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Conformal invariances of two-dimensional Finsler spaces with isotropic main scalar

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Abstract. In this paper, we study the conformal invariances of the two-dimensional Finsler spaces with isotropic main scalar. In particular, we prove that a two-dimensional Finsler space with isotropic main scalar is locally conformally flat if and only if the main scalar is constant.

1. Introduction

Compared with higher dimensional Finsler spaces, two-dimensional Finsler spaces have many special features. A fundamental fact is that any two-dimensional Finsler space is of scalar flag curvature. L. Berwald made some important pioneering work for two-dimensional Finsler spaces (see [5]).

In this paper, we mainly study the conformal invariances of the two-dimensional Finsler spaces with isotropic main scalar. The main scalar is a very important geometric quantity defined on two-dimensional Finsler spaces, which characterizes many geometrical properties of two-dimensional Finsler spaces ([1], [4], [5]).

In Finsler geometry, there are several important classes of Finsler metrics: locally Minkowski metrics, Berwald metrics, Landsberg metrics and Douglas metrics. Many Finsler geometers have studied these Finsler metrics ([6], [8], [12]). In two-dimensional case, it is proved that a two-dimensional Finsler space (M, F) with isotropic main scalar $I = I(x)$ is a Berwald metric or Landsberg metric or

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Douglas metric if and only if $I(x) = \text{constant}$ (see [4], [5]. Also see Lemma 3.1 below).

It is well known that all Finsler metrics with isotropic main scalar $I = I(x)$ on a two-dimensional manifold can be expressed as one of the following cases ([1], [5]):

$$F^2 = \beta^{2s}\gamma^{2(1-s)}, \quad (s = s(x) \neq 0, \quad s(x) \neq 1), \tag{1}$$

$$F^2 = \beta^2 e^{\frac{2\gamma}{\beta}}, \tag{2}$$

$$F^2 = (\beta^2 + \gamma^2)e^{2r\cdot\arctan\left(\frac{\beta}{\gamma}\right)}, \quad r = r(x), \tag{3}$$

where $\beta = p_i(x)y^i$ and $\gamma = q_i(x)y^i$ are two independent 1-forms. Their isotropic main scalar $I = I(x)$ are given respectively by

$$\epsilon I^2 = \frac{(2s(x)-1)^2}{s(x)(s(x)-1)}, \tag{4}$$

$$I^2 = 4, \tag{5}$$

$$I^2 = \frac{4r(x)^2}{1 + r(x)^2}, \tag{6}$$

where ϵ in (4) is the index of F satisfying that $\epsilon = 1$ if $s(x) > 1$ or $s(x) < 0$ and $\epsilon = -1$ if $0 < s(x) < 1$. If the main scalar $I = 0$, which is equivalent that $s = \frac{1}{2}$ in (1) or $r = 0$ in (3), then F is a Riemann metric.

Further, based on the classification of two-dimensional Finsler spaces with isotropic main scalar as above, L. Berwald classified all two-dimensional projectively flat Finsler spaces with isotropic main scalar ([5]). In this paper, we further characterize the locally conformal flatness and other conformal invariances of two-dimensional Finsler spaces with isotropic main scalar (see Theorem 1.1 below). It is well known that any two-dimensional Riemann metric is locally conformally flat. However, by our Theorem 1.1, this conclusion is no longer true for two-dimensional Finsler metrics.

For the related definitions and fundamental properties in Finsler conformal geometry, see [3], [7], [9], [10]. Our main theorem is as follows.

Theorem 1.1. *Let $F(x, y)$ be a two-dimensional Finsler metric with isotropic main scalar $I = I(x)$. Then the following conditions are equivalent:*

(a_1) $I = \text{constant}$.

(a_2) *$F(x, y)$ is locally conformally flat.*

(a_3) *$F(x, y)$ is locally conformal to a Finsler metric of zero flag curvature.*

(a_4) *$F(x,y)$ is locally conformal to a Berwald metric, or a Landsberg metric, or a Douglas metric.*

(a_5) *$F(x,y)$ is a Berwald metric, or a Landsberg metric, or a Douglas metric.*

The equivalence of (a_1) and (a_5) in Theorem 1.1 has actually been proved (see [4], [5]). To simplify the proof of Theorem 1.1, we note that the independent 1-forms β and γ in (1), (2) and (3) can be rewritten in the forms $\beta = p(x^1, x^2)y^1, \gamma = q(x^1, x^2)y^2$ in some local coordinate system, because of the existence of an integral factor for any 1-form $a(x^1, x^2)dx^1 + b(x^1, x^2)dx^2$.

Theorem 1.1 does not hold if the main scalar is not isotropic. For example, there are many Randers metrics which are Douglas metrics but not Berwald metrics. In this case, (a_5) does not hold. We also see the following example:

Example 1.2. Consider an n-dimensional Randers metric $F := \alpha + \beta$, where $\alpha := \delta_{ij}(x)y^i y^j = \sum_{i=1}^n (y^i)^2$ is a Riemann metric and $\beta := b_i(x)y^i$ is a closed 1-form with $\|\beta\|_\alpha < 1$. Then F is a Douglas metric. If F is locally conformally flat, then there is a local function $c(x)$ such that $e^{c(x)}F$ is locally Minkowskian. We get that $e^{c(x)}\beta$ is parallel in $e^{c(x)}\alpha$, that is,

$$(e^{c(x)}b_i)_{,j} = 0, \tag{7}$$

where the covariant derivative is taken with respect to $e^{c(x)}\alpha$. Since β is closed, it is easily verified that (7) is equivalent to

$$c_i b_j = c_j b_i, \quad \frac{\partial b_i}{\partial x^j} = -\frac{1}{2} b_r c_r \delta_{ij}, \tag{8}$$

where $c_i := \frac{\partial c}{\partial x^i}$. By (8) we get

$$\frac{\partial b_i}{\partial x^j} = 0 \quad (\forall i \neq j). \tag{9}$$

Thus there are many 1-form β's which are closed but do not satisfy (9). So we can find many Randers metrics which are Douglas metrics but not locally conformally flat. When $n = 2$, their main scalars are not isotropic.

For any two-dimensional Riemann metric, the main scalar $I = 0$. Hence, we have the following well-known result in Riemann geometry.

Corollary 1.3. *Any two-dimensional Riemann metric is locally conformally flat.*

2. Preliminaries

Let (M, F) be a Finsler manifold with the fundamental function F defined on the manifold M. In this paper, the Finsler metrics that we consider may not be positive definite.

The geodesic $x = x(t)$ of a Finsler metric F is characterized by the following system of 2nd order ordinary differential equations:

$$\frac{d^2x^i(t)}{dt^2} + 2G^i\big(x(t), \dot{x}(t)\big) = 0,$$

where

$$G^i := \frac{1}{4}g^{il}\Big\{\big(\partial_r\dot{\partial}_l(F^2)\big)y^r - \partial_l(F^2)\Big\}, \tag{10}$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, $\partial_i = \frac{\partial}{\partial x^i}$, $g_{ij} := \dot{\partial}_i\dot{\partial}_j(F^2/2)$ and $(g^{il}) := (g_{il})^{-1}$. G^i are called the *geodesic coefficients* of F.

The *Riemann curvature* $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is a family of linear transformations on tangent spaces, which is defined by

$$R^i{}_k := 2\partial_k G^i - y^r\partial_r G^i_k + 2G^r G^i_{rk} - G^i_r G^r_k, \tag{11}$$

where $G^i_{jk} := \dot{\partial}_j G^i_k$.

For a two-dimensional plane $P \subset T_pM$ and $y \in T_pM \setminus \{0\}$ such that $P = span\{y, u\}$, the pair $\{P, y\}$ is called a *flag* in T_pM. The *flag curvature* $\mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{g_y(u, \mathbf{R}_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

We say that F is of *scalar flag curvature* if for any $y \in T_pM\setminus\{0\}$, the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of P containing y. If $\mathbf{K}(P, y) =$ constant, we call $F(x, y)$ is of *constant flag curvature.* In the case of $F(x, y)$ being of scalar flag curvature we have

$$\mathbf{K}(x, y) = \frac{R^i{}_i}{(n-1)F^2}. \tag{12}$$

There are many interesting non-Riemannian quantities in Finsler geometry. The *Cartan torsion* is defined by

$$C_{ijk}(x, y) := \frac{1}{2}\dot{\partial}_k g_{ij}(x, y).$$

For a non-zero vector $y \in T_pM$, the *Berwald curvature* $\mathbf{B}_y = B^i_{hjk}dx^h \otimes dx^j \otimes dx^k \otimes \partial_i$ is defined by

$$B^i_{hjk} := \dot{\partial}_h\dot{\partial}_j\dot{\partial}_k G^i.$$

F is called a *Berwald metric* if $\mathbf{B} = 0$. The *Landsberg curvature* $\mathbf{L}_y = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := -\frac{1}{2} y_r B^r_{ijk},$$

where $y_i := g_{ir}y^r$. F is called a *Landsberg metric* if $\mathbf{L} = 0$. The *Douglas curvature* $\mathbf{D}_y = D^i_{hjk} dx^h \otimes dx^j \otimes dx^k \otimes \partial_i$ is defined by

$$D^i_{hjk} := \dot{\partial}_h \dot{\partial}_j \dot{\partial}_k \left(G^i - \frac{1}{n+1} G^r_r y^i \right),$$

where $G^j_i := \dot{\partial}_i G^j$. F is called a *Douglas metric* if $\mathbf{D} = 0$. Further, F is called a *locally Minkowski metric* if F is a Berwald metric ($\mathbf{B} = 0$) with zero flag curvature ($\mathbf{K} = 0$). It is known that a Finsler metric F is a Berwald metric if and only if

$$C_{ijk|l} = 0$$

and F is a Landsberg metric if and only if

$$C_{ijk|0} = 0,$$

where "$|$" denotes the h-covariant derivative with respect to Cartan connection (or Chern connection) and $C_{ijk|0} := C_{ijk|r}y^r$ ([1], [11]).

A Finsler metric $F(x, y)$ is said to be locally conformal to another Finsler metric $\widetilde{F}(x, y)$ on the same underlying manifold M if there exists a scalar function $c(x)$ on M such that $\widetilde{F}(x, y) = e^{c(x)} F(x, y)$. The scalar function $c(x)$ is called a *conformal factor*. If $\widetilde{F}$ is locally Minkowskian and F is locally conformal to $\widetilde{F}$, then F is called *locally conformally flat*.

In two-dimensional Finsler spaces, we define the *main scalar* $I = I(x, y)$ by

$$FC_{ijk} = I m_i m_j m_k, \tag{13}$$

where (l, m) is the Berwald frame with $l^i = y^i / F(x, y)$ and $g_{ij} m^i m^j = \varepsilon$, $g_{ij} l^i m^j = 0$. We say a two-dimensional Finsler metric has *isotropic main scalar* if $I(x, y) = I(x)$ is independent of y.

3. Some basic properties of the main scalar

In this section, we introduce some basic properties of the main scalar I in 2-dimensional Finsler spaces and prove the relations $(a_1) \Longleftrightarrow (a_4) \Longleftrightarrow (a_5)$ in Theorem 1.1

Lemma 3.1 ([4], [5]). *For two-dimensional Finsler spaces with isotropic main scalar $I = I(x)$, we have*

$$\mathbf{B} = 0 \Longleftrightarrow \mathbf{L} = 0 \Longleftrightarrow \mathbf{D} = 0 \Longleftrightarrow I = \text{constant}\,.$$

Lemma 3.2. *For two-dimensional Finsler spaces with isotropic main scalar, the main scalar is a conformal invariant.*

Proof. Let F be one of the metrics in (1), (2) and (3) and $\widetilde{F} = e^{c(x)}F$ be conformal to F for some conformal factor $c(x)$. It is easily seen that $\widetilde{F}$ is just the Finsler metric obtained from F by replacing β by $e^{c(x)}\beta$ and γ by $e^{c(x)}\gamma$. Hence the main scalar $I = I(x)$ is conformally invariant. □

Proof of $(a_1) \Longleftrightarrow (a_4) \Longleftrightarrow (a_5)$ in Theorem 1.1. By Lemma 3.1, it is easily seen that $(a_1) \Longleftrightarrow (a_5)$. It is also obvious that $(a_5) \Longrightarrow (a_4)$. Now we assume that (a_4) holds. Then by definition, the Finsler metric $\widetilde{F} := e^{c(x)}F$ for some conformal factor $c(x)$ is a metric of Berwald, or Landsberg, or Douglas type. By Lemma 3.2, the main scalar of the metric $\widetilde{F}$ is isotropic. So $\widetilde{F}$ has constant main scalar by Lemma 3.1. So F also has constant main scalar by Lemma 3.2. □

4. Two-dimensional Finsler spaces with constant main scalar

For convenience, we first make a convention in this and the next section: for an arbitrary function $f(x^1, x^2)$, define

$$f_1 := \frac{\partial}{\partial x^1} f(x^1, x^2),\ f_2 := \frac{\partial}{\partial x^2} f(x^1, x^2), \quad f_{12} := \frac{\partial^2}{\partial x^1 \partial x^2} f(x^1, x^2), etc.$$

In this section, we study two-dimensional Finsler spaces with constant main scalar and prove in Theorem 1.1 that $(a_1) \Longrightarrow (a_2) \Longrightarrow (a_3)$.

Lemma 4.1. *Let F be a two-dimensional Finsler metric with constant main scalar. Let $\beta = e^{c(x)}p(x)y^1$ and $\gamma = e^{c(x)}q(x)y^2$. Then we have*

(i) *If F is in the form*

$$F^2 = \beta^{2s}\gamma^{2(1-s)} \quad (s = \text{constant},\ s \neq 0,\ \neq 1), \tag{14}$$

then the flag curvature $\mathbf{K}$ is given by

$$\mathbf{K} = \frac{e^{-2c}}{s(s-1)pq}\Big(\frac{\gamma}{\beta}\Big)^{2s-1}\Big[c_{12} + (1-s)\frac{qq_{12} - q_1 q_2}{q^2} + s\frac{pp_{12} - p_1 p_2}{p^2}\Big]. \tag{15}$$

(ii) *If F is in the form*

$$F^2=\beta^2 e^{\frac{2\gamma}{\beta}}, \tag{16}$$

then the flag curvature $\mathbf{K}$ *is given by*

$$\mathbf{K}=-\frac{e^{-\frac{\gamma}{\beta}-2c}}{p^3q^3}\Big\{p^3qc_{22}-p^2(pq_2-qp_2)c_2 \\ +p^2(qp_{22}-p_2q_2)-p^2(qq_{12}-q_1q_2)+q^2(pp_{12}-p_1p_2)\Big\}. \tag{17}$$

(iii) *If F is in the form*

$$F^2=(\beta^2+\gamma^2)e^{2r\cdot\arctan\left(\frac{\beta}{\gamma}\right)} \quad (r=\text{constant}), \tag{18}$$

then the flag curvature $\mathbf{K}$ *is given by*

$$\mathbf{K}=-\frac{e^{-2r\cdot\arctan\left(\frac{\beta}{\gamma}\right)-2c}}{(1+r^2)p^3q^3}\Big\{pq^3c_{11}+p^3qc_{22}+q^2(pq_1-qp_1)c_1 \\ +p^2(qp_2-pq_2)c_2+pq(pp_{22}+qq_{11})-q^2p_1q_1-p^2p_2q_2 \\ +r\big[p^2(qq_{12}-q_1q_2)-q^2(pp_{12}-p_1p_2)\big]\Big\}. \tag{19}$$

Proof. The computations for (15), (17) and (19) are direct, which are from (10), (11) and (12). We just give the main details.

First we prove (15). We get by (10) and (11) that

$$G^1=\frac{1}{2}\frac{p_1qs+(1-s)pq_1+pqc_1}{pqs}(y^1)^2,$$
$$G^2=\frac{1}{2}\frac{p_2qs+(1-s)pq_2+pqc_2}{pq(1-s)}(y^2)^2,$$
$$R^1{}_1=-\frac{1}{s}\Big[c_{12}+s\frac{pp_{12}-p_1p_2}{p^2}+(1-s)\frac{qq_{12}-q_1q_2}{q^2}\Big]y^1y^2,$$
$$R^2{}_2=-\frac{1}{1-s}\Big[c_{12}+s\frac{pp_{12}-p_1p_2}{p^2}+(1-s)\frac{qq_{12}-q_1q_2}{q^2}\Big]y^1y^2.$$

Then we get by (12) that

$$\mathbf{K}=\frac{R^1{}_1+R^2{}_2}{F^2},$$

which is reduced to (15).

Next we prove (17). We have

$$G^1 = \frac{p^2c_2 + pp_2 + qp_1 - pq_1}{2pq}(y^1)^2,$$

$$G^2 = \frac{p(qc_1 - pc_2 + q_1 - p_2)}{2q^2}(y^1)^2 + \frac{(pc_2 + p_2)}{q}y^1y^2 + \frac{(pq_2 - qp_2)}{2pq}(y^2)^2,$$

$$R^1{}_1 = -\frac{y^1y^2}{p^2q^2}\Big\{p^3qc_{22} - p^2(pq_2 - qp_2)c_2 + p^2(qp_{22} - p_2q_2) - p^2(qq_{12} - q_1q_2) + q^2(pp_{12} - p_1p_2)\Big\},$$

$$R^2{}_2 = -\frac{y^1(py^1 - qy^2)}{p^2q^3}\Big\{p^3qc_{22} - p^2(pq_2 - qp_2)c_2 + p^2(qp_{22} - p_2q_2) - p^2(qq_{12} - q_1q_2) + q^2(pp_{12} - p_1p_2)\Big\},$$

Then by (12) we get (17).

Finally we prove (19). We have

$$G^1 = \frac{pqc_1 - p^2rc_2 - (pq_1 - qp_1)r^2 - pp_2r + qp_1}{2(1+r^2)pq}(y^1)^2 + \frac{qrc_1 + pc_2 + q_1r + p_2}{(1+r^2)p}y^1y^2 - \frac{q^2c_1 - pqrc_2 - q(p_2r - q_1)}{2(1+r^2)p^2}(y^2)^2$$

$$G^2 = -\frac{pqrc_1 + p^2c_2 + p(q_1r + p_2)}{2(1+r^2)q^2}(y^1)^2 + \frac{qc_1 - prc_2 - p_2r + q_1}{(1+r^2)q}y^1y^2 + \frac{q^2rc_1 + pqc_2 - (qp_2 - pq_2)r^2 + qq_1r + pq_2}{2(1+r^2)pq}(y^2)^2,$$

$$R^1{}_1 = -\frac{(qy^2 - pry^1)y^2}{(1+r^2)p^3q^2}\Big\{pq^3c_{11} + p^3qc_{22} + q^2(pq_1 - qp_1)c_1 + p^2(qp_2 - pq_2)c_2 + pq(pp_{22} + qq_{11}) - q^2p_1q_1 - p^2p_2q_2 + r\big[p^2(qq_{12} - q_1q_2) - q^2(pp_{12} - p_1p_2)\big]\Big\},$$

$$R^2{}_2 = -\frac{(py^1 + qry^2)y^1}{(1+r^2)p^2q^3}\Big\{pq^3c_{11} + p^3qc_{22} + q^2(pq_1 - qp_1)c_1 + p^2(qp_2 - pq_2)c_2 + pq(pp_{22} + qq_{11}) - q^2p_1q_1 - p^2p_2q_2 + r\big[p^2(qq_{12} - q_1q_2) - q^2(pp_{12} - p_1p_2)\big]\Big\}.$$

Then by (12) we get (19). □

Example 4.2 ([2]). For a constant $\lambda > 0$, define a Finsler metric $F(x, y)$ by

$$F(x,y) = \frac{(y^2)^{1+\frac{1}{\lambda}}}{(y^1)^{\frac{1}{\lambda}}} e^{-\alpha_1 x^1 + (\lambda+1)\alpha_2 x^2 + \alpha_3 x^1 x^2}.$$

By (4), the main scalar is given by

$$I^2 = \frac{(\lambda+2)^2}{\lambda+1}.$$

By (15), the flag curvature is given by

$$\mathbf{K}(x,y) = \frac{\lambda^2}{1+\lambda}\alpha_3 \left\{\frac{y^1}{y^2}\right\}^{\frac{2+\lambda}{\lambda}} e^{2(\alpha_1 x^1 - (\lambda+1)\alpha_2 x^2 - \alpha_3 x^1 x^2)}.$$

Example 4.3 ([2]). Define a Finselr metric $F(x, y)$ by

$$F(x,y) = \sqrt{(y^1)^2 + (y^2)^2} e^{2\frac{\alpha_1^2+\alpha_2^2}{\alpha_1+\alpha_2}\left(\alpha_1 x^1 + \alpha_2 x^2\right) + \frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\arctan\left(y^1/y^2\right)}.$$

By (6), the main scalar is given by

$$I^2 = \frac{2(\alpha_1-\alpha_2)^2}{\alpha_1^2+\alpha_2^2}.$$

By (19), the flag curvature $\mathbf{K}(x, y) = 0$. So $F(x, y)$ is locally Minkowskian.

PROOF OF $(a_1) \Longrightarrow (a_2) \Longrightarrow (a_3)$ IN THEOREM 1.1. First $(a_2) \Longrightarrow (a_3)$ is obvious. We consider the proof of $(a_1) \Longrightarrow (a_2)$.

Let F be a two-dimensional Finsler metric with the main scalar $I =$ constant. Then F is given by (1), or (2), or (3) (in which we put $\beta = p(x)y^2, \gamma = q(x)y^2$). Consider the following conformal transformation

$$\widetilde{F} = e^{c(x)} F,$$

such that for some $c(x)$, $\widetilde{F}$ is of zero flag curvature. Note that, corresponding to (1), or (2), or (3), $\widetilde{F}$ is in the form of (14), or (16), or (18) respectively. Then by Lemma 4.1, the conformal factor $c(x)$ must satisfy the following three differential equations respectively:

$$c_{12} + (1-s)\frac{qq_{12} - q_1 q_2}{q^2} + s\frac{pp_{12} - p_1 p_2}{p^2} = 0, \tag{20}$$

$$\begin{aligned} &p^3 q c_{22} - p^2(pq_2 - qp_2)c_2 + p^2(qp_{22} - p_2 q_2) \\ &\qquad - p^2(qq_{12} - q_1 q_2) + q^2(pp_{12} - p_1 p_2) = 0, \end{aligned} \tag{21}$$

$$pq^3c_{11} + p^3qc_{22} + q^2(pq_1 - qp_1)c_1 + p^2(qp_2 - pq_2)c_2 + pq(pp_{22} + qq_{11}) \\ - q^2p_1q_1 - p^2p_2q_2 + r\big[p^2(qq_{12} - q_1q_2) - q^2(pp_{12} - p_1p_2)\big] = 0. \tag{22}$$

It is easily seen that there is a local solution $c(x)$ satisfying (20) or (21). Then we consider (22). This equation is a 2-order PDE which is linear, elliptic and in which, the coefficient of c is zero. Now according to the theory of PDEs, there exists a local solution $c(x)$ satisfying (22). Thus for such a solution $c(x)$ of (20), or (21), or (22), the corresponding Finsler metric $\widetilde{F}$ in (14), or (16), or (18), is of zero flag curvature. On the other hand, it is easily seen that $\widetilde{F}$ is also a Berwald metric by Lemma 3.1 and Lemma 3.2. So $\widetilde{F}$ is a locally Minkowski metric. Thus the Finsler metric F is locally conformally flat. □

5. Two-dimensional Finsler spaces with isotropic main scalar

In this section, we study two-dimensional Finsler spaces with isotropic main scalar and prove in Theorem 1.1 that $(a_3) \Longrightarrow (a_1)$. After this is proved, the proof of Theorem 1.1 is completed.

Since the metric in (2) has constant main scalar $I^2 = 4$, we only consider in the following the metrics in (1) and (3).

Lemma 5.1. *Let*

$$F^2(x, y) = \beta^{2s}\gamma^{2(1-s)}, \tag{23}$$

where

$$\beta = p(x)y^1, \quad \gamma = q(x)y^2, \quad s = s(x)\ (s \neq 0, s \neq 1).$$

Then the flag curvature $K(x, y)$ is given by

$$\mathbf{K}(x, y) = \frac{\beta^{-2s}\gamma^{2s-2}}{\big[2s(s-1)pq\big]^2}\Big\{\big[A_{11}(y^1)^2 + A_{12}y^1y^2 + A_{22}(y^2)^2\big]ln\Big|\frac{y^2}{y^1}\Big| \\ + B_{11}(y^1)^2 + B_{12}y^1y^2 + B_{22}(y^2)^2\Big\}, \tag{24}$$

where

$$A_{11} = 2p^2q^2(s-1)(s_1)^2, \quad A_{22} = 2p^2q^2s(s_2)^2,$$

$$A_{12} = -4p^2q^2\big[s_1s_2 - s(2s_1s_2 + s_{12}) + s^2s_{12}\big],$$

$$B_{11} = pq\Big\{-pq\big[1 + 2s + 2(s-1)ln\Big|\frac{p}{q}\Big|\big](s_1)^2 \\ + 2(s-1)\big[pqss_{11} - pq_1s_1 + (q_1p - qp_1)ss_1\big]\Big\},$$

$$B_{12} = 4pq\Big\{q\big[-ps_2 - sp_2 + p\cdot ln\Big|\frac{p}{q}\Big|(1-2s)s_2\big]s_1 - pq_1(s-1)s_2$$
$$+ pq\ln\Big|\frac{p}{q}\Big|s(s-1)s_{12}\Big\}$$
$$+ 4s(s-1)\big[sq^2(pp_{12} - p_1p_2) - (s-1)p^2(qq_{12} - q_1q_2)\big],$$
$$B_{22} = p^2q^2\Big(2s - 3 - 2s\ln\Big|\frac{p}{q}\Big|\Big)(s_2)^2 + 2pqs\big[(s-1)pq_2 - sqp_2\big]s_2$$
$$- 2p^2q^2s(s-1)s_{22}.$$

Proof. This proof is a direct computation, similar to that in Lemma 4.1. We only give the results of G^1 and G^2 as follows:

$$G^1 = \frac{pqs_1\big[ln|\frac{\beta}{\gamma}| - 1\big] + s(qp_1 - pq_1) + pq_1}{2pqs}(y^1)^2 - \frac{s_2}{2s}y^1y^2,$$
$$G^2 = -\frac{s_1}{2(s-1)}y^1y^2 + \frac{pqs_2\big[ln|\frac{\beta}{\gamma}| + 1\big] + s(qp_2 - pq_2) + pq_2}{2pq(1-s)}(y^2)^2. \qquad \square$$

Lemma 5.2. *Let*

$$F^2(x,y) = (\beta^2 + \gamma^2)e^{2r(x)\arctan\left(\frac{\beta}{\gamma}\right)}, \tag{25}$$

where

$$\beta = p(x)y^1, \quad \gamma = q(x)y^2.$$

Then the flag curvature $\mathbf{K}(x,y)$ *is given by*

$$\mathbf{K}(x,y) = \frac{e^{-2r\cdot\arctan\left(\frac{\beta}{\gamma}\right)}}{(1+r^2)^2p^3q^3(\beta^2+\gamma^2)}\Big\{\big[C_{11}(y^1)^2 + C_{12}y^1y^2 + C_{22}(y^2)^2\big]\arctan\Big(\frac{\beta}{\gamma}\Big)$$
$$+ D_{11}(y^1)^2 + D_{12}y^1y^2 + D_{22}(y^2)^2\Big\},$$

where

$$C_{11} = p^2\Big\{-pq(1+r^2)\big(q^2r_{11} + p^2r_{22}\big) + pqr\big[3q^2(r_1)^2 + p^2(r_2)^2\big]$$
$$+ 2p^2q^2r_1r_2 + (1+r^2)\big[q^2(p_1q - pq_1)r_1 - p^2(qp_2 - pq_2)r_2\big]\Big\},$$
$$C_{12} = 2p^2q^2\Big(p^2(r_2)^2 + 2pqrr_1r_2 - q^2(r_1)^2\Big),$$
$$C_{22} = q^2\Big\{-pq(1+r^2)\big(q^2r_{11} + p^2r_{22}\big) + pqr\big[q^2(r_1)^2 + 3p^2(r_2)^2\big]$$
$$- 2p^2q^2r_1r_2 + (1+r^2)\big[q^2(p_1q - pq_1)r_1 - p^2(qp_2 - pq_2)r_2\big]\Big\},$$

and

$$\begin{aligned} D_{11} &= -p^4q^2(1+r^2)r_{12} + 2p^3q^3(r_1)^2 + p^4q^2rr_1r_2 \\ &\quad + p^3q\big[3q(rq_1+p_2)r_1 + p(rp_2-q_1)r_2\big] \\ &\quad - p^3q(1+r^2)\big(pp_{22}+qq_{11}\big) + p^3qr(1+r^2)\big(qp_{12}-pq_{12}\big) \\ &\quad + (1+r^2)p^2(rp^2q_1q_2 - rq^2p_1p_2 + p^2p_2q_2 + q^2p_1q_1), \\ D_{12} &= p^2q^2(1+r^2)(q^2r_{11}-p^2r_{22}) - p^2q^4r(r_1)^2 \\ &\quad + pq^3\big[4p^2r_2 + (r^2pq_1 - r^2qp_1 + 2rpp_2 - pq_1 - p_1q)\big]r_1 \\ &\quad + p^4q^2r(r_2)^2 + p^3q(r^2pq_2 - r^2qp_2 + pq_2 + qp_2 + 2rqq_1)r_2, \\ D_{22} &= p^2q^4(1+r^2)r_{12} + 2p^3q^3(r_2)^2 - p^2q^4rr_1r_2, \\ &\quad + pq^3\big[q(rq_1+p_2)r_1 + 3p(rp_2-q_1)r_2\big] \\ &\quad - pq^3(1+r^2)(pp_{22}+qq_{11}) + pq^3r(1+r^2)(qp_{12}-pq_{12}) \\ &\quad + (1+r^2)q^2(rp^2q_1q_2 - rq^2p_1p_2 + p^2p_2q_2 + q^2p_1q_1). \end{aligned}$$

Proof. This proof is a direct computation, similar to that in Lemma 4.1. We only give the results of G^1 and G^2 as follows:

$$\begin{aligned} G^1 &= \frac{1}{2p^2q(1+r^2)}\big\{[a_{11}(y^1)^2 + a_{12}y^1y^2 + a_{22}(y^2)^2]\arctan\left(\frac{\beta}{\gamma}\right) \\ &\quad + b_{11}(y^1)^2 + b_{12}y^1y^2 + b_{22}(y^2)^2\big\}, \\ G^2 &= \frac{1}{2(1+r^2)pq^2}\big\{[c_{11}(y^1)^2 + c_{12}y^1y^2 + c_{22}(y^2)^2]\arctan\left(\frac{\beta}{\gamma}\right) \\ &\quad + d_{11}(y^1)^2 + d_{12}y^1y^2 + d_{22}(y^2)^2\big\}, \end{aligned}$$

where

$$a_{11} = p^2(qr_1 - prr_2), \quad a_{12} = 2pq(pr_2 + qrr_1), \quad a_{22} = -q^2(qr_1 - prr_2),$$

$$b_{11} = p[(qp_1 - pq_1)r^2 - p(p_2 + qr_1)r + qp_1], \quad b_{12} = pq(2p_2 + 2q_1r - prr_2 + qr_1),$$

$$b_{22} = -q^2(q_1 - p_2r - pr_2), \quad c_{11} = -p^2(qrr_1 + pr_2), \quad c_{12} = 2pq(qr_1 - prr_2),$$

$$c_{22} = q^2(pr_2 + qrr_1), \quad d_{12} = pq(2q_1 - 2p_2r - pr_2 - qrr_1),$$

$$d_{11} = -p^2(q_1r + p_2 + qr_1), \quad d_{22} = q[(pq_2 - qp_2)r^2 + q(q_1 - pr_2)r + pq_2]. \quad \square$$

Lemma 5.3. *Let F be a two-dimensional Finsler metric with isotropic main scalar $I = I(x)$. If the flag curvature $\mathbf{K}(x,y) = 0$, then $I(x) = $ constant.*

Proof. By assumption, the flag curvature $\mathbf{K}(x,y)=0$.

If F is given by (23), then by (24) we have

$$A_{11}=0,\ A_{12}=0,\ A_{22}=0,\ B_{11}=0,\ B_{12}=0,\ B_{22}=0.$$

Now it is easily seen that $A_{11}=0$ implies that $s_1=0$ and $A_{22}=0$ implies that $s_2=0$. Thus $s(x)=$ constant and the main scalar of the metric in (23) is a constant by (4).

Next we assume that F is given by (25), then by (5) we have

$$C_{11}=0, C_{12}=0, C_{22}=0, D_{11}=0, D_{12}=0, D_{22}=0.$$

Now it is easily seen that

$$\begin{aligned}0=q^2C_{11}-p^2C_{22}&=p^2q^2\Big\{2pqr\big[q^2(r_1)^2-p^2(r_2)^2\big]+4p^2q^2r_1r_2\Big\}\\&=2p^3q^3\Big\{r\big[q^2(r_1)^2-p^2(r_2)^2\big]+2pqr_1r_2\Big\}.\end{aligned}$$

Then we have

$$r\big[q^2(r_1)^2-p^2(r_2)^2\big]+2pqr_1r_2=0. \tag{26}$$

Substitute (26) into C_{12}, then we have

$$2p^2q^2(1+r^2)\big[p^2(r_2)^2-q^2(r_1)^2\big]=0.$$

Therefore,

$$p^2(r_2)^2=q^2(r_1)^2. \tag{27}$$

Substituting (27) into (26), we get $r_1=0$ or $r_2=0$. Then by (27) we obtain $r_1=r_2=0$. This fact means that $r(x)=$ constant and the main scalar of the metric in (25) is a constant by (6). □

Now by Lemma 3.1 and Lemma 5.3 we easily get

Corollary 5.4. *Let F be a two-dimensional Finsler metric with isotropic main scalar $I=I(x)$. Then F is locally Minkowskian if and only if the flag curvature $\mathbf{K}(x,y)=0$.*

Proof of $(a_3)\Longrightarrow(a_1)$ in Theorem 1.1. Suppose that the Finsler metric F is locally conformal to a Finsler metric of zero flag curvature. Then there is a $c(x)$ such that the Finsler metric $\widetilde{F}(x,y):=e^{c(x)}F(x,y)$ is of zero flag curvature. Then we conclude from Lemma 5.3 and Lemma 3.2 that the main scalar $I(x)$ of the Finsler metric F is a constant. □

References

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, The theory of sprays and Finsler spaces with applications in physics and biology, *Kluwer Academic Publishers*, 1993.

[2] P. Antonelli, B. Han and J. Modayil, New results on 2-dimensional constant sprays with an application to heterochrony, *World Scientific Press*, 1998.

[3] S. Bácsó and X. Cheng, Finsler conformal transformations and the curvature invariances, *Publ. Math. Debrecen* **70** (2007), 221–231.

[4] S. Bácsó and M. Matsumoto, Reduction theorems of certain Landsberg spaces to Berwald spaces, *Publ. Math. Debrecen* **48** (1996), 357–366.

[5] L. Berwald, On Finsler and Cartan geometries III, Two-dimensional Finsler spaces with rectilinear extremals, *Ann. of Math.* **42** (1941), 84–112.

[6] X. Cheng and Z. Shen, On Douglas metrics, *Publ. Math. Debrecen* **66** (2005), 503–512.

[7] H. Izumi, Conformal transformations of Finsler spaces. II. An h-conformally flat Finsler space, *Tensor, N.S.* **34** (1980), 337–358.

[8] B. Li, Y. Shen and Z. Shen, On a class of Douglas metrics, *Studia Sci. Math. Hungarica* **46(3)** (2009), 355–365.

[9] M. Matsumoto,, A special class of locally Minkowski spaces with an (α, β)-metric and conformally flat Kropina spaces, *Tensor, N.S.* **50** (1991), 202–207.

[10] M. Matsumoto, On h-recurrent Finsler connections and conformally Minkowski spaces, *Tensor, N.S.* **49** (1990), 18–23.

[11] Z. Shen, Differential geometry of spray and Finsler spaces, *Kluwer Academic Publishers, Dordrecht*, 2001.

[12] Z. Shen, On a class of Landsberg metrics in Finsler geometry, *Canadian J. of Math.* **61(6)** (2009), 1357–1374.

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