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The closedness of some generalized curvature 2-forms on a Riemannian manifold I

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Abstract. In this paper we study the closedness properties of generalized curvature 2-forms, which are said to be Riemannian, Conformal, Projective, Concircular and Conharmonic curvature 2-forms, associated to each generalized curvature tensors on a Riemannian manifold. Corresponding to each curvature tensors, such generalized curvature 2-forms are the associated curvature 2-forms.

In particular, we focus on the closedness of differential 2-forms associated to the divergence of generalized curvature tensors, which is weaker than the notion of harmonic curvature. In this case, we give an algebraic condition involving the Riemann curvature tensor and the Ricci tensor arising from an old identity due to Lovelock.

1. Introduction

Let M be a smooth n-dimensional Riemannian manifold endowed with the operator of covariant differentiation ∇ with respect to the metric g_{kl}. Let ${R_{jkl}}^m$ the Riemann curvature tensor of type $(1,3)$. It satisfies the two Bianchi identities

$${R_{jkl}}^m + {R_{klj}}^m + {R_{ljk}}^m = 0,$$

and

$$\nabla_i {R_{jkl}}^m + \nabla_j {R_{kil}}^m + \nabla_k {R_{ijl}}^m = 0.$$

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The previous identities are valid in a torsion-free connection [11]. In this paper we define the Ricci tensor to be $R_{kl} = -R_{mkl}{}^m$ [21] and the scalar curvature $R = g^{ij}R_{ij}$. It is well known that in a metric connection the Ricci tensor is symmetric [11]. Contracting the second identity, we get $\nabla_m R_{jkl}{}^m = \nabla_k R_{jl} - \nabla_j R_{kl}$. From this, a Riemannian manifold is said to have a *harmonic curvature tensor* if $\nabla_m R_{jkl}{}^m = 0$ [1], or if the Ricci tensor is of *Codazzi type*([1], [7]).

Now, in the language of differential forms, there exist a *Riemannian curvature 2-form* associated to the Riemann curvature tensor, precisely one defines ([1], [11]):

$$\Omega_l^m = -\frac{1}{2} R_{jkl}{}^m dx^j \wedge dx^k. \tag{1.1}$$

Moreover, we may define another *curvature 2-form* associated to the divergence of the Riemann curvature tensor, that is [12]:

$$\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k. \tag{1.2}$$

Finally, a *Ricci 1-form* associated to the Ricci tensor may be defined in the following way [18]:

$$\Lambda_l = R_{kl} dx^k. \tag{1.3}$$

Now we consider the class of curvature tensors $K_{jkl}{}^m$ with the usual symmetries of the Riemann curvature tensor satisfying the first Bianchi identity. Specifically, we admit a generalized curvature tensor satisfying the following relations (see [12] and [19]:

$$\begin{aligned} &\text{a)} \qquad K_{jkl}{}^m + K_{klj}{}^m + K_{ljk}{}^m = 0, \quad K_{jkl}{}^m = -K_{kjl}{}^m, \\ &\text{b)} \qquad \nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m, \end{aligned} \tag{1.4}$$

where $B_{ijkl}{}^m$ is a tensor source in the second Bianchi identity. Moreover, we may define also a completely covariant $(0,4)$-type tensor K with the following further properties [18]:

$$K_{jklm} = -K_{kjlm} = -K_{jkml}, \qquad K_{jklm} = K_{lmjk}. \tag{1.5}$$

In this way the contraction $K_{kl} = -K_{mkl}{}^m$ defines a symmetric generalized Ricci tensor [18]. It is worthwhile to see that in this general case the second Bianchi identity admits a nonzero source tensorial term B. An n-dimensional Riemannian manifold is said to be *K flat* if $K_{jkl}{}^m = 0$, *K-symmetric* if $\nabla_i K_{jkl}{}^m = 0$, and *K-harmonic* if $\nabla_m K_{jkl}{}^m = 0$ ([12]).

Now the *curvature 2-form* associated to this tensor may be defined in the following manner:

$$\Omega_{(K)l}{}^m = K_{jkl}{}^m dx^j \wedge dx^k. \tag{1.6}$$

Consequently, the 2-form associated to the divergence of this tensor is defined as:

$$\Pi_{(K)l} = \nabla_m K_{jkl}{}^m dx^j \wedge dx^k. \tag{1.7}$$

If we consider the symmetric contraction $K_{kl} = -K_{mkl}{}^m$ a generalized *Ricci 1-form* may be defined [18] as:

$$\Lambda_{(K)l} = K_{kl} dx^k. \tag{1.8}$$

Among the tensors K, for example we may consider the conformal curvature tensor [13] whose local components are given by:

$$\begin{aligned} C_{jkl}{}^m = R_{jkl}{}^m &+ \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j^m g_{kl} - R_k^m g_{jl}) \\ &- \frac{R}{(n-1)(n-2)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}). \end{aligned} \tag{1.9}$$

It can be easily seen that the Weyl tensor vanishes identically for $n = 3$ [13]. A Riemannian manifold of dim $n > 3$ is said to be *conformally flat* if $C_{jkl}{}^m = 0$ [13] , *conformally symmetric* if $\nabla_i C_{jkl}{}^m = 0$ [17], and has a *harmonic conformal curvature tensor* if $\nabla_m C_{jkl}{}^m = 0$ [1]. Because of the general definitions the following *conformal curvature 2-form* associated to the conformal curvature tensor can be defined on a Riemannian manifold as follows:

$$\Omega_{(C)l}{}^m = C_{jkl}{}^m dx^j \wedge dx^k. \tag{1.10}$$

Obviously, the 2-form associated to the divergence of the conformal curvature tensor becomes:

$$\Pi_{(C)l} = \nabla_m C_{jkl}{}^m dx^j \wedge dx^k. \tag{1.11}$$

In the same way, we may consider other well known curvature tensors such as the *projective curvature tensor* $P_{jkl}{}^m$ (see [8] and [17]), the *concircular curvature tensor* $\tilde{C}_{jkl}{}^m$ (see [16] and [20]) and the *conharmonic curvature tensor* $N_{jkl}{}^m$ (see [9] and [17]).

First, the local components of the *projective curvature tensor* are defined as (see [8] and [17]):

$$P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j^m R_{kl} - \delta_k^m R_{jl}). \tag{1.12}$$

As a second, the local components of the *concircular curvature tensor* can be defined by (see [16] and [20]):

$$\tilde{C}_{jkl}{}^m = R_{jkl}{}^m + \frac{R}{n(n-1)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}). \tag{1.13}$$

Finally, the local components of the *conharmonic curvature tensor* are defined by (see [9] and [17]):

$$N_{jkl}{}^{m} = R_{jkl}{}^{m} + \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j^m g_{kl} - R_k^m g_{jl}). \tag{1.14}$$

It is worthwhile to note that all the previous tensors are built from the Riemann curvature tensor and the Ricci tensor. Consequently, we may define the associated curvature 2-forms $\Omega_{(P)l}{}^{m}$, $\Omega_{(\tilde{C})l}{}^{m}$, $\Omega_{(N)l}{}^{m}$, which are said to be *projective, concircular, conformal* and *conharmonic curvature* 2*-forms* respectively, and the corresponding 2-forms associated to the divergence of such tensors $\Pi_{(P)l}$, $\Pi_{(\tilde{C})l}$, $\Pi_{(N)l}$. The closedness of such forms gives a great geometric importance which of makes specific restrictions on the Riemann curvature tensor and the Ricci tensor.

In Section 2, we will examine the closedness conditions of such forms. Specially, we quote some known results about the closedness of the *projective, concircular, conformal* and *conharmonic curvature* 2*-forms*. We will show that particular differential structures built from the generalized tensor K give rise to the closedness of the form $\Omega_{(K)l}{}^{m}$. Moreover, concerning the closedness of $\Pi_{(K)l}$, we give a general algebraic condition involving the Riemann curvature tensor and the Ricci tensor that arises from an old identity due to Lovelock. Thus the closedness of the form $\Pi_{(K)l}$ is weaker than the notion of harmonic curvature.

2. Closedness properties for curvature 2-forms

In this section the closedness of the previously defined curvature 2-form will be investigated. The coefficients of such forms are, in usual, the components of a general tensor. Now let us recall some facts for the differentials of a general 2-form as follows:

$$\Phi_l^m = \frac{1}{2} A_{jkl}{}^{m} dx^j \wedge dx^k. \tag{2.1}$$

Let M be an n-dimensional Riemannian manifold. As is well known to us, one can consider two kinds of the differentials acting on the previous form. The first one is the exterior covariant derivative D defined in [1], [2] and [11] as follows:

$$D\Phi_l^m = \frac{1}{2} \nabla_i A_{jkl}{}^{m} dx^i \wedge dx^j \wedge dx^k. \tag{2.2}$$

The form is said to be closed if $D\Phi_l^m = 0$. The second differential is said to be codifferential and is given by [1] and [2]:

$$\delta\Phi_l^m = -\nabla^j A_{jkl}{}^{m} dx^k. \tag{2.3}$$

The form is said to be *coclosed* if $\delta\Phi_l^m = 0$. It is interesting to note that $D^2\Phi_l^m$ is non zero in general unlike the case of the ordinary exterior differential (see [1], page 24). A form which is closed and coclosed is said to be *harmonic* [2]. Thus it is easy to verify that the second Bianchi identity for the Riemann curvature tensor represents the closedness of the Riemann curvature 2-form (1.1) in the absence of torsion (see [1] and [11]).

In fact, if the operator of exterior covariant derivative is applied to (1.1), we obtain (see [11] Section 5.2):

$$\begin{aligned} D\Omega_l^m &= -\frac{1}{2}\nabla_i R_{jkl}{}^m dx^i \wedge dx^j \wedge dx^k = -\frac{1}{2\cdot 3!}\nabla_i R_{jkl}{}^m \delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t \\ &= -\frac{1}{2}\sum_{r<s<t} \nabla_i R_{jkl}{}^m \delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t, \end{aligned} \tag{2.4}$$

where we have used $dx^i \wedge dx^j \wedge dx^k = \frac{1}{3!}\delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t$.

Now it may be scrutinized (see [11], Section 5.1) that $D\Omega_l^m = 0$ if and only if $\nabla_i R_{jkl}{}^m \delta^{ijk}_{rst} = 2[\nabla_r R_{stl}{}^m + \nabla_s R_{trl}{}^m + \nabla_t R_{rsl}{}^m] = 0$. In fact, the basis elements $ds^r \wedge dx^s \wedge dx^t$ with $r < s < t$ are linearly independent and $\nabla_i R_{jkl}{}^m \delta^{ijk}_{rst}$ is completely skew-symmetric in such indices. From this, we assert that the curvature 2-form Ω_l^m is always closed.

Now we recall that when the manifold has harmonic curvature tensor, that is, $\nabla_m R_{jkl}{}^m = 0$ [1], the form Ω_l^m becomes coclosed (see [18] and [19]). If we focus now on the closedness condition for the Ricci 1-form, it is not difficult to find:

$$D\Lambda_l = \nabla_i R_{kl} dx^i \wedge dx^k = \frac{1}{2!}\nabla_i R_{kl}\delta^{ik}_{rs} dx^r \wedge dx^s = \sum_{r<s} \nabla_i R_{kl}\delta^{ik}_{rs} dx^r \wedge dx^s. \tag{2.5}$$

So $D\Lambda_l = 0$ if and only if $\nabla_i R_{kl}\delta^{ik}_{rs} = \nabla_r R_{sl} - \nabla_s R_{rl} = \nabla_m R_{srl}{}^m = 0$, that is, if and only if the Ricci tensor becomes a Codazzi tensor(in this case the manifold has a harmonic curvature tensor [7]). Then we have the following

Lemma 2.1. *The curvature 2-form Ω_l^m is coclosed if and only if the Ricci tensor is of Codazzi type.*

Obviously we have $D\Lambda_{(K)l} = 0$ if and only if $\nabla_i K_{kl}\delta^{ik}_{rs} = \nabla_r K_{sl} - \nabla_s K_{rl} = 0$, that is, if and only if the generalized Ricci tensor is of Codazzi type.

It might be realized that for a generalized curvature tensor the presence of a source term $B_{ijkl}{}^m$ in the second Bianchi identity prevents in general the correspondent $\Omega_{(K)l}{}^m$ to be closed. In this way the 2-form is not closed unless we have

$B_{ijkl}{}^m = 0$. This may happen for example when the manifold is endowed with some differential structures. Thus if the manifold is K symmetric $\nabla_i K_{jkl}{}^m=0$ ([10], [12] and [17]), we have simply $B_{ijkl}{}^m = 0$. This may happen also with more involved differential structures such as pseudo K-symmetric manifolds, in which the tensor K satisfies [5]:

$$\nabla_i K_{jkl}{}^m = 2A_i K_{jkl}{}^m + A_j K_{ikl}{}^m + A_k K_{jil}{}^m + A_l K_{jki}{}^m + A^m K_{jkli}. \tag{2.6}$$

If the previous equation holds for $K_{jkl}{}^m = R_{jkl}{}^m$, then the manifold is called pseudo symmetric [3], if it holds for $K_{jkl}{}^m = C_{jkl}{}^m$, then the manifold is called pseudo conformally symmetric [6](or conformally quasi recurrent [14]), if holds for $K_{jkl}{}^m = P_{jkl}{}^m$, then the manifold is called pseudo projective symmetric [4]. The covector A_i is said to be an associated 1-form. From the equation (2.6) it may be easily verified that $\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m = 0$, and so that the symmetric generalized Ricci tensor K_{kl} satisfies $\nabla_m K_{jkl}{}^m = \nabla_k K_{jl} - \nabla_j K_{kl}$, that is, a second contracted Bianchi identity for the Ricci tensor. Thus the closedness condition of the form $D\Omega_{(K)l}{}^m$ represents a proper generalization of a K symmetric manifold. Obviously, when $\nabla_m K_{jkl}{}^m = 0$ the form $\Omega_{(K)l}{}^m$is said to be *coclosed* ([18], [19]). In particular, for a pseudo conformally symmetric manifold, we get $\nabla_i C_{jkl}{}^m+\nabla_j C_{kil}{}^m+\nabla_k C_{ijl}{}^m = B_{ijkl}{}^m = 0$ and by a contraction $\nabla_m C_{jkl}{}^m = 0$. Accordingly, in this case the form $\Omega_{(C)l}{}^m$ is harmonic. In general the value of $D\Omega_{(K)l}{}^m$ depends strongly on the feature of the source term and so on the K tensor.

Nevertheless, it is quite simple to give a closedness condition for the 2-form $\Omega_{(K)l}{}^m$ associated with the projective, conformal, concircular and conharmonic curvature tensors. To get the expression for the source term B we take the covariant derivative ∇_i to the local components of such tensors and sum over cyclic permutations of indices i, j, k.

First, the local components of the second Bianchi identity for the projective curvature tensor can be given by

$$\nabla_i P_{jkl}{}^m + \nabla_j P_{kil}{}^m + \nabla_k P_{ijl}{}^m = \frac{1}{n-1}\Big[\delta_j^m \nabla_p R_{kil}{}^p + \delta_i^m \nabla_p R_{jkl}{}^p + \delta_k^m \nabla_p R_{ijl}{}^p\Big]. \tag{2.7}$$

As a second, the local components of the second Bianchi identity for the conformal curvature tensor are given by:

$$\nabla_i C_{jkl}{}^m + \nabla_j C_{kil}{}^m + \nabla_k C_{ijl}{}^m = \frac{1}{n-2}\Big[\delta_j^m(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta_i^m(\nabla_k R_{jl} - \nabla_j R_{kl}) + \delta_k^m(\nabla_j R_{il} - \nabla_i R_{jl})$$

$$+ g_{il}(\nabla_j R^m_k - \nabla_k R^m_j) + g_{jl}(\nabla_k R^m_i - \nabla_i R^m_k) + g_{kl}(\nabla_i R^m_j - \nabla_j R^m_i)\Big] - \frac{1}{(n-1)(n-2)}\Big[\delta^m_j(\nabla_i R g_{kl} - \nabla_k R g_{il}) + \delta^m_i(\nabla_k R g_{jl} - \nabla_j R g_{kl}) + \delta^m_k(\nabla_j R g_{il} - \nabla_i R g_{jl})\Big]. \quad (2.8)$$

As a third, the local components of the second Bianchi identity for the concircular curvature tensor could be

$$\nabla_i \tilde{C}_{jkl}{}^m + \nabla_j \tilde{C}_{kil}{}^m + \nabla_k \tilde{C}_{ijl}{}^m = \frac{1}{n(n-1)}\Big[\delta^m_j(\nabla_i R g_{kl} - \nabla_k R g_{il}) + \delta^m_i(\nabla_k R g_{jl} - \nabla_j R g_{kl}) + \delta^m_k(\nabla_j R g_{il} - \nabla_i R g_{jl})\Big]. \quad (2.9)$$

Finally, the local components of the second Bianchi identity for the conharmonic curvature tensor are given by

$$\nabla_i N_{jkl}{}^m + \nabla_j N_{kil}{}^m + \nabla_k N_{ijl}{}^m = \frac{1}{n-2}\Big[\delta^m_j(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta^m_i(\nabla_k R_{jl} - \nabla_j R_{kl}) + \delta^m_k(\nabla_j R_{il} - \nabla_i R_{jl}) + g_{il}(\nabla_j R^m_k - \nabla_k R^m_j) + g_{jl}(\nabla_k R^m_i - \nabla_i R^m_k) + g_{kl}(\nabla_i R^m_j - \nabla_j R^m_i)\Big]. \quad (2.10)$$

Now we recall that $D\Lambda_l = 0$ if and only if the Ricci tensor becomes a Codazzi tensor. Then in such a case we say that M is of harmonic curvature, because $-\nabla_m R_{kji}{}^m = \nabla_k R_{ji} - \nabla_j R_{ki}$ from the 2nd Bianchi identity. If the Ricci tensor could be of Codazzi type, then the scalar curvature should be covariant constant.Thus the Bianchi identities of projective, conformal, concircular and conharmonic curvature tensors are satisfied with vanishing in the right side. Moreover, under these conditions according to the successive equation (2.20) we can assert that $\nabla_m P_{jkl}{}^m = 0$, $\nabla_m C_{jkl}{}^m = 0$, $\nabla_m \tilde{C}_{jkl}{}^m = 0$ and $\nabla_m N_{jkl}{}^m = 0$. So we give a more general Theorem than [18] as follows:

Theorem 2.1. *Let M be an n-dimensional Riemannian manifold with harmonic curvature tensor. Then the curvature forms $\Omega_{(P)l}{}^m$, $\Omega_{(C)l}{}^m$, $\Omega_{(\tilde{C})l}{}^m$ and $\Omega_{(N)l}{}^m$ are closed and coclosed. Thus all tensors P, C, $\tilde{C}$ and N are harmonic generalized curvature tensors.*

Remark 2.1. A deep result about the properties of harmonic generalized curvature tensors is given in Theorem 3.12 of Bourguignon's paper [2]. If C is a curvature tensor field, the following statements are equivalent:

i) C is harmonic,

ii) The irreducible components of C under the action of the orthogonal group $O(n)$ are harmonic.

From this we get a quick proof of Theorem 2.1: one just notes that the $O(n)$ components of the tensors P, C, $\tilde{C}$ and N are the same(up to scalars) with those of R.

On the other hand, let us focus on the closedness of the $\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k$ associated to the divergence of the Riemann curvature tensor.

Using the above arguments, we know that the form is closed if and only if

$$\nabla_i \nabla_m R_{jkl}{}^m \delta^{ijk}_{rst} = 2\big[\nabla_r \nabla_m R_{stl}{}^m + \nabla_s \nabla_m R_{trl}{}^m + \nabla_t \nabla_m R_{rsl}{}^m\big] = 0. \qquad (2.11)$$

This condition was stated in the paper [12]. Now an old differential identity due to Lovelock (see [11] and [12]) is pointed out in the following Lemma:

Lemma 2.2 (Lovelock's identity)**.** *Let M be an n-dimensional Riemannian manifold. Then the divergence of the Riemann curvature tensor satisfies the following identity:*

$$\begin{aligned}\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\ = -(R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m). \qquad (2.12)\end{aligned}$$

This identity and its various generalizations are referred as "the Weitzenböck formula" for *curvature-like tensors* (see equation (4.2) in [2]). It appears as the second Bianchi identity with source term for the divergence of the Riemann curvature tensor. We may thus state the following Theorem already quoted in [12]:

Theorem 2.2. *Let M be an n-dimensional Riemannian manifold. Then the 2-form $\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k$ is closed if and only if*

$$R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m = 0. \qquad (2.13)$$

When the previous equation is satisfied, the divergence of the Riemann curvature tensor satisfies the second Bianchi identity without source term. Obviously when $\nabla_m R_{jkl}{}^m = 0$, that is, M is with harmonic curvature, the Riemann curvature 2-form is closed. In paper [12] the authors pointed out other interesting cases of closedness. For example we may consider a nearly conformally symmetric manifold $(NCS)_n$ i.e. a manifold in which the following condition is satisfied:

$$\nabla_j R_{kl} - \nabla_k R_{jl} = \frac{1}{2(n-1)}\Big[\nabla_j R g_{kl} - \nabla_k R g_{jl}\Big]. \qquad (2.14)$$

This condition was introduced and studied by Roter [15] and Suh, Kwon and Yang [19]. In [12] it was proved that for such a manifold equation (2.13) is verified, thus the 2-form $\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k$ is closed. So the closedness condition of Π_l represents a proper generalization of the concept of harmonic curvature. It is worthwhile to note that equation (2.14) is equivalent to the closedness of the 1-form defined in [18] as follows:

$$\Sigma_k = \Big(R_{kl} - \frac{R}{2(n-1)} g_{kl}\Big) dx^l. \tag{2.15}$$

When the Ricci tensor satisfies equation (2.14), it is said to be a *Weyl tensor* ([18], [19]) and the corresponding form is called *Weyl form*. In this case by virtue of (2.8), a more refined version of Theorem 2.1 (see [18], Lemma 7.2) can be stated for the conformal curvature 2-form as follows:

Theorem 2.3. *Let M be an n-dimensional Riemannian manifold with closed Weyl form $D\Sigma_l = 0$. Then the conformal curvature 2-form $\Omega^m_{(C)l}$ is closed.*

Now it is worthwhile to point out that in [12] the authors proved that Lovelock's identity is left unchanged if the divergence of the Riemann tensor, that is, $\nabla_m R_{ijk}{}^m$ is replaced by the divergence of any *curvature tensor* $K_{jkl}{}^m$with the property

$$\nabla_m K_{jkl}{}^m = A \nabla_m R_{jkl}{}^m + B\Big[(\nabla_j \varphi) a_{kl} - (\nabla_k \varphi) a_{jl}\Big], \tag{2.16}$$

where A, B are non zero constants, φ is a real scalar function and a_{kl} is a symmetric $(0,2)$-type Codazzi tensor, i.e. $\nabla_i a_{kl} = \nabla_k a_{il}$ (see [1], [2] and [7]). Specifically, we have the following Theorem (see [12] Proposition 2.4):

Theorem 2.4. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor $K_{jkl}{}^m$ with the property* (2.16)*.Then*

$$\begin{aligned}\nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \\ = -A(R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m).\end{aligned} \tag{2.17}$$

This conclusion also follows from formula (4.8) in [2].

Remark 2.2. The existence of non-trival Codazzi tensor has some important geometric and topological consequences. In particular in [2] the author pointed out strong restrictions imposed on the curvature operator due to the existence of Codazzi tensors. Moreover, in [7] the authors proved that any manifold carries a C^∞ metric g_{kl} such that (M, g)admits a non-trivial Codazzi tensor a_{kl}. Now given a non-trivial Codazzi tensor we may exhibit a tensor ${K_{jkl}}^m$ satisfying the equation (2.16) as follows:

$$ {K_{(a)jkl}}^m = A{R_{jkl}}^m + B\varphi(\delta_j^m a_{kl} - \delta_k^m a_{jl}). $$

If the operator ∇_m is applied on this tensor, it is easy to see that the condition (2.16) is satisfied. On the other hand, by a straightforward calculation based on the properties of Codazzi tensors it is interesting to note that:

$$ \begin{aligned} &\nabla_i {K_{(a)jkl}}^m + \nabla_j {K_{(a)kil}}^m + \nabla_k {K_{(a)ijl}}^m \\ &= B\Big[\nabla_i\varphi\big(\delta_j^m a_{kl} - \delta_k^m a_{jl}\big) + \nabla_j\varphi\big(\delta_k^m a_{il} - \delta_i^m a_{kl}\big) + \nabla_k\varphi\big(\delta_i^m a_{jl} - \delta_j^m a_{il}\big)\Big]. \end{aligned} $$

Thus if $\nabla_i\varphi = 0$, the second Bianchi identity for such a tensor is satisfied with null source term, that is, $D{\Omega_{(K)l}}^m = 0$.

From Theorem 2.4 the following results naturally arise:

Corollary 2.1. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor ${K_{jkl}}^m$ with the property* (2.16)*. Then*

$$ D\Pi_{(K)l} = A(D\Pi_l). \tag{2.18} $$

That is, the value of the exterior covariant derivatives of these forms are proportional. We have also:

Corollary 2.2. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor ${K_{jkl}}^m$ with the property* (2.16)*. Then $\Pi_{(K)l}$ is closed if and only if Π_l is closed.*

Finally we can easily state a remarkable result that links the closedness of the 2-form $\Pi_{(K)l}$ to a simple algebraic condition involving the Riemann curvature tensor and the Ricci tensors:

Theorem 2.5. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor ${K_{jkl}}^m$ with the property* (2.16)*. Then the curvature 2-form $\Pi_{(K)l} = \nabla_m {K_{jkl}}^m dx^j \wedge dx^k$ is closed if and only if*

$$ R_{im}{R_{jkl}}^m + R_{jm}{R_{kil}}^m + R_{km}{R_{ijl}}^m = 0. \tag{2.19} $$

The algebraic condition (2.19) was used to give the closedness of the form Π_l in Theorem 2.2. In Theorem 2.5 it was also used to give the closedness of $\Pi_{(K)l}$. In this case, the divergence of the generalized K tensor satisfies the second Bianchi identity. We have also the following useful:

Corollary 2.3. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor $K_{jkl}{}^m$ with the property* (2.16). *If $D\Lambda_l = 0$, then $D\Pi_{(K)l} = 0$.*

Proof. In fact, from the 2nd Bianchi identity and the definition of the Ricci tensor we have

$$-\nabla_m R_{ijl}{}^m = \nabla_i R_{jl} - \nabla_j R_{il}.$$

On the other hand, the Ricci 1-form $D\Lambda_l = 0$ if and only if the Ricci tensor is of Codazzi type. From this, together with the above equations (2.12) and (2.13), it follows that $D\Pi_l = 0$. Then by Corollary 2.2, we get our assertion. □

Some curvature tensors $K_{jkl}{}^m$ with the property (2.16) and trivial Codazzi tensors (i.e. constant multiple of the metric) are well known. The *projective curvature tensor* $P_{jkl}{}^m$ ([8], [17]) *the conformal curvature tensor* $C_{jkl}{}^m$ [13], *the concircular curvature tensor* $\tilde{C}_{jkl}{}^m$ ([16], [20]) and *the conharmonic curvature tensor* $N_{jkl}{}^m$ [17].

In fact taking the covariant derivative ∇_m of the local components of such tensors we have:

$$\begin{aligned}
\nabla_m P_{jkl}{}^m &= \frac{n-2}{n-1}\nabla_m R_{jkl}{}^m,\\
\nabla_m C_{jkl}{}^m &= \frac{n-3}{n-3}\Big[\nabla_m R_{jkl}{}^m + \frac{1}{2(n-1)}\{(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}\}\Big],\\
\nabla_m \tilde{C}_{jkl}{}^m &= \nabla_m R_{jkl}{}^m + \frac{1}{n(n-1)}\Big\{(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}\Big\},\\
\nabla_m N_{jkl}{}^m &= \frac{n-3}{n-2}\nabla_m R_{jkl}{}^m + \frac{1}{2(n-2)}\Big\{(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}\Big\}.
\end{aligned} \tag{2.20}$$

We have thus shown that the closedness of any curvature 2-form

$$\Pi_{(K)l} = \nabla_m K_{jkl}{}^m dx^j \wedge dx^k$$

with the tensor K satisfying (2.16) depends on the same algebraic condition which gives the closedness of the form $\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k$. From the results reported in [12] we may state that the closedness of 2 form $\Pi_{(K)l}$ implies the closedness of the associated covectors in a Weakly Ricci Symmetric manifold [5]. Moreover, in a paper [18] due to Suh, Kwon and Pyo, the importance of the closedness for the associated curvature-like 2 forms corresponding to each concircular, projective and conformal curvature-like tensors defined on a semi Riemannian manifold was remarked respectively.

Finally, we may see that the differential structure (2.6) makes, in particular conditions, the closedness of the vector valued 2-form $\Pi_{(K)l}$. We write such a differential structure for the Riemann *curvature tensor* as follows:

$$\nabla_i R_{jkl}{}^m = 2A_i R_{jkl}{}^m + A_j R_{ikl}{}^m + A_k R_{jil}{}^m + A_l R_{jki}{}^m + A^m R_{jkli}. \qquad (2.21)$$

In this case a manifold is called Pseudo Symmetric $(PS)_n$ [5]. Now if we put $i = m$ and sum, the following expression easily comes out:

$$\nabla_m R_{jkl}{}^m = 3A_m R_{jkl}{}^m + A_k R_{jl} - A_j R_{kl}. \qquad (2.22)$$

The covariant derivative ∇_i is thus applied to the previous expression, and then a cyclic permutation over indices i, j, k is performed. The resulting equations are added to obtain:

$$\begin{aligned}
&\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kll}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\
&\quad = 3[\nabla_i A_m R_{jkl}{}^m + \nabla_j A_m R_{kil}{}^m + \nabla_k A_m R_{ijl}{}^m] \\
&\quad\quad + 3A_m[\nabla_i R_{jkl}{}^m + \nabla_j R_{kil}{}^m + \nabla_k R_{ijl}{}^m] \\
&\quad\quad + R_{jl}(\nabla_i A_k - \nabla_k A_i) + R_{kl}(\nabla_j A_i - \nabla_i A_j) + R_{il}(\nabla_k A_j - \nabla_j A_k) \\
&\quad\quad + A_k(\nabla_i R_{jl} - \nabla_j R_{kl}) + A_i(\nabla_j R_{kl} - \nabla_k R_{jl}) + A_j(\nabla_k R_{il} - \nabla_i R_{kl}). \qquad (2.23)
\end{aligned}$$

Now using (2.22), the second Bianchi identity and $\nabla_m R_{jkl}{}^m = \nabla_k R_{jl} - \nabla_j R_{kl}$, we easily obtain:

$$\begin{aligned}
&\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\
&= 3\Big[(\nabla_i A_m - A_i A_m) R_{jkl}{}^m + (\nabla_j A_m - A_j A_m) R_{kil}{}^m + (\nabla_k A_m - A_k A_m) R_{ijl}{}^m\Big] \\
&\quad + R_{jl}(\nabla_i A_k - \nabla_k A_i) + R_{kl}(\nabla_j A_i - \nabla_i A_j) + R_{il}(\nabla_k A_j - \nabla_j A_k). \qquad (2.24)
\end{aligned}$$

Now let us consider a concircular associated covector A_i, which satisfies the condition $\nabla_i A_m = A_i A_m + \gamma g_{im}$, being γ an arbitrary scalar function. Then we obtain that:

$$\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m = 0. \qquad (2.25)$$

In this way the vector valued 2-forms Π_l and so $\Pi_{(K)l}$ is closed. We have thus proved the following:

Theorem 2.6. *Let M be an n-dimensional pseudo symmetric Riemannian manifold with the property $\nabla_i A_m = A_i A_m + \gamma g_{im}$. Then $D\Pi_{(K)l} = 0$.*

It is easy to see that the previous Theorem is still valid for a pseudo K-symmetric manifold. In this case (2.21), (2.22), (2.23), (2.24) and (2.25) can be written for the tensor ${K_{jkl}}^m$ and for the symmetric generalized Ricci tensor satisfying the second contracted Bianchi identity. In particular, (2.24) takes the form:

$$\begin{aligned}
&\nabla_i\nabla_m {K_{jkl}}^m + \nabla_j\nabla_m {K_{kil}}^m + \nabla_k\nabla_m {K_{ijl}}^m \\
&\quad = 3\Big[(\nabla_i A_m - A_i A_m){K_{jkl}}^m + (\nabla_j A_m - A_j A_m){K_{kil}}^m + (\nabla_k A_m \\
&\qquad - A_k A_m){K_{ijl}}^m\Big] + K_{jl}(\nabla_i A_k - \nabla_k A_i) \\
&\qquad + K_{kl}(\nabla_j A_i - \nabla_i A_j) + K_{il}(\nabla_k A_j - \nabla_j A_k).
\end{aligned} \tag{2.26}$$

Now let us consider again a concircular associated covector A_i satisfying $\nabla_i A_m = A_i A_m + \gamma g_{im}$. Then one easily get:

$$\nabla_i\nabla_m {K_{jkl}}^m + \nabla_j\nabla_m {K_{kil}}^m + \nabla_k\nabla_m {K_{ijl}}^m = 0. \tag{2.27}$$

Then we can assert the following

Theorem 2.7. *Let M be an n-dimensional pseudo K-symmetric Riemannian manifold with the property $\nabla_i A_m = A_i A_m + \gamma g_{im}$. Then $D\Pi_{(K)l} = 0$.*

We have thus shown that the closeness of $\Pi_{(K)l}$ is a proper generalization of the concept of K-harmonic curvature.

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