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Averaged Riemannian metrics and connections with application to locally conformal Berwald manifolds

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Dedicated to Professor Kazunari Yamauchi on the occasion of his 65 birthday

Abstract. In this article, we investigate the class of Finsler manifolds which are locally conformal to a Berwald manifold using the so-called averaged Riemannian metric and averaged connection defined in [Ma-Ra-Tr-Ze] and [To-Et], respectively.

1. Introduction

A Finsler function for a smooth manifold M is a smooth assignment of a Minkowski norm to the tangent space T_pM at each point $p \in M$. If the function satisfies the strong convexity condition, then there exists a natural Finsler connection D in the vertical subbundle of TTM called the *Berwald connection* of (M, L). If D is induced from a symmetric linear connection on M, then (M, L) is called a *Berwald manifold.* By a clever observation of Z. I. Szabó, if (M, L) is a Berwald manifold, then its Berwald connection is induced from the Levi–Civita connection of a Riemannian metric on M (Theorem 4.1). Such a Riemannian metric is given by the so-called *averaged Riemannian metric* obtained from the given Finsler function [Ma-Ra-Tr-Ze, Vi].

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Landsberg manifolds also form a special class of Finsler manifolds, which includes Berwald manifolds. Modifying the definition in [To-Et], we shall define the *averaged connection* obtained from the Berwald connection. The first purpose of this paper is to generalize Szabó's theorem to the case of Landsberg manifolds: if (M, L) is a Landsberg manifold, then the averaged connection is the Levi–Civita connection of the averaged Riemannian metric (Theorem 5.1). Szabó's theorem is obtained as a special case of this theorem.

In a previous paper [Ai2], the author has introduced the notion of *locally conformal Berwald manifolds*, which form a special class of the so-called Wagner manifolds (see [Ha], [Sz-Sz]). The Weyl connection of the conformal class of a Riemannian metric on M plays an important role when studying these spaces. The second purpose of this paper is to show that the averaged connection is the Weyl connection of the conformal class of the averaged Riemannian metric (Theorem 6.1).

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2. Minkowski spaces and the Mazur–Ulam theorem

Let V be a real vector space of dimension n.

Definition 2.1. A function $\|\cdot\|_V : V \longrightarrow \mathbb{R}$ is called a *Minkowski norm* if it satisfies the following conditions:

(1) $\|v\|_V \geq 0$, and $\|v\|_V = 0$ if and only if $v = 0$;

(2) for every $v \in V$ and positive real number λ

$$\|\lambda \cdot v\|_V = \lambda \cdot \|v\|_V \quad \text{(positive homogeneity);} \tag{2.1}$$

(3) for all $v, w \in V$

$$\|v + w\|_V \leq \|v\|_V + \|w\|_V \quad \text{(triangle inequality).} \tag{2.2}$$

In this case the pair $(V, \|\cdot\|_V)$ is called a *Minkowski space.*

Notice that we do not assume the reversibility condition $\|v\|_V = \|-v\|_V$. Therefore the *indicatrix* $I = \{v \in V \mid \|v\|_V = 1\}$ is not necessarily symmetric around the origin of V.

If a Minkowski space $(V, \|\cdot\|_V)$ is given, we can define the *symmetrization* $\|\cdot\|_V^0$ of the norm $\|\cdot\|_V$ by

$$\|v\|_V^0 = \frac{\|v\|_V + \|-v\|_V}{2}, \quad v \in V.$$

Then $(V, \|\cdot\|_V^0)$ is a normed vector space in the usual sense, i.e., besides the conditions above the norm is also symmetric and thus absolute-homogeneous.

Definition 2.2. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two Minkowski spaces. A map $P : V \longrightarrow W$ is said to be *norm-preserving* if it satisfies

$$\|v\|_V = \|P(v)\|_W \tag{2.3}$$

for every $v \in V$. If the relation

$$\|v - w\|_V = \|P(v) - P(w)\|_W \tag{2.4}$$

holds for all $v, w \in V$, then P is called an *isometry.* If there exists an isometry P from V onto W, we say that V is *congruent* to W.

If an isometry $P : V \longrightarrow W$ satisfies $P(0) = 0$, then by substituting $w = 0$ into (2.4) we obtain (2.3), and therefore in this case P is norm-preserving. It is obvious that any norm-preserving map satisfies the condition $P(0) = 0$, but P is not an isometry in general. However, if a norm-preserving map $P : V \longrightarrow W$ is linear, then it is an isometry.

Assume that $P : V \longrightarrow W$ is an isometry. Then using the symmetrized norms we have

$$\begin{aligned}\|P(v) - P(w)\|_W^0 &= \frac{1}{2}\left\{\|P(v) - P(w)\|_W + \|-P(v) + P(w)\|_W\right\} \\ &= \frac{1}{2}\left\{\|v - w\|_V + \|-v + w\|_V\right\} = \|v - w\|_V^0,\end{aligned}$$

and thus $P : V \longrightarrow W$ is an isometry between the normed vector spaces $(V, \|\cdot\|_V^0)$ and $(W, \|\cdot\|_W^0)$. Therefore, from the Mazur–Ulam theorem (see [Ma-Ul] or Theorem 3.1.2 in [Th]) we have

Theorem 2.1 (Mazur–Ulam)**.** *If V and W are Minkowski spaces and P is an isometry from V onto W with $P(0) = 0$, then P is linear.*

3. Finsler metrics and Berwald connections

Throughout in the following, M will be a connected n-dimensional smooth manifold and $\pi : TM \to M$ its tangent bundle. We denote by $\Gamma(TM)$ the space of smooth sections of TM, i.e., the $C^\infty(M)$-module of vector fields on M. The deleted bundle for π is $E := TM \setminus o(M)$, where $o \in \Gamma(TM)$ is the zero section. If $f \in C^\infty(M)$, then $f^{\mathsf{v}} := f \circ \pi \in C^\infty(TM)$ is its vertical lift, and the function

$$f^{\mathsf{c}} : TM \to \mathbb{R}, \quad v \mapsto f^{\mathsf{c}}(v) := v(f)$$

is the complete lift of f.

We denote by $\mathcal{V}$ the vertical subbundle of TTM; it is the kernel of the tangent linear map $\pi_* : TTM \to TM$. The fibre of $\mathcal{V}$ at $v \in TM$ may be identified naturally with the tangent vector space $T_v T_{\pi(v)} M$. $\Gamma(\mathcal{V})$ and $\Gamma(TTM)$ stand for the $C^\infty(TM)$-modules of smooth sections of $\mathcal{V}$ and TTM, respectively. Generic sections in $\Gamma(TTM)$ will be denoted by Greek letters $\xi, \eta, \ldots$. For each vector field X on M there exists a unique section $X^{\mathsf{v}} \in \Gamma(\mathcal{V})$ such that

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}} \quad \text{for all } f \in C^\infty(M),$$

and we have a unique vector field $X^{\mathsf{c}} \in \Gamma(TTM)$ such that

$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}} \text{ and } X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}} \quad \text{for all } f \in C^\infty(M).$$

We say that X^{v} is the *vertical lift*, X^{c} is the *complete lift* of X.

There exists a canonical type $(1,1)$ tensor field $\mathbf{J}$ on TM such that

$$\mathbf{J} X^{\mathsf{v}} = 0, \ \mathbf{J} X^{\mathsf{c}} = X^{\mathsf{v}} \quad \text{for all } X \in \Gamma(TM).$$

$\mathbf{J}$ is called the *vertical endomorphism* of TTM. It follows immediately that

$$\mathrm{Im}(\mathbf{J}) = \mathrm{Ker}(\mathbf{J}) = \mathcal{V}, \quad \mathbf{J}^2 = 0.$$

If $(\mathcal{U}, (u^1, \ldots, u^n)) =: (\mathcal{U}, u)$ is a chart on M, then

$$\begin{cases} (\pi^{-1}(\mathcal{U}), (x^1, \ldots, x^n, y^1, \ldots, y^n)) =: (\pi^{-1}(\mathcal{U}), (x, y)), \\ x^i := (u^i)^{\mathsf{v}}, \ y^i := (u^i)^{\mathsf{c}}, \ i \in \{1, \ldots, n\} \end{cases}$$

is a chart on TM, called the chart induced by $(\mathcal{U}, u)$. Then $\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}}$, and $\Gamma(\mathcal{V})$ is locally generated by $\left(\frac{\partial}{\partial y^i}\right)_{i=1}^n$. We have a canonical section $\mathcal{E}$ in $\Gamma(\mathcal{V})$ such that

$$\mathcal{E} \restriction \pi^{-1}(\mathcal{U}) = \sum_{i=1}^n y^i \frac{\partial}{\partial y^i}.$$

Definition 3.1. A function $L : TM \to \mathbb{R}$ is called a *Finsler function* (or a *Finsler structure on M*) if

(F1) $L(v) \geq 0$ for all $v \in TM$ and $L(v) = 0$ if and only if $v = 0$;

(F2) $L(\lambda v) = \lambda L(v)$ for all $v \in TM$ and for all real number $\lambda > 0$;

(F3) L is smooth on E;

(F4) $L(v+w) \leq L(v)+L(w)$ for all $v, w \in T_pM$, $p \in M$ (triangle inequality).

Then the pair (M, L) is said to be a *Finsler manifold.* A Finsler structure on M gives rise to a Minkowski norm

$$\|v\| := L(v), \quad v \in T_pM$$

on each tangent space to M.

Let p and q be two points of a Finsler manifold (M, L), and denote by $\mathcal{C}(p, q)$ the set of all piecewise smooth, regular, oriented curve segments $\gamma : [a, b] \to M$ from p to q. Define a functional $\mathcal{F}_L : \mathcal{C}(p, q) \to \mathbb{R}$ by

$$\mathcal{F}_L(\gamma) := \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b L(\dot{\gamma}(t)) dt.$$

By the homogeneity condition (F2), $\mathcal{F}_L(\gamma)$ is well-defined; it is called the *length* of γ. The *(Finslerian) distance* of p and q is defined by

$$d_L(p, q) := \inf_{\gamma \in \mathcal{C}(p,q)} \mathcal{F}_L(\gamma).$$

Then the function

$$d_L : M \times M \to \mathbb{R}, \quad (p, q) \mapsto d_L(p, q)$$

is a *quasi-distance* on M: it satisfies the distance axioms except that $d_L(p, q)$ is not necessarily equal to $d_L(q, p)$ since we did not require the reversibility condition $L(v) = L(-v)$.

By a *geodesic* of (M, L) we mean a positive constant speed extremal of the functional $\mathcal{F}_L$, i.e., a curve $\gamma : [a, b] \to M$ such that in any induced chart $(\pi^{-1}(\mathcal{U}), (x, y))$

$$\frac{\partial L}{\partial x^i} \circ \dot{\gamma} - \left(\frac{\partial L}{\partial y^i} \circ \dot{\gamma} \right)' = 0, \quad i \in \{1, \ldots, n\}$$

(γ satisfies the Euler–Lagrange equations of $\mathcal{F}_L$), and for some positive $\lambda \in \mathbb{R}$

$$L(\dot{\gamma}(t)) = \lambda, \quad t \in [a, b].$$

The type $(0,2)$ tensor field

$$G : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to C^\infty(TM)$$

specified on the vertical lifts of vector fields on M by

$$G(X^{\mathsf{v}}, Y^{\mathsf{v}}) := \frac{1}{2} X^{\mathsf{v}}(Y^{\mathsf{v}} L^2); \quad X, Y \in \Gamma(TM)$$

is called the *metric tensor* associated to L. The components of G with respect to an induced chart are the functions

$$G_{ij} := G\left(\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}}, \left(\frac{\partial}{\partial u^j}\right)^{\mathsf{v}}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}.$$

A function $L : TM \to \mathbb{R}$ is said to be a *strongly convex* Finsler structure on M if it satisfies (F1)–(F3) and
($F4^+$) *G is fibrewise positive definite.*
In local coordinates this means that the *vertical Hessian*

$$(G_{ij}) = \frac{1}{2}\left(\frac{\partial^2 L^2}{\partial y^i \partial y^j}\right)$$

of L is positive definite at each point $v \in \pi^{-1}(\mathcal{U}) \cap E$.

It is well-known that, under (F1)–(F3), ($F4^+$) implies (F4); see e.g. [Ba-Ch-Sh]. Traditionally, a Finsler structure is defined by the stronger conditions (F1)–(F3) and ($F4^+$).

From now on all Finsler structure will be assumed strongly convex.

There exists a unique spray $S : E \to TE$ for M satisfying:

(GS1) if γ is a geodesic of (M, L), then $\dot{\gamma}$ is an integral curve of S;

(GS2) if α is an integral curve of S, then $\pi \circ \alpha$ is a geodesic of (M, L).

This spray is called the *geodesic spray* of (M, L).

The vertical subbundle $\mathcal{V}$ of TE has a canonical complementary subbundle $\mathcal{H}$ such that the $C^\infty(E)$-module $\Gamma(\mathcal{H})$ is generated by the ($\mathcal{H}$-) *horizontal lifts*

$$X^{\mathcal{H}} := \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]), \quad X \in \Gamma(TM). \tag{3.1}$$

$\mathcal{H}$ is called the *Berwald nonlinear connection* of (M, L). Specifying this nonlinear connection in E, any vector field $\xi \in \Gamma(E)$ has a unique decomposition of the form

$$\xi = \mathbf{v}\xi + \mathbf{h}\xi; \quad \mathbf{v}\xi \in \Gamma(\mathcal{V}),\ \mathbf{h}\xi \in \Gamma(\mathcal{H}).$$

This leads to two type $(1,1)$ tensors on E, the *vertical projection* $\mathbf{v}$ and the *horizontal projection* $\mathbf{h}$ associated to $\mathcal{H}$.

A fundamental property of the Berwald nonlinear connection is its *compatibility with the Finsler function* in the following sense:

$$X^{\mathcal{H}}L = 0 \quad \text{for all } X \in \Gamma(TM). \tag{3.2}$$

The $(1,1)$-tensor $\mathbf{h}$ (as every vector-valued 1-form on E) determines a graded derivation of degree 1 of the exterior algebra of E, denoted by $d_{\mathbf{h}}$, by the following rules:

(i) $(d_{\mathbf{h}}F)(\xi) := (\mathbf{h}\xi)F$, for all $F \in C^\infty(E)$ and $\xi \in \Gamma(E)$;

(ii) $d_{\mathbf{h}} \circ d = -d \circ d_{\mathbf{h}}$, where d is the standard exterior derivative.

With the help of the operator $d_{\mathbf{h}}$, relation (3.2) may be expressed in the concise form

$$d_{\mathbf{h}}L = 0. \tag{3.3}$$

The Berwald nonlinear connection induces a covariant derivative operator

$$D : \Gamma(E) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}), \quad (\xi, \mathbf{J}\eta) \mapsto D_\xi \mathbf{J}\eta \ (\eta \in \Gamma(E))$$

given by

$$D_{\mathbf{h}\xi}\mathbf{J}\eta : = \mathbf{v}[\mathbf{h}\xi, \mathbf{J}\eta], \tag{3.4}$$

$$D_{\mathbf{J}\xi}\mathbf{J}\eta : = \mathbf{J}[\mathbf{J}\xi, \eta]. \tag{3.5}$$

This covariant derivative is called the *Berwald derivative* (or *Berwald connection*) of the Finsler manifold (M, L). With the choice $\xi := X^{\mathsf{c}}$, $\eta := Y^{\mathsf{c}}$ $(X, Y \in \Gamma(TM))$ (3.4) takes the form

$$D_{X^{\mathcal{H}}}Y^{\mathsf{v}} = [X^{\mathcal{H}}, Y^{\mathsf{v}}] = \mathcal{L}_{X^{\mathcal{H}}}Y^{\mathsf{v}} \tag{3.6}$$

(since $\mathbf{h}X^{\mathsf{c}} = X^{\mathcal{H}}$, $\mathbf{J}Y^{\mathsf{c}} = Y^{\mathsf{v}}$ and $[X^{\mathcal{H}}, Y^{\mathsf{v}}] \in \Gamma(\mathcal{V})$). It may be shown that

$$D_{X^{\mathcal{H}}}Y^{\mathsf{v}} - D_{Y^{\mathcal{H}}}X^{\mathsf{v}} - [X,Y]^{\mathsf{v}} = 0 \quad (X, Y \in \Gamma(TM)); \tag{3.7}$$

this expresses that the Berwald derivative (or, more precisely, the Berwald nonlinear connection) is *torsion-free*.

4. Berwald manifolds and Landsberg manifolds

First we recall the concept of parallel displacement on a Finsler manifold with respect to the Berwald nonlinear connection $\mathcal{H}$.

Let $I \subset \mathbb{R}$ be an open interval containing 0. Let $c : I \to M$ be a (smooth) curve and $X : I \to E$ a vector field along c. X is called *$\mathcal{H}$-parallel* (or simply *parallel*) along c if

$$\dot{X}(t) \in \mathcal{H}_{\dot{c}(t)} \quad \text{for all } t \in I.$$

Given a tangent vector $v \in T_{c(0)}M \setminus \{o\}$, there exists a unique $\mathcal{H}$-parallel vector field X along c such that $\dot{X}(0) = v$. If $t \in I$, $q := c(t)$, $p := c(0)$, then the mapping

$$P_c : T_pM \to T_qM, \quad v \mapsto P_c(v) := \dot{X}(t)$$

is said to be the *parallel displacement* along c from p to q with respect to $\mathcal{H}$. It is positive-homogeneous, i.e.,

$$P_c(\lambda v) = \lambda P_c(v); \quad v \in T_pM \setminus \{o\},\ 0 < \lambda \in \mathbb{R}.$$

The compatibility of $\mathcal{H}$ and L (see (3.2) or (3.3)) implies that P_c preserves the Minkowski norms of tangent vectors, i.e.,

$$\|P_c(v)\| = \|v\| \quad \text{for all } v \in T_pM \setminus \{o\}.$$

The $\mathcal{H}$-parallel displacements, however, are not isometries in general. As it was shown in [Ic2], this extra property holds if, and only if, the Finsler manifold is a *Berwald manifold*. In our paper we take Ichijyō's characterization as a cue for giving an alternative, geometrical *definition* of Berwald manifolds.

Definition 4.1. A Finsler manifold is said to be a *Berwald manifold* if the parallel displacements with respect to the Berwald nonlinear connection are isometries between the tangent spaces *as Minkowski spaces*.

With the notation of the previous paragraph, this means that

$$\|P_c(v) - P_c(w)\| = \|v - w\| \quad \text{for all } \ v, w \in T_pM.$$

Corollary 4.1. *In a Berwald manifold the parallel displacements with respect to the Berwald nonlinear connection are* linear *isometries between the tangent spaces.*

Indeed, this is an immediate consequence of the Mazur–Ulam theorem.

Now it is not difficult to deduce the following well-known characterization of Berwald manifolds (see, e.g., [Sz-Lo-Ke]): *a Finsler manifold (M, L) is a Berwald manifold if, and only if, the 'horizontal part' of the Berwald connection D is induced by a torsion-free covariant derivative ∇ on M such that*

$$D_{X^{\mathcal{H}}} Y^{\mathsf{v}} = (\nabla_X Y)^{\mathsf{v}} \quad \textit{for all } \ X, Y \in \Gamma(TM). \tag{4.1}$$

Actually, due to the clever observation of Z. I. Szabó mentioned in the Introduction, we have more precise information on the covariant derivative ∇.

Theorem 4.1 ([Sz]). *If (M, L) is a Berwald manifold, then its Berwald derivative D is induced by the Levi–Civita derivative ∇^g of a Riemannian metric g on M such that*

$$D_{X^{\mathsf{H}}} Y^{\mathsf{v}} = (\nabla^g_X Y)^{\mathsf{v}}; \quad X, Y \in \Gamma(TM).$$

The metric tensor G of a Finsler manifold (M, L) induces a Riemannian metric $\widetilde{G}_p$, for each $p \in M$, on the punctured tangent space $T_pM \setminus \{0\}$ by the rule

$$\begin{cases} (\widetilde{G}_p)_u(v, w) := \displaystyle\sum_{i,j=1}^{n} G_{ij}(u) y^i(v) y^j(w), \\ v, w \in T_pM \cong T_u T_pM \cong \mathcal{V}_u. \end{cases}$$

Following [Ic2] again (see also [Ba]), we introduce the concept of a Landsberg manifold as follows:

Definition 4.2. A Finsler manifold is said to be a *Landsberg manifold* if the parallel displacements with respect to the Berwald nonlinear connection are local Riemannian isometries between the punctured tangent spaces *as Riemannian manifolds.*

Landsberg manifolds also have well-known tensorial characterizations:

Theorem 4.2. *A Finsler manifold is a Landsberg manifold if, and only if, its metric tensor has vanishing horizontal Berwald derivatives or, equivalently, $\mathcal{L}_{X^{\mathsf{H}}} G = 0$ for all $X \in \Gamma(TM)$.*

It has been known for a long time that every Berwald manifold is a Landsberg manifold, while the existence or the non-existence of Landsberg structures that are not of Berwald type has not been clearly decided until now.

5. Averaged Riemannian metrics and connections

The *indicatrix* of a Finsler manifold (M, L) at a point $p \in M$ is the unit sphere

$$I_p := \left\{ v \in T_pM \;\middle|\; \|v\| = L(v) = 1 \right\}$$

of the Minkowski space T_pM. The *unit sphere bundle* associated to TM is $I(M) = \bigcup_{p \in M} I_p$. We define a volume form μ in $\mathcal{V}$ by

$$\mu(X_1^{\mathsf{v}}, \dots, X_n^{\mathsf{v}}) = \sqrt{\det G(X_i, X_j)}; \quad X_i \in \Gamma(TM).$$

In an induced chart $(\pi^{-1}(\mathcal{U}), (x, y))$

$$\mu\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right) = \sqrt{\det G_{ij}} = \sqrt{\det G},$$

so in these coordinates

$$\mu = \sqrt{\det G}\ dy^1 \wedge \dots \wedge dy^n.$$

The $(n-1)$-form

$$\mu_I := i_{\mathcal{E}}\mu, \tag{5.1}$$

where $i_{\mathcal{E}}$ is the substitution operator by $\mathcal{E}$, induces a volume form in $I(M)$, which gives rise to a volume form μ_{I_p} on each indicatrix I_p, $p \in M$.

Lemma 5.1. *If (M, L) is a Landsberg manifold, then*

$$\mathcal{L}_{X^{\mathcal{H}}}\mu_I = 0 \quad \text{for all } X \in \Gamma(TM), \tag{5.2}$$

therefore the volume

$$\mathrm{vol}(I_p) = \int_{I_p} \mu_I$$

of the indicatrix I_p does not depend on the choice of the point $p \in M$.

Proof. It has already been shown in [Ai4] that if (M, L) is a Landsberg manifold, then

$$\mathcal{L}_{X^{\mathcal{H}}}\mu = 0 \quad \text{for all } X \in \Gamma(TM).$$

Hence

$$\mathcal{L}_{X^{\mathcal{H}}}\mu_I = \mathcal{L}_{X^{\mathcal{H}}} i_{\mathcal{E}}\mu = i_{\mathcal{E}}\mathcal{L}_{X^{\mathcal{H}}}\mu + i_{[X^{\mathcal{H}},\mathcal{E}]}\mu = 0,$$

taking into account that $[X^{\mathcal{H}}, \mathcal{E}] = 0$ by the homogeneity of the Berwald nonlinear connection. $\square$

Lemma 5.2 ([Ma-Ra-Tr-Ze, Vi])**.** *Let (M, L) be a Finsler manifold with metric tensor G. Then the mapping*

$$g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M),\ (X, Y) \mapsto g(X, Y)$$

given by

$$g(X, Y)(p) := \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} G(X^{\mathsf{v}}, Y^{\mathsf{v}})\mu_{I_p} \tag{5.3}$$

is a Riemannian metric on M.

The Riemannian metric given by (5.3) is said to be the *averaged Riemannian metric* for (M, L). If $g = \sum_{i,j} g_{ij} du^i \otimes du^j$ in a chart $(\mathcal{U}, (u^i)_{i=1}^n)$, then the component functions g_{ij} of the averaged Riemannian metric are given by

$$g_{ij}(p) = \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} G_{ij}\mu_{I_p}, \quad p \in \mathcal{U}.$$

Lemma 5.3. *Let g be the averaged Riemannian metric for a Finsler manifold (M,L). Then the mapping*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X,Y) \mapsto \nabla_X Y$$

given by

$$g\left(\nabla_X Y, Z\right)(p) := \frac{1}{\operatorname{vol}(I_p)} \int_{I_p} G\left(D_{X^{\mathcal{H}}} Y^{\mathsf{v}}, Z^{\mathsf{v}}\right) \mu_{I_p} \tag{5.4}$$

($Z \in \Gamma(TM)$, $p \in M$) is a torsion-free covariant derivative on the manifold M.

Proof. The mapping ∇ is obviously additive both in X and Y. Since for all $f \in C^\infty(M)$, $(fX)^{\mathcal{H}} = f^{\mathsf{v}} X^{\mathcal{H}}$ and f^{v} is fibrewise constant, it follows immediately that ∇ is tensorial in X.

We show that ∇ is a derivation in Y. Let $f \in C^\infty(M)$. Then $X^{\mathcal{H}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}}$, $(fY)^{\mathsf{v}} = f^{\mathsf{v}} Y^{\mathsf{v}}$, so we obtain

$$\begin{aligned}
g\left(\nabla_X fY, Z\right)(p) &= \frac{1}{\operatorname{vol}(I_p)} \int_{I_p} G(D_{X^{\mathcal{H}}}(fY)^{\mathsf{v}}, Z^{\mathsf{v}}) \mu_{I_p} \\
&= \frac{1}{\operatorname{vol}(I_p)} \int_{I_p} G\big((Xf)^{\mathsf{v}} Y^{\mathsf{v}} + f^{\mathsf{v}} D_{X^{\mathcal{H}}} Y^{\mathsf{v}}, Z^{\mathsf{v}}\big) \mu_{I_p} \\
&= \frac{(Xf)(p)}{\operatorname{vol}(I_p)} \int_{I_p} G(Y^{\mathsf{v}}, Z^{\mathsf{v}}) \mu_{I_p} + \frac{f(p)}{\operatorname{vol}(I_p)} \int_{I_p} G\left(D_{X^{\mathcal{H}}} Y^{\mathsf{v}}, Z^{\mathsf{v}}\right) \mu_{I_p} \\
&=: \big((Xf) g(Y,Z) + f g(\nabla_X Y, Z)\big)(p) = g\big((Xf)Z + f\nabla_X Y, Z\big)(p)
\end{aligned}$$

therefore $\nabla_X fY = (Xf)Y + f\nabla_X Y$.

Finally, the torsion-freeness of ∇ is an immediate consequence of the definition and (3.7). $\square$

We say that the torsion-free covariant derivative ∇ is the *averaged covariant derivative* obtained from (or associated to) the Berwald derivative of (M,L); cf. [To-Et].

To prepare our next result, let X be a vector field on M, and let

$$\varphi : W \subset \mathbb{R} \times M \to M, \; (t,p) \mapsto \varphi(t,p) = \varphi_t(p) = \varphi_p(t)$$

be its (local) flow. (The domain W of φ is a radial open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$.) The local flow of the horizontal lift $X^{\mathcal{H}}$ of X is $\varphi^{\mathcal{H}} : \widetilde{W} \to E$, $(t,v) \mapsto \varphi^{\mathcal{H}}(t,v) = \varphi^{\mathcal{H}}_t(v) = \varphi^{\mathcal{H}}_v$, where $\widetilde{W} = \big\{(t,v) \in \mathbb{R} \times E \big| (t,\pi(v)) \in W\big\}$ and, for any fixed $v \in E$, $\varphi^{\mathcal{H}}_v$ is the *horizontal lift* of the curve $\varphi_{\pi(v)} : t \mapsto \varphi(t,\pi(v))$ starting from v given by the conditions

$$\begin{cases} \pi \circ \varphi^{\mathcal{H}}_v = \varphi_{\pi(v)}, \\ \dot{\varphi}^{\mathcal{H}}_v(t) \in \mathcal{H}_{\varphi^{\mathcal{H}}_v(t)}, \; \varphi^{\mathcal{H}}_v(0) = v. \end{cases}$$

For a proof of this simple observation we refer to [Sz-Lo-Ke].

Theorem 5.1. *Let (M, L) be a Landsberg manifold and g the averaged Riemannian metric for (M, L). The averaged covariant derivative ∇ obtained from the Berwald derivative D of (M, L) is the Levi–Civita derivative of g.*

PROOF. As we have seen in Lemma 5.1, in a Landsberg manifold (M, L) the function

$$M \to \mathbb{R},\ p \mapsto \mathrm{vol}(I_p)$$

is constant. Keeping the notation of the previous paragraph, let X be a vector field on M and φ its flow. Relation (5.2) implies that $(\varphi_t^{\mathcal{H}})^*\mu_I = \mu_I$ for each $t \in \mathbb{R}$ such that $(t, v) \in \widetilde{W}$ for all $v \in E$. Now, for any vector fields $Y, Z \in \Gamma(TM)$ and at each point $p \in M$ we have

$$\begin{aligned}
(\varphi_t^*)(g(Y,Z))(p) = g(Y,Z)(\varphi_t(p)) &= \frac{1}{\mathrm{vol}(I_{\varphi_t(p)})}\int_{I_{\varphi_t(p)}} G(Y^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I \\
&= \frac{1}{\mathrm{vol}(\varphi_t^{\mathcal{H}}(I_p))}\int_{\varphi_t^{\mathcal{H}}(I_p)} G(Y^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I \\
&= \frac{1}{\mathrm{vol}(I_p)}\int_{I_p} (\varphi_t^{\mathcal{H}})^*\big(G(Y^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I\big) \\
&= \frac{1}{\mathrm{vol}(I_p)}\int_{I_p} \big((\varphi_t^{\mathcal{H}})^* G(Y^{\mathsf{v}}, Z^{\mathsf{v}})\big)\mu_I
\end{aligned}$$

which implies

$$X_p\, g(Y,Z) = \frac{1}{\mathrm{vol}(I_p)}\int_{I_p} \big(X^{\mathcal{H}} G(Y^{\mathsf{v}}, Z^{\mathsf{v}})\big)\mu_I. \tag{5.5}$$

Applying (5.5), (5.4) and Theorem 4.2 we obtain

$$\begin{aligned}
&(Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z))(p) \\
&\quad = \frac{1}{\mathrm{vol}(I_p)}\int_{I_p} \big(X^{\mathcal{H}} G(Y^{\mathsf{v}}, Z^{\mathsf{v}}) - G(D_{X^{\mathcal{H}}} Y^{\mathsf{v}}, Z^{\mathsf{v}}) - G(Y, D_{X^{\mathcal{H}}} Z^{\mathsf{v}})\big)\mu_I \\
&\quad = \frac{1}{\mathrm{vol}(I_p)}\int_{I_p} (D_{X^{\mathcal{H}}} G)(Y^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I = 0.
\end{aligned}$$

Thus ∇ is compatible with the metric tensor g. Since it is torsion-free at the same time, ∇ is the Levi–Civita connection of the averaged metric g. □

Having this result, Theorem 4.1 can be refined as follows.

Theorem 5.2 (see also [Vi]). *If (M, L) is a Berwald manifold, then its Berwald derivative is induced by the Levi–Civita derivative ∇^g of the* averaged Riemannian metric *g for (M, L).*

Proof. In the Berwaldian case the Berwald derivative is induced by a covariant derivative ∇' on M according to (4.1). Thus for the averaged covariant derivative ∇ obtained from the Berwald derivative we have

$$g(\nabla_X Y, Z)(p) \overset{(5.4)}{=} \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} G((\nabla'_X Y)^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I \overset{(5.3)}{=} g(\nabla'_X Y, Z)(p)$$

$(X, Y, Z \in \Gamma(TM),\ p \in M)$; hence $\nabla = \nabla'$. Since a Berwald manifold is also a Landsberg manifold, by the preceding theorem $\nabla = \nabla^g =$ the Levi–Civita derivative of g. □

Remark 5.1. Suppose (M, L) is a Landsberg manifold. Let $\widetilde{\nabla}$ be the covariant derivative on the vertical bundle $\mathcal{V}$ induced by the averaged covariant derivative ∇ for (M, L). Then, for any vector fields X, Y on M,

$$\widetilde{\nabla}_{X^{\mathsf{v}}} Y^{\mathsf{v}} = 0, \quad \widetilde{\nabla}_{X^{\mathcal{H}}} Y^{\mathsf{v}} = (\nabla_X Y)^{\mathsf{v}}.$$

Since ∇ is compatible with the averaged Riemannian metric g, applying (5.5) and (5.3) we obtain

$$\begin{aligned}
0 &= (Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z))(p) \\
&= \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} \big(X^{\mathcal{H}} G(Y^{\mathsf{v}}, Z^{\mathsf{v}}) - G\big((\nabla_X Y)^{\mathsf{v}}, Z^{\mathsf{v}}\big) - G\big(Y^{\mathsf{v}}, (\nabla_X Z)^{\mathsf{v}}\big)\big)\mu_I \\
&= \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} \big(X^{\mathcal{H}} G(Y^{\mathsf{v}}, Z^{\mathsf{v}}) - G\big(\widetilde{\nabla}_{X^{\mathcal{H}}} Y^{\mathsf{v}}, Z^{\mathsf{v}}\big) - G\big(Y^{\mathsf{v}}, \widetilde{\nabla}_{X^{\mathcal{H}}} Z^{\mathsf{v}}\big)\big)\mu_I \\
&= \frac{1}{\mathrm{vol}(I_p)} \int_{I_p} \big(\widetilde{\nabla}_{X^{\mathcal{H}}} G\big)\,(Y^{\mathsf{v}}, Z^{\mathsf{v}})\,\mu_I.
\end{aligned}$$

Thus, in a Landsberg manifold (M, L) we have

$$\int_{I_p} \big(\widetilde{\nabla}_{X^{\mathcal{H}}} G\big)(Y^{\mathsf{v}}, Z^{\mathsf{v}})\mu_I = 0 \tag{5.6}$$

for all $X, Y, Z \in \Gamma(TM)$, $p \in M$. If, in particular, (M, L) is a Berwald manifold, then (5.6) is satisfied since $\widetilde{\nabla} = D =$ the Berwald derivative of (M, L). So, to find a Landsberg structure which is not of Berwald type, one has to construct a Finsler function satisfying (5.6), but for which $\widetilde{\nabla}_{X^{\mathcal{H}}} G \neq 0$, $X \in \Gamma(TM)$.

6. L.c. Berwald manifolds and Finsler–Weyl structures

Let (M, L) be a Finsler manifold, and let ∇ be a torsion-free covariant derivative on M, whose Christoffel symbols with respect to a chart $(\mathcal{U}, u)$ are the functions γ^i_{jk} $(i, j, k \in \{1, \dots, n\})$. The covariant derivative ∇ induces a horizontal subbundle $\mathcal{H}^\nabla$ of TTM spanned in the induced chart $(\pi^{-1}(\mathcal{U}), (x, y))$ by the local vector fields

$$\frac{\partial}{\partial x^j} - \sum_{k,l} y^k \gamma^l_{jk} \frac{\partial}{\partial y^l}, \quad j \in \{1, \dots, n\}. \tag{6.1}$$

Let $\mathbf{h}^\nabla$ be the horizontal projection associated to $\mathcal{H}^\nabla$. It may easily be shown that if $d_{\mathbf{h}^\nabla} L = 0$ (c.f. (3.3)), then the Berwald derivative of (M, L) is induced by ∇. Therefore we have

Proposition 6.1. *A Finsler manifold (M, L) is a Berwald manifold if, and only if, there exists a torsion-free covariant derivative ∇ on M such that $d_{\mathbf{h}^\nabla} L = 0$. This covariant derivative is just the Levi–Civita derivative of the averaged Riemannian metric for (M, L).*

Let σ be a smooth function on M, and denote by e^σ the composite function $\exp \circ \sigma$. Consider a *conformal change*

$$L \longrightarrow \bar{L} := (e^\sigma)^{\mathsf{v}} L \tag{6.2}$$

of the Finsler function L. Then, obviously, $\bar{L}$ is also a (strongly convex) Finsler function. Under this conformal change, the metric tensor G, the indicatrix I_p at a point $p \in M$, and the volume form μ on $\mathcal{V}$ change as follows:

$$\bar{G} = (e^{2\sigma})^{\mathsf{v}}\, G, \tag{6.3}$$

$$\bar{\mu} = (e^{n\sigma})^{\mathsf{v}}\, \mu, \tag{6.4}$$

$$\bar{I}_p = \{e^{-\sigma(p)} v \in T_pM \mid v \in I_p\} = \psi(I_p), \tag{6.5}$$

where

$$\psi : E \longrightarrow E;\ v \mapsto e^{-\sigma(p)} v, \quad \text{if } v \in T_pM.$$

From relation (6.4) we obtain

$$\bar{\mu}_I = (e^{n\sigma})^{\mathsf{v}}\, \mu_I. \tag{6.6}$$

Lemma 6.1. *Under the conformal change* (6.2), *the volume of an indicatrix* I_p *changes by*

$$\operatorname{vol}(\bar{I}_p) = e^{\sigma(p)} \operatorname{vol}(I_p) \tag{6.7}$$

and the change of the averaged Riemannian metric for (M, L) *is given by*

$$\bar{g} = (e^{2\sigma})^{\mathsf{v}} g. \tag{6.8}$$

Proof. Using (6.6) and the change of variables theorem,

$$\begin{aligned}\operatorname{vol}(\bar{I}_p) :&= \int_{\bar{I}_p} \bar{\mu}_I = \int_{\psi(I_p)} (e^{n\sigma})^{\mathsf{v}} \mu_I = e^{n\sigma(p)} \int_{I_p} \psi^* \mu_I \\ &= e^{n\sigma(p)} \int_{I_p} (e^{-(n-1)\sigma})^{\mathsf{v}} \mu_I = e^{\sigma(p)} \int_{I_p} \mu_I = e^{\sigma(p)} \operatorname{vol}(I_p),\end{aligned}$$

as we claimed.

Let $\bar{g}$ be the averaged Riemannian metric for (M, L). Then for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned}\bar{g}(X, Y)(p) &:= \frac{1}{\operatorname{vol}(\bar{I}_p)} \int_{\bar{I}_p} \bar{G}(X^{\mathsf{v}}, Y^{\mathsf{v}}) \bar{\mu}_I \\ &\overset{(6.3),(6.6),(6.8)}{=} \frac{e^{-\sigma(p)}}{\operatorname{vol}(I_p)} \int_{\psi(I_p)} (e^{2\sigma})^{\mathsf{v}} \cdot (e^{n\sigma})^{\mathsf{v}} G(X^{\mathsf{v}}, Y^{\mathsf{v}}) \mu_I \\ &= \frac{e^{(n+1)\sigma(p)}}{\operatorname{vol}(I_p)} \int_{I_p} \psi^* \big(G(X^{\mathsf{v}}, Y^{\mathsf{v}}) \mu_I\big) \\ &= \frac{e^{(n+1)\sigma(p)}}{\operatorname{vol}(I_p)} \int_{I_p} \big(G(X^{\mathsf{v}}, Y^{\mathsf{v}}) \circ \psi\big) \psi^* \mu_I \\ &\overset{(*)}{=} \frac{e^{(n+1)\sigma(p)}}{\operatorname{vol}(I_p)} \int_{I_p} G(X^{\mathsf{v}}, Y^{\mathsf{v}}) (e^{-(n-1)\sigma})^{\mathsf{v}} \mu_I \\ &= \frac{e^{2\sigma(p)}}{\operatorname{vol}(I_p)} \int_{I_p} G(X^{\mathsf{v}}, Y^{\mathsf{v}}) \mu_I =: (e^{2\sigma})^{\mathsf{v}} g(X, Y)(p)\end{aligned}$$

taking into account at step $(*)$ that the function $G(X^{\mathsf{v}}, Y^{\mathsf{v}}) : E \to \mathbb{R}$ is positive-homogeneous of degree zero, and hence $G(X^{\mathsf{v}}, Y^{\mathsf{v}}) \circ \psi = G(X^{\mathsf{v}}, Y^{\mathsf{v}})$. $\square$

Motivated by Proposition 6.1, we say that a Finsler manifold (M, L) is *conformally Berwald* if there exists a conformal change of type (6.2) and a torsion-free covariant derivative ∇ on M such that $d_{\mathbf{h}^\nabla} \bar{L} = 0$.

Lemma 6.2 (cf. [Ai2]). *A Finsler manifold (M, L) is conformally Berwald if, and only if, there exists a function $\sigma \in C^\infty(M)$ and a torsion-free covariant derivative ∇ on M such that*

$$d_{\mathbf{h}^\nabla + d\sigma^{\mathsf{v}} \otimes \mathcal{E}} L = 0. \tag{6.9}$$

Proof. For any vector field X on M, we have

$$\begin{aligned}(d_{\mathbf{h}^\nabla} \bar{L})(X^{\mathsf{c}}) &= X^{\mathcal{H}^\nabla}((e^\sigma)^{\mathsf{v}} L) = \big(X^{\mathcal{H}^\nabla}(e^\sigma)^{\mathsf{v}}\big)L + (e^\sigma)^{\mathsf{v}}\big(X^{\mathcal{H}^\nabla} L\big) \\ &= (Xe^\sigma)^{\mathsf{v}} L + (e^\sigma)^{\mathsf{v}}(\mathbf{h}^\nabla X^{\mathsf{c}})L = (e^\sigma)^{\mathsf{v}}\big((X\sigma)^{\mathsf{v}} L + (\mathbf{h}^\nabla X^{\mathsf{c}})L\big) \\ &= (e^\sigma)^{\mathsf{v}}\big((X^{\mathsf{c}}\sigma^{\mathsf{v}})\mathcal{E}L + (\mathbf{h}^\nabla X^{\mathsf{c}})L\big) = (e^\sigma)^{\mathsf{v}}\big((d\sigma^{\mathsf{v}} \otimes \mathcal{E})X^{\mathsf{c}} + \mathbf{h}^\nabla X^{\mathsf{c}}\big)L \\ &= (e^\sigma)^{\mathsf{v}} d_{\mathbf{h}^\nabla + d\sigma^{\mathsf{v}} \otimes \mathcal{E}}(X^{\mathsf{c}}).\end{aligned}$$

Hence $d_{\mathbf{h}^\nabla} \bar{L} = (e^\sigma)^{\mathsf{v}} d_{\mathbf{h}^\nabla + d\sigma^{\mathsf{v}} \otimes \mathcal{E}} L$, therefore $d_{\mathbf{h}^\nabla} \bar{L}$ vanishes if, and only if, relation (6.9) is satisfied. □

Definition 6.1. A Finsler manifold (M, L) is said to be a *locally conformal Berwald (l.c. Berwald)* manifold, if there exist an open covering $(\mathcal{U}_\alpha)_{\alpha \in A}$ of M, a family $(\sigma_\alpha)_{\alpha \in A}$ of smooth functions $\sigma_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ and a torsion-free covariant derivative ∇ on M such that

$$d_{\mathbf{h}^\nabla}\big((e^{\sigma_\alpha})^{\mathsf{v}} L\big) = 0 \quad \text{for all } \alpha \in A. \tag{6.10}$$

Lemma 6.3. *If (M, L) is an l.c. Berwald manifold, then the family $(d\sigma_\alpha)_{\alpha \in A}$ of local 1-forms defines a single closed 1-form β_L on M.*

Proof. Let (by a slight abuse of notation) for each $\alpha \in A$

$$\bar{\mathbf{h}}_\alpha := \mathbf{h}^\nabla + d\sigma_\alpha^{\mathsf{v}} \otimes \mathcal{E}.$$

Then, by Lemma 6.2, $d_{\bar{\mathbf{h}}_\alpha} = 0$ $(\alpha \in A)$. So for any $\alpha, \beta \in A$ with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$ and for any vector field X on M we have

$$\begin{aligned}0 = (d_{\bar{\mathbf{h}}_\alpha} L - d_{\bar{\mathbf{h}}_\beta} L)(X^{\mathsf{c}}) &= (X^{\mathsf{c}}\sigma_\alpha^{\mathsf{v}} - X^{\mathsf{c}}\sigma_\beta^{\mathsf{v}})L \\ &= (X\sigma_\alpha - X\sigma_\beta)^{\mathsf{v}} L = \big((d\sigma_\alpha - d\sigma_\beta)X\big)^{\mathsf{v}} L\end{aligned}$$

whence $d\sigma_\alpha \restriction \mathcal{U}_\alpha \cap \mathcal{U}_\beta = d\sigma_\beta \restriction \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. So if we set $\beta_L \restriction \mathcal{U}_\alpha := d\sigma_\alpha$, $\alpha \in A$, then we obtain the desired 1-form on M. □

From Lemma 6.3 we conclude immediately

Proposition 6.2 (cf. [Ai2, Ha, Sz-Sz]). *A Finsler manifold (M, L) is an l.c. Berwald manifold if, and only if, there exists a torsion-free covariant derivative ∇ and a closed 1-form β_L on M such that $d_{\mathbf{h}^{\nabla+\beta_L\otimes\mathcal{E}}}L = 0$. Then $\bar{\nabla} := \nabla - \beta_L \otimes 1_{\Gamma(TM)}$ is a semi-symmetric covariant derivative on M (i.e., its torsion $T^{\bar{\nabla}}$ satisfies $T^{\bar{\nabla}}(X, Y) = \beta_L(Y)X - \beta_L(X)Y$), and $d_{\mathbf{h}^{\nabla}}L = 0$. Thus (M, L) (or, more accurately, the quadruple (M, L, ∇, β_L)) is a* Wagner manifold.

Notice that the covariant derivative $\bar{\nabla}$ is compatible with the averaged Riemannian metric g for (M, L), i.e., $\bar{\nabla}g = 0$.

Example 6.1 ([Ai3]). Let (M, g) be a Riemannian manifold and ξ a semi-parallel vector field on M, i.e., a vector field satisfying $g(\xi, \xi) = 1$ and

$$\nabla^g \xi = \rho(1_{\Gamma(TM)} - \beta \otimes \xi),$$

where ∇^g is the Levi–Civita derivative of (M, g), ρ is a real number, and β is the 1-form metrically equivalent to ξ, i.e.,

$$\beta(X) = g(X, \xi) \quad \text{for all} \quad X \in \Gamma(TM).$$

Now we define a Finsler function L by the rule

$$L(v) := \sqrt{g_{\pi(v)}(v, v)} + c\beta_{\pi(v)}(v), \quad v \in TM,$$

where c is a fixed scalar in $]\,0, 1\,[$. Such a Finsler function is called (by an abuse of language) a *Randers metric* on M. Assumption $0 < c < 1$ guarantees that L is indeed strongly convex (see [An-In-Ma]). It is easy to check that β is closed, and the formula

$$\nabla_X Y := \nabla^g_X Y + \rho(g(X, Y)\xi - \beta(Y)X); \quad X, Y \in \Gamma(TM)$$

defines a semi-symmetric covariant derivative on M satisfying $d_{\mathbf{h}^{\nabla}}L = 0$. Therefore this Randers manifold is an l.c. Berwald manifold and hence also a Wagner manifold.

In the rest of the paper we assume that (M, L) is an l.c. Berwald manifold with the data ∇, $(\mathcal{U}_\alpha)_{\alpha\in A}$, $(\sigma_\alpha)_{\alpha\in A}$ as in Definition 6.1.

Observe first that (M, L) remains an l.c. Berwald manifold under any conformal change of the form (6.2). Indeed, for all $\alpha \in A$ we have

$$0 = d_{\mathbf{h}^{\nabla}}((e^{\sigma_\alpha})^{\mathsf{v}}L) = d_{\mathbf{h}^{\nabla}}((e^{\sigma_\alpha - \sigma})^{\mathsf{v}}\bar{L}),$$

and the 1-form $\beta_{\bar{L}}$ described in Lemma 6.3 is given by $\beta_{\bar{L}} = \beta_L - d\sigma$.

Let $\mathcal{C}$ be the conformal class of the Finsler function L ([Ai2]), and let $A^1(M)$ denote the $C^\infty(M)$-module of 1-forms on M. If

$$\beta : \mathcal{C} \longrightarrow A^1(M), \quad \bar{L} \mapsto \beta(\bar{L}) := \beta_{\bar{L}},$$

then $(\mathcal{C}, \beta)$ is a *Finsler–Weyl structure* in the sense of [Ko]. Thus we obtain

Proposition 6.3. *On any l.c. Berwald manifold there exists a natural Finsler–Weyl structure.*

Now let g be the averaged Riemannian metric for (M, L), and let (by an abuse of notation)

$$g^\alpha := e^{2\sigma_\alpha} g \quad \text{for all } \alpha \in A.$$

By Lemma 6.1, g^α is the averaged Riemannian metric for the local Finsler function $(\sigma^\alpha)^\vee L \restriction \pi^{-1}(\mathcal{U}_\alpha)$, briefly for $(\sigma^\alpha)^\vee L$. This Finsler function is of Berwald type on its domain, so by Proposition 6.1 we have

$$0 = \nabla g^\alpha = \nabla(e^{2\sigma_\alpha} g) = e^{2\sigma_\alpha}(\nabla g + 2d\sigma^\alpha \otimes g)$$

whence $\nabla g = -2\beta_L \otimes g$. Thus we have proved

Theorem 6.1. *If (M, L) is an l.c. Berwald manifold and g is the averaged Riemannian metric for (M, L), then the torsion-free covariant derivative ∇ satisfying* (6.10) *is the Weyl connection for the conformal class of g.*

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