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Acute triangulations of double planar convex bodies

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Abstract. A (2-dimensional) *double convex body* $2K$ is a surface homeomorphic to the sphere consisting of two planar isometric compact convex bodies, K and K', with boundaries glued in the obvious way. In this note we prove that, if K admits two perpendicular axes of symmetry and bd$K$ satisfies a certain curvature condition, then $2K$ admits an acute triangulation of size 72. In particular, each double ellipse admits such a triangulation.

1. Introduction

A *triangulation* of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. The number of triangles in a triangulation is called its *size*.

In rather general two-dimensional spaces, like Alexandrov surfaces, two geodesics starting at the same point determine a well defined angle. Our interest will be focused on triangulations which are *acute*, which means that the angles of all geodesic triangles are smaller than $\frac{\pi}{2}$.

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the *Scientific American* (see [3], [4], [5]). There the question was raised whether

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a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^3$. However, their method could not give an estimate on the number of triangles used in the existing acute triangulations. In 1980, Cassidy and Lord [2] considered acute triangulations of the square. Recently, Maehara investigated acute triangulations of quadrilaterals [10] and other polygons [11], and a result on the latter was improved by Yuan [14] and C. T. Zamfirescu [19].

On the other hand, compact convex surfaces have also been triangulated. Acute triangulations of all Platonic surfaces, which are surfaces of the five well-known Platonic solids, have been investigated in [6], [8], and [9]. Furthermore, some other well-known surfaces have also been acutely triangulated, such as flat Möbius strips [15] and flat tori [7]. For a survey on acute triangulations, see [18].

However, the case of arbitrary compact convex surfaces is more difficult, even for polyhedra with small number of vertices. So, for example, even the family of all tetrahedral surfaces is far from being easy to treat.

Concerning non-polyhedral surfaces only the sphere and pieces of it have been considered so far. We shall construct here, for the first time, acute triangulations for another type of non-polyhedral surface, namely a *double convex body*, which is a surface $2K$ homeomorphic to the sphere, consisting of two planar isometric compact convex bodies, K and K', with boundaries glued according to the isometry.

Acute triangulations of double triangles [17], double quadrilaterals [16], and double pentagons [13] have been investigated. In this paper, we present a fairly small upper bound for the minimal size of an acute triangulation of $2K$ in case K has two perpendicular axes of symmetry and $\mathrm{bd}K$ is of class C^2 and satisfies a certain curvature condition.

We regard our work as a step towards a solution to the following problem first raised in [6].

Problem 1. Does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

As remarked in [9], this Problem can be extended (or restricted) to other families of surfaces (such as Riemannian surfaces), with or without boundary.

2. Main result

We consider a planar convex body K with two orthogonal axes of symmetry, say the x-axis and the y-axis, and with C^2-boundary bd$K$. Let a and b be the points of bd$K$ on the positive x-semiaxis and y-semiaxis. We may assume $\|a\| \geq \|b\|$.

Let A be the arc $\{(x, y) \in \mathrm{bd}K : x, y \geq 0\}$. This arc determines of course K. In order to obtain a relatively small acute triangulation of $2K$ we need some curvature condition on A.

Theorem. *If the curvature of A is monotone or bounded above by $2/\|b\|$, then $2K$ admits a triangulation with 72 acute triangles.*

Here, *monotone* means non-increasing or non-decreasing.

Proof. For any point $u \in \mathbb{R}^2$, let u^* be the point symmetric to u with respect to the y-axis, u_x the orthogonal projection of u on the x-axis, and u_y the orthogonal projection of u on the y-axis.

For any point $v \in K$, let v' be the image of v through the isometry between K and K'.

Choose $c \in A$ such that $\angle \mathbf{0}ac$ be slightly larger than $\pi/4$. Let k be a point of A where the tangent line to bd$K$ is parallel with ac. Consider the orthogonal projection m of k on ac.

We prove now that $mk \cup km'$ and $c_y b \cup bc'_y$ are segments on $2K$.

First assume the monotony of the curvature on A. Considering A ordered from a to b, and denoting by $\rho(u)$ the curvature radius of bd$K$ at $u \in A$, from $\|a\| \geq \|b\|$ it follows that $\rho : A \to \mathbb{R}$ is non-decreasing.

Suppose k is not a point in A closest to m. Then k is not closest to any point of the line-segment ac.

For each point $s \in ac$, let $p(s)$ be the point in A closest to s if it is unique, or the smallest subarc of A containing all points in A closest to s otherwise. The family $\{p(s) : s \in ac\}$ constitutes a partition of the subarc of A from a to c. To see why this is indeed a partition, remark that for any pair of points $s_1, s_2 \in ac$ and any choice of $t_i \in p(s_i)$ closest to s_i, the line segment s_1t_1 is orthogonal to the tangent line T_1 at t_1 to A, and therefore the angle between s_2t_1 and T_1 is acute, whence $s_1t_1 \cap s_2t_2 = \emptyset$. Thus, since k is not closest to s for any s, $k \in p(s)$ for some s, and A is externally tangent at the end-points e_1, e_2 of $p(s)$ to a circle of centre s and radius r, say. Then $\rho(e_i) \geq r$ $(i = 1, 2)$, but $\rho(e) \geq r$ cannot hold for all $e \in p(s)$. This, in turn, contradicts the monotonicity of ρ. Hence the point k is closest to m. This implies that $mk \cup km'$ is a segment in $2K$.

We already observed that ρ is non-decreasing on A. Thus, ρ attains its maximum at b, and the whole arc A is included in the osculating circle of bd$K$ at b. Let C be the arc from a to $-a$ (through b) of the circle through a, b, $-a$. The arcs A and C meet at a and b only, because the existence of any further common point obviously contradicts the monotonicity of the curvature of A. Hence $C \subset K$. Since b is the closest point of C from c_y, b is also the closest point of bd$K$ from c_y, and $c_y b \cup bc'_y$ is a segment in $2K$.

Now, assume that the curvature of A is bounded above by $2/\|b\|$.

We saw above that, if k is not closest to m, then $\rho(e) < r$, while $\|s - e_i\| = r$ $(i = 1, 2)$ for some $s \in ac$ and $e \in p(s)$; remember that e_1, e_2 are the end-points of $p(s)$. Let $h \in \mathbb{R}^2$ satisfy $h_x = a$ and $h_y = b$. Every point of ac is at distance less than $\|b\|/2$ from $ah \cup hb$, whence also from A. Hence,

$$\rho(e) < \|s - e_i\| < \|b\|/2,$$

which contradicts the upper bound on the curvature.

The argument used to show that k is closest to m can also be applied to check that b is closest to c_y. Just replace ac by cc^* and A by the arc of bd$K$ from c to c^* (through b).

Hence, $mk \cup km'$ and $c_y b \cup bc'_y$ are both segments on $2K$.

Put $\{g\} = cc_x \cap mm_y$. We have the following non-obtuse triangulation. The rhombus $c_x m_y(-c_x)(-m_y)$ can be divided into two acute triangles, if it is not a square (and in two ways into two right triangles, if it is a square). Further, cgc_y, $gc_y m_y$, cgm, $gc_x m_y$, $gc_x m_x$, $gm_x m$, $mm_x a$ are right triangles. The arc A is covered by the non-obtuse geodesic triangles $cc_y c'_y$, cmm', amm'. This partial triangulation obviously extends to a full non-obtuse triangulation $\mathcal{T}$ of $2K$, with size 72, as shown in Figure 1.

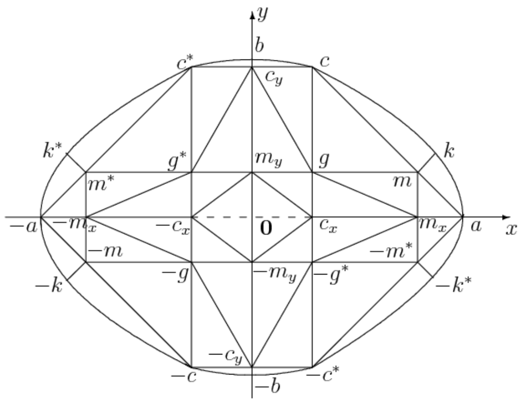

Fig. 1. Non-obtuse triangulation of Γ_d

Now, in six steps, we successively change the position of certain vertices of $\mathcal{T}$ in order to transform right angles into acute ones. Once an angle is acute, the next steps will be performed so gently that the angle remains acute.

Step 1. We replace c_y by a point on $\mathbf{0}b$ slightly closer than c_y to $\mathbf{0}$ (and replace $-c_y$ by a symmetrical point on $\mathbf{0}(-b)$). Thus, the triangles $cc_yc'_y$, cc_yg, and all the other 10 symmetrical right triangles become acute.

Step 2. Similarly, we replace m_y and $-m_y$ by symmetrical points on $b(-b)$ closer to $\mathbf{0}$. This takes care of the triangles gc_ym_y, gm_yc_x and all the other 14 symmetrical triangles. Moreover, it produces two acute triangles replacing $c_xm_y(-m_y)$ and $(-c_x)m_y(-m_y)$ in case $c_xm_y(-c_x)(-m_y)$ is a square. (The same happens in K'.)

Step 3. Then m_x and $-m_x$ are replaced by points on $a(-a)$ closer to $\mathbf{0}$.

Step 4. Similarly, we replace the points c_x and $-c_x$.

Step 5. Now replace g by a point on gg^* slightly closer to g^*, and perform the other 7 analogous replacements, to take care of cgm et al.

Step 6. Finally, replace m by a point on the line through m and k farther away from k, and make the 7 symmetrical replacements, to make all remaining right triangles acute.

Thus we obtain an acute triangulation of $2K$ of size 72. □

We mention an immediate application of our Theorem to the case of a double ellipse.

Corollary. *Every double ellipse admits an acute triangulation of size* 72.

Proof. Indeed, in this case the curvature being monotone on A, the Theorem applies. □

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