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Invariance of the naturally lifted metrics on linear frame bundles over affine manifolds

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Abstract. The natural lifts from affine manifolds to linear frame bundles have been classified by the second author in 1988. In this paper we prove the invariance Theorem for these natural lifts. But, at the beginning, we insert a short survey about some "prominent" natural lifts studied earlier by the present authors.

A short survey about some prominent natural lifts

This paper deals with the geometric concept of *naturality*. Our idea of naturality is closely related to that of A. Nijenhuis, D. B. A. Epstein, P. Stredder and others. We have used for our purposes the concepts and methods mediated by D. Krupka [21], [22] and D. Krupka and V. Mikolášová [23]. But there is a standard monograph [12] by I. Kolář, P. W. Michor and J. Slovák for the full references about naturality. (See also the booklet by D. Krupka and J. Janyška [24].) In this section, we shall make a short review of the earlier results (from the eighties) and recent results of the present authors and other geometers. For the sake of brevity, whenever we speak about natural lift, or natural transformation, we always mean that *of the first order* in the sense of [12] or [24]. (In the earlier papers of the present authors the term "of second order" was often used, but this should not cause confusion.)

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(i) Let (M, g) be a Riemannian manifold. Then all natural lifts of (M, g) to the tangent bundle TM (which are pseudo-Riemannian metrics on TM, possibly degenerate), have been classified by the present authors in [14]. All such metrics depend, in general, on six arbitrary functions of one variable. (cf. also [12] and [24] for another exposition). It is easy to restrict the notion of natural lift to the unit tangent sphere bundle T_1M, which is a hypersurface of TM. The natural lifts to T_1M then depend on four arbitrary parameters. In the new millennium, many papers appeared by M. T. K. Abbassi, G. Calvaruso, D. Perrone and others studying geometric properties of natural lifts of a Riemannian manifold (M, g) to TM, and also to T_1M. Some of these papers are related to contact geometry and some of them to harmonic maps, which are both up-to-date topics. From a rich bibliography we select just [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

(ii) In [13], the present authors have classified all natural lifts from (M, g) to the frame bundle LM. These are pseudo-Riemannian metrics on LM, possibly degenerate, which depend on $n(n^3+3n^2+n+1)/2$ arbitrary functions of $n(n+1)/2$ variables. The well-known example of such a lift is the so-called Sasaki–Mok metric on LM (see [11] and the literature cited here). These natural lifts can be easily restricted to the orthonormal frame bundle OM. The study of the geometry of the restricted Sasaki–Mok metric gives interesting results about curvature (see [18] and [19] as related papers).

(iii) In [27], the second author has classified all natural transformations of affine manifolds (M, ∇) (with a symmetric connection) to the cotangent bundle T^*M, which are pseudo-Riemannian metrics of the balanced signature (n, n), depending on two real parameters. The most important case of such a natural transformation is the so-called Riemann extension, see *e.g.* [25], [26], [29], [30], [31].

(iv) Finally, in [28], the second author has classified all natural transformations of symmetric affine manifolds (M, ∇) to linear frame bundles LM. These are again pseudo-Riemannian metrics on LM, possibly degenerate, depending on $n(n^3 + 3n^2 + n + 1)/2$ arbitrary constants.

The authors have realized already in [14], that general naturally lifted metrics should have the "invariance property", *i.e.,* the property that properly constructed lifts of (local) isometries of (M, g) or (local) affine transformations of (M, ∇) are (local) isometries of lifted metrics. This is based on the results published in [21], [22], [23], [24] and, in a more modern and abstract form, in the monograph [12]. Yet, only two decades later, the authors decided to publish independent proofs of invariance for each particular case (i)–(iii) in [16], [17], [20]. There were two motivations for doing so: 1) Each particular case offers a geometrically interesting and meaningful independent proof. 2) The successful proof of the invariance is, in each case, a strong check of the correctness of the classification result itself.

The proof of the invariance property in the case (iv) is the main topic of the present paper.

1. Preliminaries

Let M be a smooth and connected manifold M of dimension n. Then the linear frame bundle LM over M consists of all pairs (x, u), where x is a point of M and u is a basis for the tangent space M_x of M at x. We denote by p the natural projection of LM to M defined by $p(x, u) = x$. If $(\mathfrak{U};\ x^1, x^2, \ldots, x^n)$ is a system of local coordinates in M, then a basis $u = (u_1, u_2, \ldots, u_n)$ for M_x can be expressed in the unique way in the form

$$u_\rho = \sum_{i=1}^{n} u_\rho^i \left(\frac{\partial}{\partial x^i} \right)_x$$

for all indices $\rho = 1, 2, \ldots, n$, and hence $(p^{-1}(\mathfrak{U});\ x^1, x^2, \ldots, x^n, u_1^1, u_1^2, \ldots, u_n^n)$ is a system of local coordinates in LM.

Let ∇ be an affine connection of M. Then the tangent space $(LM)_{(x,u)}$ of LM at $(x, u) \in LM$ splits into the horizontal and vertical subspace $H_{(x,u)}$ and $V_{(x,u)}$ with respect to ∇:

$$(LM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If a point $(x, u) \in LM$ and a vector $X \in M_x$ are given, then there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_*(X^h) = X$. We call X^h the *horizontal lift* of X to LM at (x, u). If ω is a one-form on M, then $\iota_\rho\omega$, $\rho = 1, 2, \ldots, n$, are functions on LM defined by $(\iota_\rho\omega)(x, u) = \omega(u_\rho)$ for all $(x, u) = (x, u_1, u_2, \ldots, u_n) \in LM$. The *vertical lifts* $X^{v,\kappa}$, $\kappa = 1, 2, \ldots, n$, of $X \in M_x$ to LM at (x, u) are the n vectors such that $X^{v,\kappa}(\iota_\rho\omega) = \omega(X)\delta_\rho^\kappa$, $\kappa, \rho = 1, 2, \ldots, n$, hold for all one-forms ω on M, where δ_ρ^κ denotes the Kronecker's delta. The n vertical lifts are always uniquely determined, and they are linearly independent if $X \neq 0$. They are expressed in a local coordinate system as

$$X_{(x,u)}^{v,\kappa} = \sum_{i=1}^{n} \xi^i \left(\frac{\partial}{\partial u_\kappa^i} \right)_{(x,u)}$$

for all $\kappa = 1, 2, \ldots, n$, where $X = \sum_{i=1}^{n} \xi^i (\partial/\partial x^i)_x$ (*cf.* [11]).

In an obvious way we can define horizontal and vertical lifts of vector fields on M. These are uniquely defined vector fields on LM. The *canonical vertical vector fields* on LM are vector fields U_λ^κ, $\kappa, \lambda = 1, 2, \ldots, n$, defined, in terms of local coordinates, by $U_\lambda^\kappa = \sum_{i=1}^{n} u_\lambda^i \partial/\partial u_\kappa^i$. Here U_λ^κ's do not depend on the choice of local coordinates and they are defined globally on LM.

For a vector $u_\rho = \sum_{i=1}^{n} u_\rho^i (\partial/\partial x^i)_x \in M_x$, $\rho = 1, 2, \ldots, n$, we see that

$$(u_\rho)^h_{(x,u)} = \sum_{i=1}^{n} u_\rho^i \left(\frac{\partial}{\partial x^i} \right)^h_{(x,u)}, \tag{1.1}$$

$$(u_\rho)^{v,\kappa}_{(x,u)} = \sum_{i=1}^{n} u_\rho^i \left(\frac{\partial}{\partial x^i} \right)^{v,\kappa}_{(x,u)} = \sum_{i=1}^{n} u_\rho^i \Big(\frac{\partial}{\partial u_\kappa^i} \Big)_{(x,u)} = U_\rho^\kappa(x,u). \tag{1.2}$$

For any functions f of M and vector fields X, Y tangent to M, we have

$$\begin{aligned} &(fX)^h = (f \circ p) X^h, \\ &X^h (f \circ p) = (Xf) \circ p, \\ &(fX)^{v,\kappa} = (f \circ p) X^{v,\kappa}, \\ &X^{v,\kappa} (f \circ p) = 0. \end{aligned} \tag{1.3}$$

2. The naturally lifted metrics and the invariance theorem

Let ∇ be a symmetric affine connection of M. Then the metrics $\boldsymbol{G}^\nabla$ on LM which comes from a natural transformation of ∇ have been described by the second author in 1988, see [28]. Each such metric $\boldsymbol{G}^\nabla$ is defined at $(x,u) = (x, (u_1, u_2, \ldots, u_n)) \in LM$, in terms of classical lifts, by

$$\begin{aligned} \boldsymbol{G}^\nabla_{(x,u)}(X^h, Y^h) &= \sum_{\alpha,\beta} C_{\alpha\beta} \omega^\alpha(X) \omega^\beta(Y), \\ \boldsymbol{G}^\nabla_{(x,u)}(X^h, Y^{v,\lambda}) &= \sum_{\alpha,\beta} C_{\alpha\beta}^{\;\lambda} \omega^\alpha(X) \omega^\beta(Y), \\ \boldsymbol{G}^\nabla_{(x,u)}(X^{v,\kappa}, Y^{v,\lambda}) &= \sum_{\alpha,\beta} C_{\alpha\beta}^{\kappa\lambda} \omega^\alpha(X) \omega^\beta(Y) \end{aligned} \tag{2.1}$$

for all $X, Y \in M_x$, where $(\omega^1, \omega^2, \ldots, \omega^n)$ is the basis for the cotangent space M_x^* at $x \in M$ dual to $u = (u_1, u_2, \ldots, u_n)$, $C_{\alpha\beta}$, $C_{\alpha\beta}^{\;\lambda}$ and $C_{\alpha\beta}^{\kappa\lambda}$ are constants satisfying symmetry conditions $C_{\alpha\beta} = C_{\beta\alpha}$ and $C_{\alpha\beta}^{\kappa\lambda} = C_{\beta\alpha}^{\lambda\kappa}$ for all $\alpha, \beta, \kappa, \lambda = 1, 2, \ldots, n$.

Let ϕ be a (local) transformation of a manifold M. Then we define a transformation Φ of LM by

$$\Phi(x,u) = (\phi x, \phi_{*x}u_1, \phi_{*x}u_2, \dots, \phi_{*x}u_n) \tag{2.2}$$

for all $(x,u) = (x, u_1, u_2, \dots, u_n) \in LM$.

Proposition 2.1. *Let ϕ be an affine transformation (or, a local affine transformation) of a manifold M with an affine connection ∇ and let Φ be the lift of ϕ to LM defined as above. Then we have*

$$\Phi_*(X^h) = (\phi_*X)^h, \quad \Phi_*(X^{v,\kappa}) = (\phi_*X)^{v,\kappa} \tag{2.3}$$

for all $X \in \mathfrak{X}(M)$ and all indices $\kappa = 1, 2, \dots, n$. In particular, for the canonical vertical vector fields, we have

$$\Phi_*(U^\kappa_\rho) = U^\kappa_\rho \tag{2.4}$$

for all indices $\kappa, \rho = 1, 2, \dots, n$.

Proof. We use the formula $p \circ \Phi = \phi \circ p$. For all $X \in \mathfrak{X}(M)$ and functions f of M we calculate at $(x,u) \in LM$:

$$\begin{aligned}(p_{*\Phi(x,u)}(\Phi_{*(x,u)}(X^h_{(x,u)})))f &= X^h_{(x,u)}(f \circ p \circ \Phi) = X^h_{(x,u)}(f \circ \phi \circ p)\\ &= (p_{*(x,u)}(X^h_{(x,u)}))(f \circ \phi) = X_x(f \circ \phi) = (\phi_{*x}X_x)f.\end{aligned}$$

Since Φ preserves the horizontal distribution, we have

$$\Phi_{*(x,u)}(X^h_{(x,u)}) = (\phi_{*x}X_x)^h_{\Phi(x,u)}.$$

Next, using the formula $\iota_\mu\omega \circ \Phi = \iota_\mu(\phi^*\omega)$ for all one-forms ω on M and all indices $\mu = 1, 2, \dots, n$, we calculate at $(x,u) \in LM$ for all $X \in \mathfrak{X}(M)$:

$$\begin{aligned}(\Phi_{*(x,u)}(X^{v,\kappa}_{(x,u)}))(\iota_\mu\omega) &= (X^{v,\kappa}_{(x,u)})((\iota_\mu\omega) \circ \Phi) = (X^{v,\kappa}_{(x,u)})(\iota_\mu(\phi^*\omega))\\ &= (\phi^*\omega)_x(X_x)\delta^\kappa_\mu = \omega_{\phi(x)}(\phi_{*x}X_x)\delta^\kappa_\mu = (\phi_{*x}X_x)^{v,\kappa}(\iota_\mu\omega).\end{aligned}$$

Since Φ preserves the vertical distribution, we have $\Phi_{*(x,u)}(X^{v,\kappa}_{(x,u)}) = (\phi_{*x}X)^{v,\kappa}_{\Phi(x,u)}$ for all $\kappa = 1, 2, \dots, n$. In particular, we have $\Phi_{*(x,u)}(u_\rho{}^{v,\kappa}) = (\phi_{*x}u_\rho)^{v,\kappa}$ for all $\kappa, \rho = 1, 2, \dots, n$, that is, by (1.2), we have (2.4). □

Theorem 2.2. *Let ϕ be an affine diffeomorphism (or, a local affine diffeomorphism, respectively) of a manifold M with a symmetric affine connection ∇. Then the metric $\boldsymbol{G}^\nabla$ on LM which comes from a natural transformation of ∇ is invariant by the lift Φ of ϕ defined by (2.2). In other words, Φ is an isometry (or, a local isometry, respectively) of $(LM, \boldsymbol{G}^\nabla)$.*

Proof. Now $(\omega_1, \omega_2, \dots, \omega_n)$ is the basis for the cotangent space M_x^* at $x \in M$ dual to a basis $u = (u_1, u_2, \dots, u_n)$ for the tangent space M_x at x. So, for any (local) difeomorphism ϕ of M, $((\phi^{-1})^*\omega_1, (\phi^{-1})^*\omega_2, \dots, (\phi^{-1})^*\omega_n)$ is a basis for $M_{\phi(x)}^*$ dual to a basis $(\phi_{*x}u_1, \phi_{*x}u_2, \dots, \phi_{*x}u_n)$ for $M_{\phi(x)}$. Hence, by Proposition 2.1, we obtain for all vectors X and Y tangent to M at x that

$$\begin{aligned}
&\boldsymbol{G}^{\nabla}_{\Phi(x,u)}(\Phi_{*(x,u)}(X^h), \Phi_{*(x,u)}(Y^h)) = \boldsymbol{G}^{\nabla}_{\Phi(x,u)}((\phi_{*x}X)^h, (\phi_{*x}Y)^h) \\
&\quad = \sum_{\alpha,\beta} C_{\alpha\beta}\big((\phi^{-1})^*\omega^\alpha\big)(\phi_{*x}X)\big((\phi^{-1})^*\omega^\beta\big)(\phi_{*x}Y) \\
&\quad = \sum_{\alpha,\beta} C_{\alpha\beta}\omega^\alpha(X)\omega^\beta(Y) = \boldsymbol{G}^{\nabla}_{(x,u)}(X^h, Y^h), \\
&\boldsymbol{G}^{\nabla}_{\Phi(x,u)}(\Phi_{*(x,u)}(X^h), \Phi_{*(x,u)}(Y^{v,\lambda})) = \boldsymbol{G}^{\nabla}_{\Phi(x,u)}((\phi_{*x}X)^h, (\phi_{*x}Y)^{v,\lambda}) \\
&\quad = \sum_{\alpha,\beta} C_{\alpha}{}^{\lambda}_{\beta}\big((\phi^{-1})^*\omega^\alpha\big)(\phi_{*x}X)\big((\phi^{-1})^*\omega^\beta\big)(\phi_{*x}Y) \\
&\quad = \sum_{\alpha,\beta} C_{\alpha}{}^{\lambda}_{\beta}\omega^\alpha(X)\omega^\beta(Y) = \boldsymbol{G}^{\nabla}_{(x,u)}(X^h, Y^{v,\lambda})
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{G}^{\nabla}_{\Phi(x,u)}(\Phi_{*(x,u)}(X^{v,\kappa}), \Phi_{*(x,u)}(Y^{v,\lambda})) = \boldsymbol{G}^{\nabla}_{\Phi(x,u)}((\phi_{*x}X)^{v,\kappa}, (\phi_{*x}Y)^{v,\lambda}) \\
&\quad = \sum_{\alpha,\beta} C^{\kappa\lambda}_{\alpha\beta}\big((\phi^{-1})^*\omega^\alpha\big)(\phi_{*x}X)\big((\phi^{-1})^*\omega^\beta\big)(\phi_{*x}Y) \\
&\quad = \sum_{\alpha,\beta} C^{\kappa\lambda}_{\alpha\beta}\omega^\alpha(X)\omega^\beta(Y) = \boldsymbol{G}^{\nabla}_{(x,u)}(X^{v,\kappa}, Y^{v,\lambda}).
\end{aligned}$$

This completes the proof. □

References

[1] M. T. K. Abbassi and G. Calvaruso, g-natural contact metrics on unit tangent sphere bundles, *Monatsh. Math.* **151**, no. 2 (2007), 89–109.

[2] M. T. K. Abbassi and G. Calvaruso, Curvature properties of g-natural contact metric structures on unit tangent sphere bundles, *Beiträge Algebra Geom.* **50**, no. 1 (2009), 155–178.

[3] M. T. K. Abbassi, G. Calvaruso and D. Perrone, Some examples of harmonic maps for g-natural metrics, *Ann. Math. Blaise Pascal* **16**, no. 2 (2009), 305–320.

[4] M. T. K. Abbassi, G. Calvaruso and D. Perrone, Harmonicity of unit vector fields with respect to Riemannian g-natural metrics, *Differ. Geom. Appl.* **27**, no. 1 (2009), 157–169.

[5] M. T. K. Abbassi, G. Calvaruso and D. Perrone, Harmonic maps defined by the geodesic flow, *Houston J. Math.* **36**, no. 1 (2010), 69–90.

[6] M. T. K. Abbassi, G. Calvaruso and D. Perrone, Harmonic sections of tangent bundles equipped with g-natural Riemannian metrics, *Q. J. Math.* **62**, no. 2 (2011), 259–288. DOI:10.1093/qmath/hap040

[7] M. T. K. Abbassi and O. Kowalski, Naturality of homogeneous metrics on Stiefel manifolds $\mathrm{SO}(m+1)/\mathrm{SO}(m-1)$, *Differ. Geom. Appl.* **28**, no. 2 (2010), 131–139.

[8] M. T. K. Abbassi and O. Kowalski, On Einstein Riemannian g-natural metrics on unit tangent sphere bundles, *Ann. Global. Anal. Geom.* **38**, no. 1 (2010), 11–20.

[9] M. T. K. Abbassi and M. Sarih, On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds, *Differ. Geom. Appl.* **22**, no. 1 (2005), 19–47.

[10] G. Calvaruso and D. Perrone, Harmonic morphisms and Riemannian geometry of tangent bundles, *Ann. Global Anal. Geom.* **39**, no. 2 (2011), 187–213.

[11] L. A. Cordero and M. de León, On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold, *J. Math. Pures Appl.* **65**, no. 1 (1986), 81–91.

[12] I. Kolář, P. W. Michor and J. Slovák, Natural Operations in Differential Geometry, *Springer-Verlag*, *Berlin – Heiderberg – New York*, 1993.

[13] O. Kowalski and M. Sekizawa, Natural transformations of Riemannian metrics on manifolds to metrics on linear frame bundles – a classification, Differential geometry and its applications (Brno, 1986), Math. Appl. (East European Ser.), **Vol. 27**, *Reidel, Publ. Comp.*, *Dordrecht*, 1987, 149–178.

[14] O. Kowalski and M. Sekizawa, Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles – a classification, *Bull. Tokyo Gakugei Univ.* (4) **40** (1988), 1–29.

[15] O. Kowalski and M. Sekizawa, On curvatures of linear frame bundles with naturally lifted metrics, *Rend. Semin. Mat. Univ. Politec. Torino* **63**, no. 3 (2005), 283–295.

[16] O. Kowalski and M. Sekizawa, Invariance of g-natural metrics on tangent bundles, Differential Geometry and its Applications, Proc. Conf., in Honour of Leonhard Euler, Olomouc, August 2007, *World Sci. Publ.*, *Hackensack, NJ*, 2008, 171–181.

[17] O. Kowalski and M. Sekizawa, Invariance of g-natural metrics on linear frame bundles, *Arch. Math. (Brno)* **44**, no. 2 (2008), 139–147.

[18] O. Kowalski and M. Sekizawa, On the geometry of orthonormal frame bundles, *Math. Nachr.* **281**, no. 12 (2008), 1799–1809, DOI 10.1002/mana.200610715.

[19] O. Kowalski and M. Sekizawa, On the geometry of orthonormal frame bundles II, *Ann. Global Anal. Geom.* **33**, no. 4 (2008), 357–371.

[20] O. Kowalski and M. Sekizawa, On natural Riemann extensions, *Publ. Math. Debrecen* **78** (2011), 709–721, DOI10.5486/PMD.2011.4992.

[21] D. Krupka, Elementary theory of differential invariants, *Arch. Math. (Brno)* **14**, no. 4 (1978), 207–214.

[22] D. Krupka, Differential invariants, Lecture Notes, *Faculty of Science, Purkyně University*, *Brno*, 1979.

[23] D. Krupka and V. Mikolášová, On the uniqueness of some differential invariants: d, $[\ ,\ ]$, ∇, *Czechoslovak Math. J.* **34(109)**, no. 4 (1984), 588–597.

[24] D. Krupka and J. Janyška, Lectures on Differential Invariants, *University J. E. Purkyně in Brno*, 1990.

[25] E. M. Patterson and A. G. Walker, Riemann extensions, *Quart. J. Math., Oxford Ser.* (2) **3** (1952), 19–28.

[26] H. S. Ruse, A. G. Walker and T. J. Willmore, Harmonic spaces, *Consiglio Nazionale delle Ricerche Monografie Matematiche, 8 Edizioni Cremonese*, *Rome*, 1961.

[27] M. Sekizawa, Natural transformations of affine connections on manifolds to metrics on cotangent bundles, Proceedings of the 14th Winter School on Abstract Analysis (Srni, 1986), *Rend. Circ. Mat. Palermo* (2) *Suppl.* **14** (1987), 129–142.

[28] M. Sekizawa, Natural transformations of symmetric affine connections on manifolds to metrics on linear frame bundles: a classification, *Monatsh. Math.* **105**, no. 3 (1988), 229–243.

[29] L. Vanhecke and T. J. Willmore, Riemann extensions of D'Atri spaces, *Tensor (N.S.)* **38** (1982), 154–158.

[30] T. J. Willmore, An introduction to differential geometry, *Clarendon Press*, *Oxford*, 1959.

[31] K. Yano and E. M. Paterson, Vertical and complete lifts from a manifold to its cotangent bundle, *J. Math. Soc. Japan* **19** (1967), 91–113.

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