Publ. Math. Debrecen
81/1-2 (2012), 167–177
DOI: 10.5486/PMD.2012.5178

Nonlinear connections for conformal gauge theories on path-spaces and duality

By MIRCEA CRASMAREANU (Iaşi)

Abstract. Weyl structures and compatible nonlinear connections are introduced in the geometry of semisprays as a natural generalization of similar notions from Riemannian geometry. The existence and formula for the set of all compatible nonlinear connections are derived by using the Obata tensors naturally associated to a fixed metric in the given conformal class; this formula is also expressed in terms of dual nonlinear connections which generalize the Norden's notion of dual linear connections. A geometric meaning for pairs (Weyl structure, compatible nonlinear connection) is provided in terms of gauge conformal invariance.

1. Introduction

Soon after the creation of general theory of relativity, Hermann Weyl attempted in [11] an unification of gravitation and electromagnetism in a model of space-time geometry combining conformal and projective structures.

Let $\mathcal{G}$ be a conformal structure on the smooth manifold M i.e. an equivalence class of Riemannian metrics: $g \sim \overline{g}$ if there exists a smooth function $f \in C^{\infty}(M)$ such that $\overline{g} = e^{2f} g$. Denoting by $\Omega^1(M)$ the $C^{\infty}(M)$-module of 1-forms on M a (Riemannian) Weyl structure is a map $W : \mathcal{G} \to \Omega^1(M)$ such that $W(\overline{g}) = W(g) + 2df$. In [5] it is proved that for a Weyl manifold $(M, \mathcal{G}, W)$ there exists an unique torsion-free linear connection ∇ on M such that for every $g \in \mathcal{G}$:

$$\nabla g = W(g) \otimes g. \tag{$*$}$$

Mathematics Subject Classification: 53C60; 58A30, 34A26, 37C10; 53C15.
Key words and phrases: semispray, conformal class of metrics, Weyl structure; compatible nonlinear connection, conformal path-gauge invariance; duality for nonlinear connections.

The parallel transport induced by ∇ preserves the given conformal class $\mathcal{G}$. For other physical meanings of Weyl structures see [6] and an interesting generalization to statistical geometry appears in [9].

The aim of present paper is to extend the Weyl structures and compatible connections $(*)$ in the framework of systems of second order differential equations on M. More precisely, given such a system S, on short *semispray*, we can derive a type of differential ∇ if S is considered as a vector field on the tangent bundle TM. A necessary tool in the definition of ∇ is given by a splitting of the iterated tangent bundle $T(TM)$ provided by a distribution N on TM. Such an object N is called *nonlinear connection*. A remarkable result is that every S yields such a nonlinear connection, $\overset{c}{N}$, indexed by us with c from *canonical* and on this way we recover the above Riemannian case $(*)$. Let us point out that two previous generalizations of Weyl structures in the tangent bundle geometry are: i) for Finsler metrics, in [1]–[2], [7]–[8], ii) for (generalized) Lagrange geometry in [4].

2. Nonlinear connections and semisprays on tangent bundles

Let M be a smooth, n-dimensional manifold for which we denote: $C^\infty(M)$-the algebra of smooth real functions on M, $\mathcal{X}(M)$-the Lie algebra of vector fields on M, $T^r_s(M)$-the $C^\infty(M)$-module of tensor fields of (r,s)-type on M.

A local chart $x = (x^i) = (x^1, \dots, x^n)$ on M lifts to a local chart on the tangent bundle TM given by: $(x,y) = (x^i, y^i)$. If $\pi : TM \to M$ is the canonical projection then the kernel of the differential of π is an integrable distribution $V(TM)$ with local basis $\left(\frac{\partial}{\partial y^i}\right)$. An important element of $V(TM)$ is the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$. $V(TM)$ is called *the vertical distribution* and its elements are *vertical vector fields*.

The tensor field $J \in T^1_1(TM)$ given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called *the tangent structure*. Two of its properties are: the nilpotence $J^2 = 0$ and $\operatorname{im} J (= \ker J) = V(TM)$.

A well-known notion in the tangent bundles geometry is:

Definition 2.1 ([3, p. 336]). A supplementary distribution N to the vertical distribution $V(TM)$:

$$T(TM) = N \oplus V(TM) \tag{2.1}$$

is called *horizontal distribution* or *nonlinear connection*. A vector field belonging to N is called *horizontal*.

A nonlinear connection has a local basis:

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} \tag{2.2}$$

and the functions $(N_j^i(x,y))$ are called *the coefficients* of N. So, a basis of $\mathcal{X}(TM)$ adapted to the decomposition (2.1) is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ called *Berwald basis*. The dual of the Berwald basis is: $(dx^i, \delta y^i = dy^i + N_j^i dx^j)$.

A second remarkable structure on TM is provided by:

Definition 2.2 ([3, p. 336]). $S \in \mathcal{X}(TM)$ is called *semispray* if:

$$J(S) = \mathbb{C}. \tag{2.3}$$

In canonical coordinates:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i} \tag{2.4}$$

and the functions $(G^i(x,y))$ are *the coefficients* of S. The flow of S is a system of second order differential equations: $\frac{d^2x^i}{dt^2} = 2G^i\left(x, \frac{dx}{dt}\right)$ and then the pair (M,S) will be called *path-space*.

An important result is that a nonlinear connection $N = (N_j^i)$ yields an unique horizontal semispray denoted $S(N)$ with:

$$G^i = \frac{1}{2} N_j^i y^j \tag{2.5}$$

In other words:

$$S(N) = y^i \frac{\delta}{\delta x^i}. \tag{2.6}$$

The converse of this result is that a semispray S yields a nonlinear connection $\overset{c}{N}$ given by:

$$\overset{c}{N}{}_j^i = \frac{\partial G^i}{\partial y^j}. \tag{2.7}$$

Definition 2.3. A semispray S for which the coefficients (G^i) are homogeneous of degree 2 with respect to the variables (y^i) will be called *spray*.

Locally this means, via Euler theorem:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{2.8}$$

and then $\overset{c}{N}$ is 1-homogeneous:

$$\overset{c}{N}{}_j^i = y^a \frac{\partial \overset{c}{N}{}_j^i}{\partial y^a} \tag{2.9}$$

which yields that S is horizontal with respect to $\overset{c}{N}$ i.e. S has the expression (2.7).

3. Weyl structures and conformal path-gauge invariance

Let us fix a semispray $S = (G^i)$ and a nonlinear connection $N = (N^i_j)$. Following [3, p. 337] let us consider:

Definition 3.1. The *dynamical derivative* associated to the pair (S, N) is the map $\overset{SN}{\nabla} : V(TM) \to V(TM)$ given by:

$$\overset{SN}{\nabla} X = \overset{SN}{\nabla} \left(X^i \frac{\partial}{\partial y^i} \right) := (S(X^i) + N^i_j X^j) \frac{\partial}{\partial y^i}. \tag{3.1}$$

Properties:

I) $\overset{SN}{\nabla} \left(\frac{\partial}{\partial y^i} \right) = N^j_i \frac{\partial}{\partial y^j}$,

II) $\overset{SN}{\nabla} (X + Y) = \overset{SN}{\nabla} X + \overset{SN}{\nabla} Y$,

III) $\overset{SN}{\nabla} (fX) = S(f)X + f \overset{SN}{\nabla} X$.

It is straightforward to extend the action of $\overset{SN}{\nabla}$ to general vertical tensor fields by requiring to preserves the tensor product and the Leibniz rule. Moreover, we will extend $\overset{SN}{\nabla}$ to a special class of tensor fields:

Definition 3.2. A *d-tensor field* (d from distinguished) on TM is a tensor field whose change of components, under a change of canonical coordinates $(x, y) \to (\widetilde{x}, \widetilde{y})$ on TM, involves only factors of type $\frac{\partial \widetilde{x}}{\partial x}$ and (or) $\frac{\partial x}{\partial \widetilde{x}}$.

Example 3.3. i) $\left(\frac{\delta}{\delta x^i} \right)$ and $\left(\frac{\partial}{\partial y^i} \right)$ are components of d-tensor fields of $(1, 0)$-type.

ii) (dx^i) and (δy^i) are components of d-tensor fields of $(0, 1)$-type.

iii) (G^i) are not components of a d-tensor field since a change of coordinates implies:

$$2\widetilde{G}^i = 2 \frac{\partial \widetilde{x}^i}{\partial x^j} G^j - \frac{\partial \widetilde{y}^i}{\partial x^j} y^j$$

but it results that given two semisprays $\overset{1}{S}$ and $\overset{2}{S}$ their difference $X = \overset{2}{S} - \overset{1}{S}$ is a vertical (and then d-) vector field.

iv) (N^i_j) are not components of a d-tensor field since a change of coordinates implies:

$$\frac{\partial \widetilde{x}^j}{\partial x^k} N^k_i = \widetilde{N}^j_k \frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}.$$

It follows that given two nonlinear connections $\overset{1}{N}$ and $\overset{2}{N}$ their difference $F = \overset{2}{N} - \overset{1}{N} = \left(F^i_j = \overset{2}{N}{}^i_j - \overset{1}{N}{}^i_j \right)$ is a d-tensor field of $(1, 1)$-type.

Definition 3.4. A *metric* g on TM is a d-tensor field of $(0,2)$-type which is symmetric and non-degenerated.

It results for the components $g_{ij} = g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ the following properties:

1) (symmetry) $g_{ij} = g_{ji}$,
2) (non-degeneration) $\det(g_{ij}) \neq 0$; then there exists the d-tensor field of $(2,0)$-type $g^{-1} = (g^{ij})$.

The name is justified from the fact that $g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$ is a Riemannian metric on TM for which N and $V(TM)$ are orthogonal distributions.

Using the Leibniz rule we arrive at:

Definition 3.5. The *dynamical derivative of metric* g with respect to the pair (S,N) is $\overset{SN}{\nabla} g : V(TM) \times V(TM) \to C^\infty(TM)$ given by:

$$\overset{SN}{\nabla} g(X,Y) = S(g(X,Y)) - g\Big(\overset{SN}{\nabla} X, Y\Big) - g\Big(X, \overset{SN}{\nabla} Y\Big). \tag{3.2}$$

One of the main notions of this section is:

Definition 3.6. Two metrics $g, \overline{g}$ are called *conformal equivalent* if there exists $f \in C^\infty(TM)$ such that $\overline{g} = e^{2f} g$.

In the following let $\mathcal{G}$ be a conformal structure i.e. an equivalence class of conformal equivalent metrics. Our generalization of classical Weyl structures is:

Definition 3.7. A *Weyl structure* on the path-space (M,S) is a map $W : \mathcal{G} \to C^\infty(TM)$ such that for every $g, \overline{g} \in \mathcal{G}$:

$$W(\overline{g}) = W(g) + 2df(S). \tag{3.3}$$

The data $(M, S, \mathcal{G}, W)$ will be called *path-Weyl manifold*.

Another main notion is:

Definition 3.8. Let $(M, S, \mathcal{G}, W)$ be a path-Weyl manifold. The nonlinear connection N is called *compatible with* $g \in \mathcal{G}$ if:

$$\overset{SN}{\nabla} g = W(g)g. \tag{3.4}$$

An important result is:

Proposition 3.9. *If N is compatible with $g \in \mathcal{G}$ then N is compatible with the whole class $\mathcal{G}$.*

Proof. From the Leibniz rule and (3.4) we get:

$$\overset{SN}{\nabla}\ \overline{g} = S(e^{2f})g + e^{2f}\ \overset{SN}{\nabla}\ g = 2df(S)\overline{g} + e^{2f}W(g)g = (W(g) + 2df(S))\overline{g} = W(\overline{g})\overline{g}$$

which means the conclusion. □

Let us end this section with a geometrical interpretation for pairs (Weyl structure, compatible nonlinear connection). In addition to the pair (S, N) let us consider a pair (metric g, $F \in C^\infty(TM)$) and define, inspired by [9, p. 109], the map: $\mathcal{C}_{SN}(g, F) : V(TM) \times V(TM) \to C^\infty(TM)$:

$$\mathcal{C}_{SN}(g, F) = \overset{SN}{\nabla}\ g - Fg. \tag{3.5}$$

Then, $\mathcal{C}_{SN}(g, F)$ is, in fact, a d-tensor field of $(0, 2)$-type and a pair (Weyl structure, compatible nonlinear connection) is characterized by the vanishing of $\mathcal{C}_{SN}(g, W(g))$.

Definition 3.10. A function $f \in C^\infty(TM)$ induces *the conformal path-gauge transformation*:

$$(g, F) \to (g', F') := (e^{2f}g, F + 2df(S)). \tag{3.6}$$

Proposition 3.11. *The d-tensor field $\mathcal{C}_{SN}(g, F)$ is not conformal path-gauge invariant but a pair (Weyl structure, compatible nonlinear connection) is a conformal path-gauge invariant notion.*

Proof. A calculus similar to that of the previous Proof above gives:

$$\mathcal{C}_{SN}(g', F') = e^{2f}\mathcal{C}_{SN}(g, F) \tag{3.7}$$

which get the all conclusions. □

4. The general expression of a compatible nonlinear connection

The aim of this section is to find all nonlinear connections which are compatible with a given Weyl structure. In order to answer at this question, a look at example 3.3 iv) gives necessary a study of two operators, called *Obata* in the following, acting on the space of d-tensor fields of $(1, 1)$-type:

$$O^{ij}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - g^{ij}g_{kl}), \quad \overset{*}{O}{}^{ij}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l + g^{ij}g_{kl}). \tag{4.1}$$

The Obata operators are supplementary projectors on the space of tensor fields of $(1,1)$-type:

$$O^{ia}_{bj}\overset{*}{O}{}^{bk}_{la}=\overset{*}{O}{}^{ia}_{bj}O^{bk}_{la}=0,\qquad O^{ia}_{bj}O^{bk}_{la}=O^{ik}_{lj},\qquad \overset{*}{O}{}^{ia}_{bj}\overset{*}{O}{}^{bk}_{la}=\overset{*}{O}{}^{ik}_{lj}\tag{4.2}$$

and the tensorial equations involving these operators has solutions as follows:

Proposition 4.1. *The system of equations:*

$$\overset{*}{O}{}^{ia}_{bj}(X^b_a)=A^i_j,\qquad (O^{ia}_{bj}(X^b_a)=A^i_j)\tag{4.3}$$

with X as unknown has solutions if and only if:

$$O^{ia}_{bj}(A^b_a)=0,\qquad \Big(\overset{*}{O}{}^{ia}_{bj}(A^b_a)=0\Big)\tag{4.4}$$

and then, the general solution is:

$$X^i_j=A^i_j+O^{ia}_{bj}(Y^b_a),\qquad \Big(X^i_j=A^i_j+\overset{*}{O}{}^{ia}_{bj}(Y^b_a)\Big)\tag{4.5}$$

with Y an arbitrary d-tensor field of $(1,1)$-type.

We are ready for the main results of paper:

Theorem 4.2. *Let $(M,S,\mathcal{G},W)$ be a path-Weyl manifold. The family $\mathcal{N}(S,\mathcal{G},W)$ of all compatible nonlinear connections is infinite. More precisely, $\mathcal{N}(S,\mathcal{G},W)$ is a $C^\infty(TM)$-affine module over the $C^\infty(TM)$-module of d-tensor fields of $(1,1)$-type.*

Proof. Fix $g\in\mathcal{G}$ and search (N^i_j) of the form:

$$N^i_j=\overset{c}{N}{}^i_j+F^i_j\tag{4.6}$$

with (F^i_j) a d-tensor field of $(1,1)$-type to be determined. The local expression of equation (3.4) is:

$$S(g_{uv})-g_{um}N^m_v-g_{mv}N^m_u=W(g)g_{uv}\tag{4.7}$$

and inserting (4.6) in (4.7) gives:

$$S(g_{uv})-g_{um}\overset{c}{N}{}^m_v-g_{mv}\overset{c}{N}{}^m_u=g_{um}F^m_v+g_{mv}F^m_u+W(g)g_{uv}.$$

Multiplying the last relation with g^{ku} we get:

$$g^{ku}S(g_{uv}) - \overset{c}{N}{}^k_v - g^{ku}g_{mv}\overset{c}{N}{}^m_u - W(g)\delta^k_v = F^k_v + g^{ku}g_{mv}F^m_u = 2\overset{*}{O}{}^{kb}_{av}(F^a_b). \tag{4.8}$$

Let us search for the condition (4.4):

$$O^{kb}_{av}(g^{am}S(g_{mb}) - \overset{c}{N}{}^a_b - g^{am}g_{bl}\overset{c}{N}{}^l_m - W(g)\delta^a_b)$$
$$= g^{km}S(g_{mv}) - \overset{c}{N}{}^k_v - g^{km}g_{vl}\overset{c}{N}{}^l_m - g^{km}S(g_{mv}) + g^{km}g_{vl}\overset{c}{N}{}^l_m + \overset{c}{N}{}^k_v = 0.$$

It follows:

$$F^i_j = \frac{1}{2}g^{\mathrm{im}}S(g_{mj}) - \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a - \frac{W(g)}{2}\delta^i_j + O^{ib}_{aj}(X^a_b)$$

and returning to (4.6) we have the conclusion:

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a + \frac{1}{2}g^{ia}S(g_{aj}) - \frac{W(g)}{2}\delta^i_j + O^{ia}_{bj}(X^b_a) \tag{4.9}$$

with $X = (X^b_a)$ an arbitrary d-tensor field of $(1,1)$-type. □

In the spray case the equation (4.9) admits a simplification:

Proposition 4.3. *If S is a spray then the set $\mathcal{N}(S, \mathcal{G}, W)$ is:*

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^b_a + \frac{1}{2}g^{ia}y^m\frac{\delta g_{aj}}{\delta x^m} - \frac{W(g)}{2}\delta^i_j + O^{ia}_{bj}(X^b_a). \tag{4.10}$$

Example 4.4 Classical Weyl structures

Let us consider $g = g(x)$ a Riemannian metric on M and let S be its corresponding spray i.e. S gives the geodesics of g. Recall also that a symmetric linear connection on M with coefficients $(\Gamma^i_{jk}(x))$ yields the nonlinear connection with the coefficients:

$$N^i_j = \Gamma^i_{ja}y^a. \tag{4.11}$$

and then the associated semispray $S(N)$ is a spray:

$$G^i = \frac{1}{2}\Gamma^i_{jk}y^jy^k. \tag{4.12}$$

In order to work on M we consider, from 1-homogeneity reasons, $W(g)$ to be the function $W(g)(x,y) = W(g)_a(x)y^a$.

It is well known that the solution of $(*)$ is the unique *Weyl connection* [5, p. 147]:

$$\Gamma^i_{jk} = \overset{c}{\Gamma}{}^i_{ij} + \frac{1}{2}(W(g)^i g_{jk} - \delta^i_j W(g)_k - \delta^i_k W(g)_j) \qquad (*W)$$

where $\overset{c}{\Gamma}$ is the Levi-Civita connection of g and $W(g)^i = g^{ia}W(g)_a$ is the g-contravariant version of the basic 1-form $W_M(g) = W(g)_a dx^a$ which is exactly the 1-form of $(*)$. We recover this last formula from (4.9) with:

$$X^i_j = -W(g)_j y^i \qquad (4.13)$$

which is a d-tensor field of $(1,1)$-type since: $\widetilde{y}^a = \frac{\partial \widetilde{x}^a}{\partial x^b} y^b$ while $W_M(g)$ is a tensor field on the base manifold M. In fact, X is the tensor product $X = -W_M(g) \otimes \mathbb{C}$; also the basic 1-form $W_M(g)$ admits the lift $W_{TM}(g) = W(g)_a dy^a$ to TM and then:

$$W(g) = W_{TM}(g)(\mathbb{C}). \qquad (4.14)$$

5. Dual nonlinear connections in metric path-spaces

A natural question about the general formula (4.9) is to find a geometric meaning for some remarkable choices of X. The aim of this section is to provide an answer to the case $X = 0$:

$$\overset{0}{N}{}^i_j = \frac{1}{2} \overset{c}{N}{}^i_j - \frac{1}{2} g^{ia} g_{jb} \overset{c}{N}{}^b_a + \frac{1}{2} g^{ia} S(g_{aj}) - \frac{W(g)}{2} \delta^i_j. \qquad (5.1)$$

In order to explain more geometrically this relation let us recall the notion of dual connections introduced by A. P. Norden:

Definition 5.1 ([10, p. 913]). Two linear connections ∇, ∇^* on the Riemannian manifold (M, g) are called *dual* (or *g-conjugated*) if, for all vector fields X, Y, Z:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z). \qquad (5.2)$$

We generalize this notion to:

Definition 5.2. Let (M, S, g) be a metric path-space and N a nonlinear connection on M. The nonlinear connection $\overset{Sg}{N}$ is called *dual* or (S,g)-*conjugated* to N if:

$$S(g(X,Y)) = g\Big(\overset{SN}{\nabla} X, Y\Big) + g\Big(X, \overset{S\overset{Sg}{N}}{\nabla} Y\Big) \qquad (5.3)$$

for all vector fields X, Y on TM.

Let us remark that $\overset{Sg}{N}$ exists since g is non-degenerated. In local coefficients, the last formula becomes:

$$S(g_{uv}) = N^a_u g_{av} + \left(\overset{Sg}{N} \right)^a_v g_{ua} \tag{5.4}$$

and then:

$$\left(\overset{Sg}{N} \right)^a_b = g^{au} S(g_{ub}) - g^{au} g_{bv} N^v_u. \tag{5.5}$$

A straightforward computation gives that the dual of $\overset{Sg}{N}$ is exactly N, a result well-known for dual linear connections, [10, p. 913].

Denoting with I the Kronecker tensor we derive a global formula for compatible nonlinear connections and comparing (5.1) and (5.5) we get:

Theorem 5.3. *Let $(M, S, \mathcal{G}, W)$ be a path-Weyl structure. The family $\mathcal{N}(S, \mathcal{G}, W)$ of all compatible nonlinear connections is given by:*

$$N = \frac{1}{2}\left(\overset{c}{N} + \overset{Sg,c}{N} \right) - \frac{W(g)}{2} I + O(X). \tag{5.6}$$

In the particular case of $W(g) = 0$ we obtain a global expression for the Theorem 2.4. of [3, p. 339]: the family of all *metric* nonlinear connections on (M, S, g) is given by:

$$N = \frac{1}{2}\left(\overset{c}{N} + \overset{Sg,c}{N} \right) + O(X) \tag{5.7}$$

and, on this way, we generalize the fact that for a pair (∇, ∇^*) of g-conjugate linear connections, the mean linear connection $\frac{1}{2}(\nabla + \nabla^*)$ is a metric connection, [10, p. 913].

References

[1] T. Aikou, Locally conformal Berwald spaces and Weyl structures, *Publ. Math. Debrecen* **49** (1996), 113–126.

[2] T. Aikou and Y. Ichijyo, Finsler-Weyl structures and conformal flatness, *Rep. Fac. Sci. Kagoshima Univ.* **23** (1990), 101–109.

[3] I. Bucataru, Metric nonlinear connections, *Diff. Geom. Appl.* **25** (2007), 335–343.

[4] M. Crasmareanu, Generalized Lagrange–Weyl structures and compatible connections, *Sci. Ann. Univ. Agric. Sci. Vet. Med.* **48** (2005), 273–278.

[5] G. B. Folland, Weyl manifolds, *J. Diff. Geom.* **4** (1970), 145–153.

[6] G. S. Hall, Weyl manifolds and connections, *J. Math. Phys.* **33** (1992), 2633–2638.

[7] L. Kozma, On Finsler–Weyl manifolds and connections, *Rend. Circ. Mat. Palermo* (2) **43** (1996), 173–179.

[8] L. Kozma, Finslerian Weyl structures in the tangent bundle, *Bull. Soc. Sci. Lett. Lódz Sér. Rech. Déform.* **49** (2006), 19–26.

[9] H. Matsuzoe, Geometry of semi-Weyl manifolds and Weyl manifolds, *Kyushu J. Math.* **55** (2001), 107–117.

[10] U. Simon, Affine differential geometry, Chapter 9 of Handbook of differential geometry, **Vol. I**, *North-Holland, Amsterdam*, 2000, 905–961.

[11] H. Weyl, Space, time, matter, 4th ed., rep., *Dover Publications, New York*, 1951.

MIRCEA CRASMAREANU
FACULTY OF MATHEMATICS
UNIVERSITY “AL. I.CUZA”
IAŞI, 700506
ROMANIA

E-mail: mcrasm@uaic.ro
URL: http://www.math.uaic.ro/∼mcrasm

(Received March 16, 2011)