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A class of non-recurring sequences over a Galois field

By HASSAN AL-ZAID (Kuwait) and SURJEET SINGH (Kuwait)

Abstract. Let F be a Galois field and $\Gamma(F)$ be the set of all sequences $(s_k)_{k\geq 0}$ over F. For any non-zero polynomial $f(D)$ over F, the set $\Omega(f(D))$ of those $S\in\Gamma(F)$ of which $f(D)$ is a charasteristic polynomial has been extensively studied by many authors for the recurrence properties of its members and for its module theoretic properties. However $\Gamma(F)$ has uncountably many non-recurring sequences. For any $f(D)\neq 0$ in $F[D]$ the concept of a pseudo-periodic sequence having $f(D)$ as its pseudo-charasteristic polynomial is introduced. The set $\bar{\Omega}(f(D))$ of all such sequences in $\Gamma(F)$ contains uncountably many non-recurring sequences. The set $\bar{\Omega}(F(D))$ is found to have interesting module theoretic properties. The lattice $L(F)$ of these $\bar{\Omega}(f(D))$ is investigated. In this investigation $\bar{\Omega}(1)$ is found to play a crucial role.

Introduction

Let F be a Galois field and $F[D]$ be the ring of polynomials over F in the indeterminate D. The vector space $\Gamma(F)$ of all sequences over F is a divisible $F[D]$–module [3]. For any $f(D)\neq 0$ in $F[D]$

$$\Omega(f(D))=\{S\in\Gamma(F):f(D)S=0\}$$

is a finite $F[D]$–module, whose members are recurring sequences. The sum $W(F)$ of such $\Omega(f(D))$ is the torsion submodule of $\Gamma(F)$. There are uncountably many non-recurring sequences in $\Gamma(F)$. One of the simplest example of a non-recurring sequence is a sequence $S=(s_k)$ which is not eventually zero and in which between any two consecutive non-zero terms s_k, s_ℓ, $k<\ell$, the number $\ell-k-1$ of zero terms strictly increases. This example has motivated the definition of a pseudo-periodic sequence and its pseudo-characteristic polynomial, given in section 3. The definition depends upon that of a sparse set of natural numbers given in section 2. The concept of a sparse set is a generalization of that of lacunary sets used in investigating power series. Some results on sparse sets that may also be of independent interest are proved in section 2. For any $f(D)\neq 0$ in $F[D]$, the set $\bar{\Omega}(f(D))$ of pseudo-periodic sequences with pseudo-characteristic polynomial $f(D)$ is investigated in section 3. The class $L(F)$ of these $\bar{\Omega}(f(D))$ is shown to be closed under finite intersections and sums. Let $\bar{W}(F)$ be the sum of all $\bar{\Omega}(f(D))$'s. Let L be any injective hull of $\bar{\Omega}(1)$ in $\Gamma(F)$. Beside other results it is shown that $\bar{W}(F) = W(F) + L$ and $L \cap W(F) = \Omega(D^\infty)$.

1. Preliminaries

Throughout F is a Galois field. For any $S = (s_n)_{n\geq 0}$ in $\Gamma(F)$ and $f(D) = \sum\limits_{i=0}^{k} a_i D^i \in F[D]$, define $f(D)\cdot S = (w_n)$ such that $w_n = \sum_i a_i s_{n+i}$. This makes $\Gamma(F)$ a divisible left $F[D]$–module [3]. For $f(D) \in F[D]$ of degree $k \geq 0$,

$$\Omega(f(D)) = \{S \in \Gamma(F) : f(D) \cdot S = 0\}$$

is a submodule of $\Gamma(F)$, whose dimension over F is k. $DS = (w_n)$, with $w_n = s_{n+1}$. The set $\Omega(f(D)^\infty)$ equals $\bigcup\limits_{n\geq 1} \Omega(f(D)^n)$; it is the smallest divisible (hence injective) submodule of $\Gamma(F)$ containing $\Omega(f(D))$. For any module $M, N \subset' M$ denotes that N is an essential submodule of M. For basic concepts on rings and modules one may refer to [1] and for recurring sequences to [2].

2. Sparse subsets

Throughout, $\mathbb{N}$ denotes the set of natural numbers.

Definition 2.1. An infinite subset A of $\mathbb{N}$ is called a sparse set if there exists an integer $t \geq 2$, depending on A, with the property that given $k > 0$, there exists $m \geq 0$ such that for any $m_i \in A$, $1 \leq i \leq t$, satisfying

$$m_1 > m_2 > \cdots > m_t \geq m$$

one has $m_1 - m_t \geq k$. The smallest t satisfying the above condition is called the sparsity of A and is denoted by $s(A)$.

Let $S(\mathbb{N})$ denote the set of all sparse subsets of $\mathbb{N}$. Each $A \in S(N)$ will be also written as an infinite sequence $(m_i)_{i\geq 0}$ such that $m_i < m_{i+1}$. We define $DA = (n_i)$ with $n_i = m_{i+1}$. Further m_{i+1} is called the successor of m_i in A and m_i is called the predecessor of m_{i+1}. For any $r \geq 0$, m_{i+r} is called the r-th successor of m_i in A. Finally $\{m_i, m_{i+1}\}$ is called a consecutive pair in A.

Lemma 2.2. (i) $S(\mathbb{N})$ *is closed under finite union. For any* $A_1, A_2 \in S(\mathbb{N})$, $s(A_1 \cup A_2) \leq s(A_1) + s(A_2)$.

(ii) *For any $A \in S(\mathbb{N})$ any subset B of A is either finite or $B \in S(\mathbb{N})$ with $s(B) \leq s(A)$.*

(iii) *For any $A \in S(\mathbb{N})$, $DA \in S(\mathbb{N})$ with $s(DA) = s(A)$.*

(iv) *For any positive integer k the set $S_k(\mathbb{N})$ of those $A = (m_i) \in S(\mathbb{N})$ with $m_{i+1} - m_i \geq k$ for every i, is uncountable.*

Proof. Let $s(A_1) = t$, $s(A_2) = u$. Given any $k \geq 1$, there exists $m \in \mathbb{N}$ such that for any t members of A_1 or u members of A_2, all greater than m, the difference between the largest and the smallest among them is at least k. Consider any $t + u$ members of $A_1 \cup A_2$, all greater than m. Then either at least t of them are in A_1 or at least u of them are in A_2. Consequently the difference between the largest and the smallest among them is at least k. This proves that $A_1 \cup A_2 \in S(\mathbb{N})$ and $s(A_1 \cup A_2) \leq s(A_1) + s(A_2)$. This proves (i). Further (ii) and (iii) are obvious. Finally (iv) follows from (ii).

Lemma 2.3. *Let $A \in S(\mathbb{N})$ with $s(A) = t$ and k be any positive integer. Then there exists $m \in A$ such that for any $p, q \in A$ satisfying $p > q \geq m$, $p - q \geq tk$, there exist consecutive members $r, s \in A$ such that $q \leq s < r \leq p$ and $r - s \geq k$.*

Proof. By definition there exists $m \in A$ such that given

$$m_1 > m_2 > \cdots > m_t \geq m$$

in A, $m_1 - m_t \geq tk$. Let the result be false for some $p, q \in A$ with $p - q \geq tk$ and $p > q \geq m$. We get a sequence

$$q = u_0 < u_1 < u_2 < \cdots < u_t \leq p$$

in A with each u_i a successor of u_{i-1} and $u_i - u_{i-1} < k$. This gives $u_t - u_0 < tk$. This is a contradiction. This proves the result.

Lemma 2.4. *Let $A \in S(\mathbb{N})$ with $s(A) = t$ and a be a positive integer. Define $A' \subseteq \mathbb{N}$ such that $x \in A'$ if and only if either $x \in A$ or x is the smallest or the largest multiple of a between two consecutive members n, m of A. Then $A' \in S(\mathbb{N})$ with $s(A') \leq 3t + 1$.*

Proof. Observe that given two consecutive members $u < v$ of A, there cannot be more than four members of A' between u and v; two of these are u and v and the other two are of the type pa, where p is the smallest or the largest integer satisfying $u \leq pa \leq v$. Consider any $k \geq 1$. There exists $m \in A$ such that given any t members of A all $\geq m$, the difference between the largest and the smallest among them is at least k. Consider any $3t + 1$ members

$$m \leq m_1 < m_2 < \cdots < m_{3t+1}$$

of A'. Let p_1 be the largest member of A such that $p_1 \leq m_1$. Then $m \leq p_1 \leq m_1$. Let p_2 be the successor of p_1 in A. The observation above given shows that $m_1 \le p_2 \le m_4$. By continuing this process we get a successor sequence

$$p_1 < p_2 < \cdots < p_{t+1}$$

in A such that $p_i \le m_{3(i-1)+1}$. Thus $p_{t+1} \le m_{3t+1}$. As $m_1 \le p_2 < p_{t+1} \le m_{3t+1}$ it is immediate that $m_{3t+1} - m_1 \ge k$.

Lemma 2.5. *Let $A \in S(\mathbb{N})$ with $s(A) = t$. Let k be a fixed positive integer. The set A' consisting of those $x \in \mathbb{N}$ for which either $x \in A$ or $x = n - k$ for some consecutive members n, m of A with $n - m > k$, is a sparse set with $s(A') \le 2t - 1$.*

Proof. Observe that for any $x \in A' - A$, the successor of x in A' is $x + k \in A$. Consider any $x > 0$. There exists $m \in \mathbb{N}$ such that for any

$$m \le m_1 < m_2 < \cdots < m_t$$

with $m_i \in A$ we have $m_t - m_1 \ge x + k$. Consider

$$m \le n_1 < n_2 < \cdots < n_{2t-1}$$

with $n_i \in A'$. This gives

$$m_1 < m_2 < \cdots < m_t$$

in A such that $m_i = n_{2i-1}$ if $n_{2i-1} \in A$ or $m_i = n_{2i-1} + k$ if $n_{2i-1} \notin A$. Then $m \le m_1$ and $m_t - m_1 \ge x + k$. By using this, it follows that $n_{2t-1} - n_1 \ge x$. Hence A' is a sparse set with $s(A') \le 2t - 1$.

We end this section by the remark that given two infinite subsets A, B of $\mathbb{N}$, their sum $C = \{x+y : x \in A, y \in B\}$ is not a sparse set. Suppose the contrary and let C be a sparse set with sparsity v. Let $A = (a_i),\ B = (b_i)$ with $a_i < a_{i+1},\ b_i < b_{i+1}$. Choose $k > b_v - b_1$. By definition there exists $m \in \mathbb{N}$ such that given $z_1 < z_2 < \cdots < z_v$ in C with $m \le z_1$, we have $z_v - z_1 \ge k$. For some s, $a_s \ge m$. This gives $a_s + b_i \ge m$. Consequently $b_v - b_1 = (a_s + b_v) - (a_s + b_1) \ge k$. This is a contradiction. Hence C is not a sparse set. In particular the sum of two sparse sets is never a sparse set.

3. Pseudo periodic sequences

Let $S = (s_k)$ be any sequence. For any $n \ge m \ge 0$, $[s_m, s_n]$ denotes the ordered $n - m + 1$–tuple $(s_m, s_{m+1}, \ldots, s_n)$ and is called a section of S of length $n - m$. Further $[s_m, s_n] = 0$ means that $s_t = 0$ for $m \le t \le n$. Any section of the form $[s_0, s_n]$ is called an initial section. Let F be a Galois field and $f(D) \in F[D]$ with $\deg f(D) \ge 0$. Write $f(D) = D^u g(D)$ for some $u \ge 0$ and $g(D) \in F[D]$ satisfying $g(0) \ne 0$. Then u is called the index of $f(D)$ and is denoted by $i(f(D))$. Further the order of $f(D)$ denoted by $O(f(D))$ is the smallest positive integer k such that $g(D)$ divides $D^k - 1$ [2]. The sum $i(f(D)) + O(f(D))$ is called the quasi-order of $f(D)$ and is denoted by $O'(f(D))$. For any $S \in \Omega(f(D))$ either $D^u S$ is zero or else it is a non-zero periodic sequence of least period a factor of $O(f(D))$. If $D^u S$ has a zero section of length $\geq O(f(D))$, then $D^u S = O$. For any non-zero $f(D),\ g(D) \in F[D]$, $\Omega(f(D)) + \Omega(g(D)) = \Omega(h(D))$ and $\Omega(f(D)) \cap \Omega(g(D)) = \Omega(h'(D))$ where $h(D)$ and $h'(D)$ are the lcm and gcd respectively of $f(D)$ and $g(D)$ [2]. These observations give the following essentially known result:

Lemma 3.1. *Let $f(D)$ be a non-zero member of $F[D]$.*

(i) *If $S \in \Omega(f(D))$ and an initial section of S of length $\geq O'(f(D))$ is zero, then $S = 0$.*

(ii) *Let $\deg f(D) = k > 0$, $S \in \Gamma(F)$ and $w_0, w_1, \ldots, w_{k-1}$ be any k members of F. Then there exists a unique $S' = (s'_n) \in \Gamma(F)$, such that $s'_n = w_n$ for $0 \leq n \leq k-1$ and $f(D)S' = S$.*

(iii) *Given $S_1, S_2 \in \Omega(f(D))$ such that some section of S_1 of length $\geq O'(f(D))$ equals a section of S_2, we have $D^r S_1 = D^s S_2$ for some $r, s \geq 0$.*

(iv) *$\Omega(D^k f(D) \subseteq \Omega(D^\infty) + \Omega(f(D))$.*

Lemma 3.2. *Let $S = (s_k) \in \Gamma(F)$ and $f(D),\ g(D)$ be two non-zero members of $F(D)$ with $\deg f(D) = r$. Let $f(D)S = S' = (s'_k)$. Consider any $n, m \in \mathbb{N}$ with $n - m \geq (r-1)$. If $[s'_m, s'_n]$ is a section of a member of $\Omega(g(D))$, then $[s_m, s_{n+r}]$ is a section of a member of $\Omega(f(D)g(D))$.*

Proof. The hypothesis gives $T = (t_p) \in \Omega(g(D))$ such that $[s'_m, s'_n] = [t_0, t_{n-m}]$. By (3.1), there exists a unique $T' = (t'_k)$ with $t'_i = s_{m+i}$ for $0 \leq i \leq r-1$ and $f(D) \cdot T' = T$. Clearly $T' \in \Omega(f(D)g(D))$. In $f(D)S = S'$, $[s'_m, s'_n]$ sorresponds to $[s_m, s_{n+r}]$. In $f(D)T' = T$, $[t_0, t_{n-m}]$ corresponds to $[t'_0, t'_{n-m+r}]$. By comparing T' with S, we get $[t'_0, t'_{n-m+r}] = [s_m, s_{n+r}]$. Hence $[s_m, s_{n+r}]$ is a section of a member of $\Omega(f(D)g(D))$.

Definition 3.3. A sequence $S = (s_n) \in \Gamma(F)$ is called a pseudo-periodic sequence if there exists a sparse set A, a positive integer u and $f(D) \neq 0$ in $F[D]$ such that for any consecutive members n, m of A with $n - m \geq u$, $[s_m, s_n]$ is a section of a member of $\Omega(f(D)); u$ is called a pseudo-period of S, $f(D)$ is called a pseudo-characteristic polynomial of S and A is called a sparse set associated with S. Any such triple $(A, u, f(D))$ is called a companion of S.

Let $\bar{W}(F)$ denote the set of all pseudo-periodic sequences in $\Gamma(F)$. For any $A \in S(\mathbb{N})$, the sequence (s_n) with $s_n = 1$ if $n \in A$ and $s_n = 0$ otherwise, is a member of $\bar{W}(F)$, with associated sparse set A' such that $x \in A'$ if and only if $x \geq 0$ and $x = n \pm 1$ for some $n \in A$. As $S(\mathbb{N})$ is uncountable, $\bar{W}(F)$ is uncountable. For any $f(D) \neq 0$ in $F[D]$, $\bar{\Omega}(f(D))$ denotes the set of those $S \in \bar{W}(F)$ for which $f(D)$ is a pseudo-characteristic polynomial of S.

Consider $S \in \bar{\Omega}(1)$. By definition there exists a sparse set A associated with S and a positive integer u such that for any two consecutive members n, m of A with $n - m \geq u$ one has $[s_m, s_n] = 0$. Because of this, each member of $\bar{\Omega}(1)$ is called a pseudo-zero sequence. $\bar{\Omega}(1)$ is also uncountable.

Lemma 3.4. *Let $(A, u, f(D))$ be a companion of an $S \in \bar{W}(F)$ and $g(D)$ be any multiple of $f(D)$. Then:*

(i) *For any $v \geq u$, $(A, v, g(D))$ is a companion of S.*

(ii) *Given $A' \in S(N)$ such that $A \subseteq A'$, there exists $w \geq u$ such that $(A', w, g(D))$ is a companion of S.*

Proof. That $\Omega(f(D)) \subseteq \Omega(g(D))$ gives (i). Let a be the smallest member of A. Let $w = a + u$. Let n, m be any two consecutive members of A' such that $n - m \geq w$. This gives $n > a$. As $A \subseteq A'$, the fact that n, m are consecutive in A', gives $m \geq a$. We get two consecutive members, p, q of A such that $p \leq m < n \leq q$. By (i) $[s_p, s_q]$ is a section of a member of $\Omega(g(D))$. This yields that $[s_m, s_n]$ is a section of a member of $\Omega(g(D))$. Hence $(A', w, g(D))$ is a companion of S.

Lemma 3.5. *Let $S \in \Gamma(F)$, $0 \neq f(D) \in F[D]$ with $f(D)S \in \bar{W}(F)$. Then $S \in \bar{W}(F)$. Further if $(A, u, g(D))$ is a companion of $f(D)S$ with $u > \deg f(D)$, then $(A, u, f(D)g(D))$ is a companion of S.*

Proof. Let $(A, u, g(D))$ be companion of $f(D)S$ with $u > r = \deg f(D)$. Consider any two consecutive members n, m of A with $n - m \geq u$. Let $f(D)S = S' = (s'_k)$. Then $[s'_m, s'_n]$ is a section of a member of $\Omega(g(D))$. By (3.2) $[s_m, s_n]$ is a section of a member of $\Omega(f(D)g(D))$. This completes the proof.

Proposition 3.6. *For any Galois field F, $\bar{W}(F)$ is a divisible submodule of $\Gamma(F)$. For any $f(D) \neq 0$ in $F[D]$, $\bar{\Omega}(f(D))$ is a submodule of $\bar{W}(F)$.*

Proof. Let $S_1, S_2 \in \bar{W}(F)$. By using (2.2) (i) and (3.4) we get a triple $(A, u, f(D))$ which is a companion of both S_1 and S_2. Then obviously $(A, u, f(D))$ is also a companion of $S_1 + S_2$, aS_1 for any $a \in F$. Further $A' \in S(\mathbb{N})$ such that $x \in A'$ iff $x = n - 1$ for some $n > 0$ in A, is a sparse set such that $(A', u, f(D))$ is a companion of DS_1. This proves that $\bar{W}(F)$ is a submodule of $\Gamma(F)$. Now $\Gamma(F)$ is an injective $F[D]$–module. So $\bar{W}(F)$ has an injective hull E in $\Gamma(F)$. Consider any $S \in E$. Then for some $0 \neq f(D) \in F[D]$, $f(D)S \in \bar{W}(F)$. So by (3.5) $S \in \bar{W}(F)$. Consequently $\bar{W}(F)$ itself is injective. The last part is obvious.

Lemma 3.7. *Let $S = (s_n) \in \bar{W}(F)$ have a companion $(A, u, f(D))$ with $u > \deg f(D)$. Then for any factor $h(D)$ of $f(D)$, $(A', u, f(D)/h(D))$ is a companion of $h(D)S$ for some A' containing A.*

Proof. Let $h(D) = \sum_{i=0}^{k} a_i D^i,\ k = \deg h(D)$. Then $S' = h(D)S = (w_t)$ with $w_t = \sum_{i=0}^{k} a_i s_{t+i}$. Consider any two consecutive members n, m of A with $n - m \geq u + k$. The formula for S' shows that $[w_m, w_{n-k}]$ is a section of a member of $\Omega(f(D)/h(D))$. Define A' such that $x \in A'$ if and only if either $x \in A$ or for some consecutive members p, q of A with $p - q > k,\ x = p - k$. Then by (2.5) A' is a sparse set. By what has been proved above it follows that $(A', u, f(D)/h(D))$ is a companion of S'.

The following is an immediate consequence of (3.5) and (3.7).

Lemma 3.8. *For any $f(D) \neq 0$ in $F[D]$ any factor $h(D)$ of $f(D)$ in $F[D]$,*

(i) $h(D)\,\bar{\Omega}(f(D)) = \bar{\Omega}(f(D)/h(D))$

(ii) $f(D)\,\bar{\Omega}(f(D)) = \bar{\Omega}(1)$

(iii) $\bar{\Omega}(f(D))/\bar{\Omega}(1)$ *is the annihilator of $f(D)$ in $\Gamma(F)/\bar{\Omega}(1)$.*

For any torsion module M over $F[D]$, given any non-zero members $f(D),\ g(D)$ of $F[D]$,

$$\operatorname{ann}_M(f(D)) + \operatorname{ann}_M(g(D)) = \operatorname{ann}_M(\ell(D))$$
$$\operatorname{ann}_M(f(D)) \cap \operatorname{anm}_M(g(D)) = \operatorname{ann}_M(d(D))$$

where $\ell(D)$ and $d(D)$ are the lcm and gcd respectively of $f(D)$ and $g(D)$. This observation and (3.8) (iii) give the following result analogous to that for the $\Omega(f(D))$'s.

Theorem 3.9. *For any two non-zero polynomials $f(D),\ g(D)$ in $F[D]$*

$$\bar{\Omega}(f(D)) + \bar{\Omega}(g(D)) = \bar{\Omega}(\ell(D))$$
$$\bar{\Omega}(f(D)) \cap \bar{\Omega}(g(D)) = \bar{\Omega}(d(D))$$

where $\ell(D)$ and $g(D)$ are the lcm *and* gcd *respectively of $f(D)$ and $g(D)$.*

Observe that as every member of $\Omega(D^\infty)$ is eventually zero we have $\Omega(D^\infty) \subseteq \bar{\Omega}(1)$. The following result describes the torsion submodule of any $\bar{\Omega}(f(D))$.

Theorem 3.10. *For any non-zero $f(D) \in F[D]$,*

$$\bar{\Omega}(f(D)) = (\Omega(D^{\infty}) + \Omega(f(D)) \oplus L$$

for some torsion-free $F[D]$–module L of $\bar{\Omega}(f(D))$. Further $\bar{\Omega}(f(D))$ is divisible by D.

Proof. First of all we prove that $W(F) \cap \bar{\Omega}(f(D)) = \bar{\Omega}(D^{\infty}) + \Omega(f(D))$. It is obvious that $\Omega(D^{\infty}) + \Omega(f(D))$ is contained in $W(F) \cap \bar{\Omega}(f(D))$. Conversely, let $S = (s_n) \in \bar{\Omega}(f(D)) \cap W(F)$. Then, for some $g(D) \neq 0$ in $F[D]$, $g(D)S = 0$. Consider any companion $(A, u, f(D))$ of S. By using (2.3) we can find two consecutive members n, m of A such that $n - m$ is greater than u as well as $O'(f(D)g(D))$. Then $[s_m, s_n]$ is a section of a member S' of $\Omega(f(D))$ and clearly both S and S' are in $\Omega(f(D)g(D))$. By (3.1) (iii)

$$D^r S = D^s S' \in \Omega(h(D))$$

for some positive integers r, s, where $h(D)$ is the gcd of $f(D)$ and $g(D)$. Consequently $f(D)D^r S = 0$. So by (3.1) (iv), $S \in \Omega(D^{\infty}) + \Omega(f(D))$. This proves $W(F) \cap \bar{\Omega}(f(D)) = \Omega(D^{\infty}) + \Omega(f(D))$, the torsion submodule of $\bar{\Omega}(f(D))$. As $\Omega(D^{\infty})$ is injective,

$$\bar{\Omega}(f(D)) = \Omega(D^{\infty}) \oplus L'$$

for some submodule L' of $\bar{\Omega}(f(D))$. Then the torsion submodule L'' if L' is a homomorphic image of $\Omega(f(D))$, is finitely generated; consequently L'' is a summand of L'. This gives

$$\begin{aligned}\bar{\Omega}(f(D)) &= \Omega(D^{\infty}) \oplus (L'' \oplus L) \\ &= (\Omega(D^{\infty}) + \Omega(f(D)) \oplus L\end{aligned}$$

where L is torsion free.

Proposition 3.11. *For any $f(D) \in F[D]$ with $f(0) \neq 0$ and $\deg f(D) > 0$, $\bar{\Omega}(f(D)) \neq \bar{\Omega}(1) + \Omega(f(D))$. Further $\bar{\Omega}(1) + \Omega(f(D)) \subset \bar{\Omega}(f(D))$.*

Proof. Clearly $\bar{\Omega}(1) + \Omega(f(D)) \subseteq \bar{\Omega}(f(D))$. Suppose the contrary and let $\bar{\Omega}(f(D)) = \bar{\Omega}(1) + \Omega(f(D))$. Consider any sparse set $A = (n_i)_{i \geq 0}$ such that $n_{2i} + 1 = n_{2i+1}$, $n_{2i+2} - n_{2i+1} > O(f(D))$ and $n_{2j+2} - n_{2j+1} > n_{2i+2} - n_{2i+1}$ for $j > i \geq 0$. We can construct a $T = (t_k)$ in $\bar{\Omega}(f(D))$ such that $[t_{n_{2i+1}}, t_{n_{2i+2}}]$ is a section of non-zero member of $\Omega(f(D))$ for i odd, and is zero for i even. Then for some $S = (s_k) \in \Omega(f(D))$, $T - S \in \bar{\Omega}(1)$. We can find a sparse set A' containing A and a $u > O(f(D))$ such that $(A, u, 1)$ is a companion of $T - S$. Let $t = s(A')$. We can find an m such that for $i \geq m$, $n_{2i+2} - n_{2i+1} > ut$. By (2.3) we choose m such that given $i \geq m$, we have consecutive members, $a_i < b_i$ of A' such that

$n_{2i+1} \le a_i < b_i < n_{2i+2}$ and $b_i - a_i \ge u$. Then $[t_{a_i} - s_{a_i}, t_{b_i} - s_{b_i}] = 0$. If $S \ne 0$, then $[s_{a_i}, s_{b_i}] \ne 0$ and hence $[t_{n_{2i+1}}, t_{n_{2i+2}}] \ne 0$. If $S = 0$, then $[t_{a_i}, t_{b_i}] = 0$; in this cae $[t_{n_{2i+1}}, t_{n_{2i+2}}] = 0$, as it has a zero subsection of length greater than $O(f(D))$. Thus either all the $[t_{n_{2i+1}}, t_{n_{2i+2}}]$ are non-zero or all of them are zero for $i \ge m$. This contradicts the construction of T. This proves that $\bar{\Omega}(1) + \bar{\Omega}(f(D)) \ne \bar{\Omega}(f(D))$. Finally consider any T' in $\bar{\Omega}(f(D))$ such that $T' \notin \Omega(f(D))$. Then $f(D)T' \ne 0$. But $f(D)T' \in \bar{\Omega}(1) + \Omega(f(D))$ by (3.8). Hence

$$\bar{\Omega}(1) + \Omega(f(D)) \subset' \Omega(f(D)) .$$

This completes the proof.

Theorem 3.12.

(i) *For any injective hull L of $\bar{\Omega}(1) + \Omega(f(D))$ in $\Gamma(F)$, $\bar{\Omega}(f(D)) \subseteq L$.*

(ii) *For any injective hull K of $\bar{\Omega}(1)$ in $\Gamma(F)$, $K + \Omega(f(D)^\infty)$ is an injective hull of $\bar{\Omega}(f(D))$.*

Proof. Let $\bar{\Omega}(f(D)) \not\subset L$. Then $\deg f(D) > 0$ and L is a proper summand of $L + \bar{\Omega}(f(D))$. This gives $S = S_1 + S_2$ with $S_1 \in L$ and $S_2 \in \bar{\Omega}(f(D))$ such that $S_2 \ne 0$ and $L \cap F[D]S = 0$. Now $f(D)S = f(D)S_1 + f(D)S_2$ with $f(D)S_2 \in \bar{\Omega}(1)$. This yields $f(D)S \in L$. Hence $f(D)S \in L \cap F[D]S = 0$. Consequently $S \in \Omega(f(D))$. But $\Omega(f(D)) \subseteq L$, so that $S \in L$. This is a contradiction. This proves (i). Consider any injective hull K of $\bar{\Omega}(1)$ in $\Gamma(F)$. Then $L = K + \Omega(f(D)^\infty)$, being a sum of two injective submodules, is injective. So by (i) $\bar{\Omega}(f(D)) \subseteq L$. Consider $0 \ne S \in L$. If $S \in K$ or $S \in \Omega(f(D)^\infty)$, then by using the fact that $\bar{\Omega}(1) \subseteq \bar{\Omega}(f(D))$ and (3.8) (iii), we get a $g(D) \in F[D]$ such that $0 \ne g(D)S \in \bar{\Omega}(f(D))$. So let $S \notin K$ and $S \notin \Omega(f(D)^\infty)$. Now $S = S_1 + S_2$ for some $S_1 \in K$ and $S_2 \in \Omega(f(D)^\infty)$. Also, for some $k \ge 1$, $f(D)^k S_2 = 0$. This gives $0 \ne f(D)^k S = f(D)^k S_1 \in K$. Thus for some $g(D) \in F[D]$, $0 \ne g(D)f(D)^k S \in \bar{\Omega}(1) \subseteq \bar{\Omega}(f(D))$. Hence $\bar{\Omega}(f(D)) \subset' K + \Omega(f(D)^\infty)$. This proves (ii).

Theorem 3.13. *Let L be any injective hull of $\bar{\Omega}(1)$ in $\Gamma(F)$, then $\bar{W}(F) = W(F) + L$ and $L \cap W(F) = \Omega(D^\infty)$.*

Proof. For any $f(D) \in F[D]$ of positive degree with $f(0) \ne 0$, $\bar{\Omega}(1) \cap \Omega(f(D)) = 0$ and $\Omega(D^\infty) \subseteq \bar{\Omega}(1)$ gives $L \cap W(F) = \Omega(D^\infty)$. By (3.12) $\bar{\Omega}(f(D)) \subseteq L + \Omega(f(D)^\infty) \subseteq L + W(F)$. Hence $\bar{W}(F) = L + W(F)$.

We now discuss some divisibility properties of an $\bar{\Omega}(g(D))$. Given two relatively prime polynomials $f(D)$, $g(D)$ in $F[D]$, it is well known that $f(D) \cdot \Omega(g(D)) = \Omega(g(D))$. This need not be true for $\bar{\Omega}(g(D))$. We start with the following

Lemma 3.14. *Let $f(D) \in F[D]$ with $\deg f(D) = k > 0$ and $f(0) \neq 0$. Let $\beta > O(f(D))$. Consider any sparse set A constituted by $n_i, m_i \in \mathbb{N}$ such that $n_0 = 0$, $n_{i+1} - m_i > \beta$, $0 \leq m_i - n_i < \deg f(D)$. Let $S = (s_n) \in \Gamma(F)$ be such that for some i, $[s_{n_{i+1}}, s_{m_{i+1}}] \neq 0$, but $[s_{m_i} + 1, s_{n_{i+1}} - 1] = 0 = [s_{m_{i+1}} + 1, s_{n_{i+2}} - 1]$. Let $S' = (w_n) \in \Gamma(F)$ be such that $f(D)S' = S$. If $[w_{m_i+1}, w_{n_{i+1}+k-1}] = 0$, then $[w_{m_{i+1}+1}, w_{n_{i+2}+k-1}]$ is a non-zero section of a member of $\Omega(f(D))$.*

Proof. It is enough to take $i = 0$. Let $f(D) = \sum\limits_{i=0}^{k} a_i D^i$ with $a_k \neq 0$. Then

$$s_{n_{1+j}} = \sum_{i=0}^{k} a_1 w_{n_{1+j+i}} \tag{1}$$

By the hypothesis $[s_{n_1}, s_{m_1}] \neq 0$ and $[w_{m_0+1}, w_{n_1+k-1}] = 0$. Consequently by (1) $[w_{n_1+k}, w_{m_1+k}] \neq 0$. As $n_1 + k > m_1$ and $m_1 + k \leq n_2 - 1 < n_2 + k - 1$, we get $[w_{m_1+1}, w_{n_2+k-1}] \neq 0$. In the equation $f(D)S' = S$, $[s_{m_1+1}, s_{n_2-1}]$ corresponds to $[w_{m_1+1}, w_{n_2+k-1}]$. As $[s_{m_1+1}, s_{n_2-1}] = 0$, it follows that $[w_{m_1+1}, w_{n_2+k-1}]$ is an initial section of a member of $\bar{\Omega}(f(D))$.

Theorem 3.15. *$\bar{\Omega}(1)$ is not divisible by any $f(D) \in F[D]$ of positive degree such that $f(0) \neq 0$. Indeed, given any $g(D) \neq 0$ in $F[D]$ with $\deg g(D) < \deg f(D)$, there exists $S \in \bar{\Omega}(1)$ such that $g(D)S$ is not divisible by $f(D)$ in $\bar{\Omega}(1)$.*

Proof. Let $\deg g(D) = u$, $\deg f(D) = k$. Consider any $\beta > O(f(D)) + u + 1$. Let $A = (n_i)$ be a sparse set such that

$$\beta < (n_{i+1} - n_i) < (n_{i+2} - n_{i+1})$$

for every i. Consider a sequence $S = (s_n) \in \Gamma(F)$ such that $s_n = 1$ for $n \in A$ and $s_n = 0$ otherwise. Let $g(D)S = S' = (s'_n)$. Then for any $i \geq 1$, $[s'_{n_i-u}, s'_{n_i}] \neq 0$ and for $0 < n \notin [n_i - u, n_i]$, $s'_n = 0$. Suppose the contrary and for some $S_1 = (w_n) \in \bar{\Omega}(1)$ let $f(D)S_1 = S'$. By using (2.5) and (3.4) we get a companion $(A', v, 1)$ of S_1 such that $A \subseteq A'$, $n_i - u \in A'$ for $i > 0$, and $v > \beta$. Let $t = s(A')$. We can find $j > 0$ such that $n_{j+1} - u - n_j > tv$. By (2.3) we choose j, such that for $i \geq j$ we have consecutive members $a_i, b_i \in A$ satisfying

$$n_i < a_i < b_i \leq n_{i+1} - u\,,$$

$b_i - a_i > v$. As $[s'_{n_j+1}, s'_{n_{j+1}-u-1}] = 0$ we get that $[w_{n_j+1}, w_{n_j-u+k-1}]$ is a section of a member T of $\Omega(f(D))$. As $[w_{a_j}, w_{b_j}] = 0$ and $b_j - a_j > O(f(D))$ we get $T = 0$ and hence $[w_{n_j+1}, w_{n_j-u+k-1}] = 0$. Then for every $i \geq j$ we have $[w_{n_i+1}, w_{n_i-u+k-1}] = 0$. This contradicts (3.14) and hence the result follows.

Corollary 3.16. *For any non-zero $f(d), g(D) \in F[D]$ with* $\deg(g(D))>0$ *and* $g(0) \neq 0$, $\bar{\Omega}(f(D))$ *is not divisible by* $g(D)$.

PROOF. Suppose the contrary and let $g(D)\bar{\Omega}(f(D)) = \bar{\Omega}(f(D))$. Then $g(D)f(D)\bar{\Omega}(f(D)) = f(D)\bar{\Omega}(f(D))$ i. e. $g(D)\bar{\Omega}(1) = \bar{\Omega}(1)$. This contradicts (3.15). This proves the result.

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HASSAN AL–ZAID
DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY
P. O. BOX 5969–13060 SAFAT
KUWAIT

SURJEET SINGH
DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY
P. O. BOX 5969–13060 SAFAT
KUWAIT

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