaussian curvature must be parabolic. Therefore, h is constant on Σ^2, that is, Σ^2 is a slice M_t^2, for some $t \in \mathbb{R}$. □

Remark 5.4. In Proposition 13 of [21], S. Montiel have proved that when Σ^n is a complete spacelike hypersurface immersed with constant mean curvature $H \ge 1$ in $\mathbb{S}_1^{n+1}$, by supposing that the hyperbolic image of Σ^n is contained in the closure of the interior domain enclosed by a horosphere, then we have that $H = 1$. When $n = 2$, from the main theorem of [2] (see also [25]), this implies that Σ^2 is also an umbilical surface and its hyperbolic image is exactly a horosphere.

6. Entire vertical graphs in $-\mathbb{R}\times_{e^t} M^n$

Let $\Omega \subseteq M^n$ be a connected domain of M^n. A *vertical graph* over Ω is determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$ and it is given by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset -\mathbb{R}\times_{e^t} M^n.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma^n(u)$ is

$$\langle\, ,\rangle = -du^2 + e^{2u}\langle\, ,\rangle_{M^n}. \tag{6.1}$$

The graph is said to be entire if $\Omega = M^n$. It can be easily seen that a graph $\Sigma^n(u)$ is a spacelike hypersurface if and only if $|Du|^2_{M^n} < e^{2u}$, Du being the gradient of u in Ω and $|Du|_{M^n}$ its norm, both with respect to the metric $\langle\, ,\rangle_{M^n}$ in Ω. Observe that by Lemma 3.1 in [9], in the case where M^n is a simply connected manifold, every complete spacelike hypersurface Σ^n bounded away from the infinity of $-\mathbb{R}\times_{e^t} M^n$ is an entire spacelike graph in such space. However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric (6.1) is not necessarily complete on M^n.

In this context, by using the ideas of [4], we obtain the following non-parametric result

Corollary 6.1. *Let M^n be a complete Riemannian manifold with nonnegative sectional curvature and let $\Sigma^n(u)$ be an entire spacelike vertical graph bounded away from the infinity of $-\mathbb{R}\times_{e^t} M^n$ and with bounded mean curvature $1 \leq H \leq \alpha$, for some constant α. If*

$$|Du|^2_{M^n} \leq e^{2u}\left(1 - \sup_{\Sigma(u)} \frac{1}{H^2}\right), \tag{6.2}$$

then $\Sigma^n(u)$ is a slice.

Proof. Observe first that, under the assumptions of the theorem, $\Sigma^n(u)$ is a complete hypersurface. In fact, from (6.1) and the Cauchy–Schwarz inequality we get

$$\langle X, X\rangle = -\langle Du, X\rangle^2_{M^n} + e^{2u}\langle X, X\rangle_{M^n} \geq \left(e^{2u} - |Du|^2_{M^n}\right)\langle X, X\rangle_{M^n},$$

for every tangent vector field X on $\Sigma^n(u)$. Therefore,

$$\langle X, X\rangle \geq c\langle X, X\rangle_{M^n}$$

for the positive constant $c = e^{2\inf_{\Sigma(u)} u} \sup_{\Sigma(u)} \frac{1}{H^2}$. This implies that $L \geq \sqrt{c} L_{M^n}$, where L and L_{M^n} denote the length of a curve on $\Sigma^n(u)$ with respect to the Riemannian metrics $\langle\, , \rangle$ and $\langle\, , \rangle_{M^n}$, respectively. As a consequence, as M^n is complete by assumption, the induced metric on $\Sigma^n(u)$ from the metric of $-\mathbb{R} \times_{e^t} M^n$ is also complete.

On the other hand, we have that

$$N = -\langle N, \partial_t \rangle \partial_t + N^*, \tag{6.3}$$

where N^* denotes the projection of N onto the fiber M^n. Consequently, from (4.3) and (6.3), we obtain

$$N^{*\top} = -\langle N, \partial_t \rangle \nabla h \tag{6.4}$$

and

$$|\nabla h|^2 = e^{2h} \langle N^*, N^* \rangle_{M^n}. \tag{6.5}$$

Moreover, with a straightforward computation we verify that

$$N = \frac{e^u}{\sqrt{e^{2u} - |Du|^2_{M^n}}} \left(\partial_t + \frac{1}{e^{2u}} Du \right). \tag{6.6}$$

Thus, from (6.4), (6.5) and (6.6) we get

$$|\nabla h|^2 = \frac{|Du|^2_{M^n}}{e^{2u} - |Du|^2_{M^n}}. \tag{6.7}$$

Therefore, taking into account equations (4.4) and (6.7), we easily see that the hypothesis (6.2) guarantees that $\cosh\theta \leq \inf_\Sigma H$, and the result follows from Theorem 5.1. □

Following the same ideas of the proof of Corollary 6.1, we also obtain a non-parametric version of Theorem 5.3

Corollary 6.2. *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature and let $\Sigma^2(u)$ be an entire spacelike vertical graph bounded away from the past infinite of $-\mathbb{R} \times_{e^t} M^2$. Suppose that $\Sigma^2(u)$ has nonnegative Gaussian curvature and mean curvature $H \geq 1$. If*

$$|Du|^2_{M^2} \leq e^{2u} \left(1 - \frac{1}{H^2} \right),$$

then $\Sigma^2(u)$ is a slice.

Remark 6.3. In [5], the second author jointly with A. Albujer and F. Camargo obtained uniqueness results concerning to complete spacelike hypersurfaces with constant mean curvature immersed in a RW spacetime. As an application of such uniqueness results for the case of vertical graphs in a RW spacetime, they also get non-parametric rigidity results.

References

[1] R. Aiyama, On the Gauss map of complete space-like hypersurfaces of constant mean curvature in Minkowski space, *Tsukuba J. Math.* **16** (1992), 353-361.

[2] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, *Math. Z.* **196** (1987), 13–19.

[3] A. L. Albujer and L.J. Alías, Spacelike hypersurfaces with constant mean curvature in the steady state space, *Proc. Amer. Math. Soc.* **137** (2009), 711-721.

[4] A. L. Albujer, F. E. C. Camargo and H. F. de Lima, Complete spacelike hypersurfaces with constant mean curvature in $-\mathbb{R}\times\mathbb{H}^n$, *J. Math. Anal. Appl.* **368** (2010), 650–657.

[5] A. L. Albujer, F. E. C. Camargo and H. F. de Lima, Complete spacelike hypersurfaces in a Robertson-Walker spacetime, *Math. Proc. Camb. Phil. Soc.* **151** (2011), 271–282.

[6] J. A. Aledo and L. J. Alías, On the curvatures of bounded complete spacelike hypersurfaces in the Lorentz–Minkowski space, *Manuscripta Math.* **101** (2000), 401–413.

[7] J. A. Aledo and L. J. Alías, On the volume and the Gauss map image of complete spacelike hypersurfaces in de Sitter space, *Proc. Amer. Math. Soc.* **130** (2001), 1145–1151.

[8] L. J. Alías and A. G. Colares, Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker spacetimes, *Math. Proc. Cambridge Philos. Soc.* **143** (2007), 703-729.

[9] L. J. Alías, A. Romero and M. Sánchez, Uniqueness of complete spacelike hypersurfaces with constant mean curvature in Generalized Robertson-Walker spacetimes, *Gen. Relativity Gravitation* **27** (1995), 71–84.

[10] H. Bondi and T. Gold, On the generation of magnetism by fluid motion, *Monthly Not. Roy. Astr. Soc.* **108** (1948), 252-270.

[11] A. Caminha and H. F. de Lima, Complete Vertical Graphs with Constant Mean Curvature in Semi-Riemannian Warped Products, *Bull. of the Belgian Math. Soc. Simon Stevin* **16** (2009), 91–105.

[12] A. G. Colares and H. F. de Lima, Spacelike Hypersurfaces with Constant Mean Curvature in the Steady State Space, *Bull. of the Belgian Math. Soc. Simon Stevin* **17** (2010), 287–302.

[13] G. J. Galloway, Cosmological spacetimes with $\Lambda > 0$, *Contemp. Math.* **359** (2004), 87-101.

[14] A. J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, *Math. Proc. Cambridge Phil. Soc.* **82** (1977), 489–495.

[15] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, *Cambridge Univ. Press*, *Cambridge*, 1973.

[16] F. Hoyle, A new model for the expanding universe, *Monthly Not. Roy. Astr. Soc.* **108** (1948), 372-382.

[17] A. Huber, On subharmonic functions and differential geometry in the large, *Comment. Math. Helv.* **32** (1957), 13–72.

[18] H. F. de Lima, Spacelike hypersurfaces with constant higher order mean curvature in de Sitter space, *J. Geom. Phys.* **57** (2007), 967–975.

[19] S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, *Indiana Univ. Math. J.* **37** (1988), 909–917.

[20] S. Montiel, Uniqueness of Spacelike Hypersurfaces of Constant Mean Curvature in foliated Spacetimes, *Math. Ann.* **314** (1999), 529–553.

[21] S. Montiel, Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces, *J. Math. Soc. Japan* **55** (2003), 915–938.

[22] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan* **19** (1967), 205–214.

[23] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, *Academic Press, London*, 1983.

[24] B. Palmer, The Gauss map of a spacelike constant mean curvature hypersurface in Minkowski space, *Comment. Math. Helv.* **65** (1990), 52-57.

[25] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space, *Indiana Univ. Math. J.* **36** (1987), 349–359.

[26] S. Weinberg, Gravitation and cosmology: Principles and applications of the general theory of relativity, *John Wiley & Sons, New York*, 1972.

[27] Y. L. Xin, On the Gauss image of a spacelike hypersurface with constant mean curvature in Minkowski space, *Comment. Math. Helv.* **66** (1991), 590-598.

[28] S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Appl. Math.* **28** (1975), 201–228.

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