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Finite groups with hall Schmidt subgroups

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Abstract. A Schmidt group is a non-nilpotent group whose every proper subgroup is nilpotent. We study the properties of a non-nilpotent group G in which every Schmidt subgroup is a Hall subgroup of G.

1. Introduction

A non-nilpotent finite group whose proper subgroups are all nilpotent is called a Schmidt group. O. Yu. Schmidt pioneered the study of such groups [9]. In a series of Chunihin's papers, Schmidt groups were applied in order to find criterions of nilpotency and generalized nilpotency, and also to find non-nilpotent subgroups, (see [2]). A whole paragraph from Huppert's monography is dedicated to Schmidt groups, (see [4], III.5). Review of the results on Schmidt groups and perspectives of its application in group theory as of 2001 are provided in paper [7].

Let S be a Schmidt group. Then the following properties hold: S contains a normal Sylow subgroup N such that S/N is a primary cyclic subgroup; the derived subgroup of S is nilpotent; the derived length of S does not exceed 3; non-normal Sylow subgroup Q of S is cyclic and every maximal subgroup of Q is contained in $Z(S)$; every normal primary subgroup of S other than a Sylow subgroup of S is contained in $Z(S)$.

In this paper the properties of a non-nilpotent group G in which every Schmidt subgroup is a Hall subgroup of G are studied. In particular, for such groups a number of properties of Schmidt groups are applicable. We prove the following theorem.

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Theorem. *Let G be a finite non-nilpotent group in which every Schmidt subgroup is a Hall subgroup of G. Then the following statements hold:*

1) *if P is a non-normal Sylow p-subgroup of G, then P is cyclic and every maximal subgroup of P is contained in $Z(G)$;*
2) *if P is a normal Sylow p-subgroup of G and G is not p-decomposable, then P is either minimal normal in G or non-abelian, $Z(P) = P' = \Phi(P)$, and $P/\Phi(P)$ is minimal normal in $G/\Phi(P)$;*
3) *if P_1 is a normal p-subgroup of G, P_1 is not a Sylow p-subgroup of G, and G is not p-decomposable, then P_1 is contained in $Z(G)$;*
4) *if $Z(G) = 1$, then G has a normal abelian Hall subgroup A in which every Sylow subgroup is minimal normal in G, G/A is cyclic and $|G/A|$ is squarefree.*

Corollary. *Let G be a finite non-nilpotent group in which every Schmidt subgroup is a Hall subgroup of G. Then G contains a nilpotent Hall subgroup H such that G/H is cyclic. In particular, $G/\Phi(G)$ is metabelian.*

2. Preliminary results

Throughout this article, all groups are finite. The terminology and notation are standart, as in [4] and [8]. Recall that a p-closed group is a group with a normal Sylow p-subgroup and a p-nilpotent group is a group of order $p^a m$, where p does not divide m, with a normal subgroup of order m. A group is called p-decomposable if it is p-closed and p-nilpotent simultaneously. A group whose order is divisible by a prime p is a pd-group. We denote by $Z(G)$, G', $\Phi(G)$, $F(G)$, G_p the center, the derived subgroup, the Frattini subgroup, the Fitting subgroup, and a Sylow p-subgroup of G respectively. We use $G = [A]B$ to denote the semidirect product of A and B, where A is a normal subgroup of G. The set of prime divisors of the order of G is denoted by $\pi(G)$. As usual, A_n and S_n are the alternating and the symmetric groups of degree n respectively. We use E_{p^n} to denote an elementary abelian group of order p^n and Z_m to denote a cyclic group of order m. Let G be a group of order $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. We say that G has a Sylow tower if there exists a series $1 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{k-1} \subset G_k = G$ of normal subgroups of G such that for each $i = 1, 2, \dots, k$, G_i/G_{i-1} is isomorphic to a Sylow subgroup of G. Recall that a positive integer n is said to be squarefree if n is not divisible by the square of any prime number. A group is called metabelian if it contains a normal abelian subgroup such that the corresponding quotent group is also abelian. If H is a subgroup of a group G, then $\mathrm{Core}_G H = \bigcap_{x \in G} x^{-1}Hx$ is called the core of H in G. If a group G has a maximal subgroup M such that $\mathrm{Core}_G M = 1$, then G is called a primitive group, and M is called a primitivator of G.

We use $\mathfrak{H}$ to denote the class of all groups G such that each Schmidt subgroup S of G is a Hall subgroup of G. It is clear that all nilpotent groups, all Schmidt groups, and all squarefree groups belong to $\mathfrak{H}$. If T is biprimary non-nilpotent and $T \in \mathfrak{H}$, then T is a Schmidt group. Below the other examples of groups of this class are given.

Example 1. Let $G = A \times B$, $(|A|, |B|) = 1$, $A \in \mathfrak{H}$, $B \in \mathfrak{H}$. Then, evidently, $G \in \mathfrak{H}$.

Example 2. Let P be extraspecial of order 409^3. It is clear that $\Phi(P) = Z(P) = P'$ has prime order 409 and $P/\Phi(P)$ is elementary abelian of order 409^2. The automorphism group of $P/\Phi(P)$ is $GL(2, 409)$. By Theorem II.7.3 [4], $GL(2, 409)$ has a cyclic subgroup Z_{210} of order $5 \cdot 41$. Since Z_{210} acts irreducibly on $P/\Phi(P)$, there is $T = [P]Z_{210}$ such that $\Phi(P) = Z(T)$. The group T possesses exactly three maximal subgroups: $[P]Z_5$ is a Schmidt group; $[P]Z_{41}$ is a Schmidt group; $\Phi(P) \times Z_{210}$ is a nilpotent subgroup. Therefore $\pi(T) = \{p, q, r\}$, where p, q, r are distinct primes and every Schmidt subgroup of T is a Hall subgroup of T.

Example 3. Let $G = [(E_4 \times E_{25} \times E_7 \times E_{121} \times E_{169} \times \cdots)]Z_3$, where $[E_4]Z_3$, $[E_{25}]Z_3$, $[E_7]Z_3$, $[E_{121}]Z_3$, $[E_{169}]Z_3, \ldots$ are Schmidt groups in which all proper subgroups are primary. Let K be a proper subgroup of G. If 3 does not divide $|K|$, then K is nilpotent. Now suppose that 3 divides $|K|$. Since G is p-closed for any $p \neq 3$, it follows that K is p-closed too and there exists $[K_p]Z_3$. By Hall's theorem, $[K_p]Z_3 \subseteq [G_p]Z_3$. However all proper subgroups of $[G_p]Z_3$ are primary. Thus $[K_p]Z_3 = [G_p]Z_3$ is either Hall in G or $K_p = 1$. Since p is an arbitrary prime number, $p \neq 3$, we see that K is a Hall subgroup of G and $G \in \mathfrak{H}$.

Lemma 1 ([7], [9])**.** *Let S be a Schmidt group. Then the following statements hold:*

1) *$S = [P]\langle y \rangle$, where P is a normal Sylow p-subgroup, $\langle y \rangle$ is a non-normal cyclic Sylow q-subgroup, p and q are distinct primes, $y^q \in Z(S)$;*
2) *$|P/P'| = p^m$, where m is the order of p modulo q;*
3) *if P is abelian, then P is an elementary abelian p-group of order p^m and P is a minimal normal subgroup of S;*
4) *if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^m$;*

5) *if* P_1 *is a non-trivial normal* p*-subgroup of* S *such that* $P_1 \neq P$*, then* P *is non-abelian and* $P_1 \subseteq Z(P)$*;*

6) $Z(S) = \Phi(S) = \Phi(P) \times \langle y^q \rangle$*;* $S' = P$*,* $P' = (S')' = \Phi(P)$*;*

7) *if* N *is a proper normal subgroup of* S*, then* N *does not contain* $\langle y \rangle$ *and either* $P \subseteq N$ *or* $N \subseteq \Phi(S)$*.* □

We denote by $S_{\langle p,q \rangle}$-group a Schmidt group with a normal Sylow p-subgroup and a cyclic Sylow q-subgroup.

Lemma 2 ([6], Lemma 2)**.** *If* K *and* D *are subgroups of* G *such that* D *is normal in* K *and* K/D *is an* $S_{\langle p,q \rangle}$*-subgroup, then each minimal supplement* L *to* D *in* K *has the following properties:*

1) L *is a* p*-closed* $\{p, q\}$*-subgroup;*

2) *all proper normal subgroups of* L *are nilpotent;*

3) L *contains an* $S_{\langle p,q \rangle}$*-subgroup* $[P]Q$ *such that* D *does not contain* Q *and* $L = ([P]Q)^L = Q^L$*.* □

Lemma 3. *If* $G \in \mathfrak{H}$*, then every subgroup of* G *and every quotient of* G *belongs to* $\mathfrak{H}$*.*

Proof. Let $V \leq G \in \mathfrak{H}$. If V is non-nilpotent, then it contains a Schmidt subgroup S. Since $G \in \mathfrak{H}$, we can easily observe that S is a Hall subgroup of G. It is clear that S is a Hall subgroup of V, hence $V \in \mathfrak{H}$.

Let D be a normal subgroup of G and K/D is a Schmidt subgroup of G/D. By the previous lemma, minimal supplement L to D in K has an $S_{\langle p,q \rangle}$-subgroup $[P]Q$ such that D does not include Q. By Lemma 1, $[P]QD/D$ is a Schmidt subgroup, hence $[P]QD/D = K/D$. Since $G \in \mathfrak{H}$, it follows that $[P]Q$ is a Hall subgroup of G. Therefore $[P]QD/D = K/D$ is a Hall subgroup of G/D and $G/D \in \mathfrak{H}$. □

Remark 1. The class $\mathfrak{H}$ is not closed under direct products. For example, $S_3 \in \mathfrak{H}$, $Z_2 \in \mathfrak{H}$ but $S_3 \times Z_2 \notin \mathfrak{H}$. This shows that $\mathfrak{H}$ is neither a formation nor a Fitting class.

Lemma 4. 1) *If* G *is not* p*-nilpotent, then* G *has a* p*-closed Schmidt* pd*-subgroup.*

2) *If* G *is not* 2*-closed, then* G *has a* 2*-nilpotent Schmidt subgroup of even order.*

3) *If a* p*-solvable group* G *is not* p*-closed, then* G *has a* p*-nilpotent Schmidt* pd*-subgroup.*

Proof. 1. The proof of this part follows directly from the Frobenius theorem (see, [4], Theorem IV.5.4).

2. In [1] there is a proof based on Suzuki's theorem on simple groups with independent Sylow 2-subgroups. Let us show another proof. By induction, all proper subgroups of G are 2-closed. It follows that G is not biprimary, (see part 1 of the lemma). If G is solvable, then all biprimary Hall subgroups of G are 2-closed and G is also 2-closed, a contradiction. Thus G is not solvable. It is clear that $G/\Phi(G)$ is a simple group. Let X be the conjugacy class of involutions of $G/\Phi(G)$. By Theorem IX.7.8 [5], there exists involutions $x, y \in X$ such that $\langle x, y\rangle$ is not 2-group. It is well known that $\langle x, y\rangle$ is the dihedral group of order $2|xy|$, (see [8], Theorem 2.49). It is not 2-closed, a contradiction.

3. By Theorem 5.3.13 [10], G is a $D_{\{p,q\}}$-group for any $q \in \pi(G)$. Suppose that G is not p-closed. Then G contains a Hall $\{p, q\}$-subgroup H such that H is not p-closed for some prime $q \in \pi(G)$. It is clear that H is not q-nilpotent. By part 1 of the lemma, H has a p-nilpotent Schmidt pd-subgroup. □

For any odd prime p assertion 2 of Lemma 4 is false. If $p = 3$, then the counterexamples are $SL(2, 2^n)$ for any odd n and $PSL(2, p)$ for $p \geq 5$.

Lemma 5. *If $G \in \mathfrak{H}$, then G possesses a Sylow tower.*

Proof. First of all, we prove that if $G \in \mathfrak{H}$ and p is the smallest prime dividing $|G|$, then G is either p-closed or p-nilpotent. Let $p = 2$. If G is not 2-closed, then, by Lemma 4 (2), G has a 2-nilpotent Schmidt subgroup S of even order. Any Sylow 2-subgroup of S is cyclic. Since $G \in \mathfrak{H}$, we deduce that S is a Hall subgroup of G and a Sylow 2-subgroup of G is cyclic. Thus G is 2-nilpotent by Theorem IV.2.8 [4]. Now suppose that $p > 2$. Then G is solvable. If G is not p-closed, then, by Lemma 3(3), G has a p-nilpotent Schmidt pd-subgroup T. A Sylow p-subgroup P of T is cyclic. Since $G \in \mathfrak{H}$, it follows that T is a Hall subgroup of G and P is a Sylow p-subgroup of G. Thus G is p-nilpotent by Theorem IV.2.8 [4].

Therefore if $G \in \mathfrak{H}$ and p is the smallest prime dividing $|G|$, then G is either p-closed or p-nilpotent. We use induction on $|G|$. Prove that G possesses a Sylow tower. Let p be the smallest prime dividing $|G|$. If a Sylow p-subgroup P is normal in G, then, by Lemma 3, $G/P \in \mathfrak{H}$ and, by induction, G/P possesses a Sylow tower. Thus G possesses a Sylow tower. If G is p-nilpotent, then G contains a normal subgroup K such that G/K is isomorphic to a Sylow p-subgroup of G. By Lemma 3, $K \in \mathfrak{H}$ and, by induction, K possesses a Sylow tower. Therefore G possesses a Sylow tower. □

Lemma 6. *Let $G \in \mathfrak{H}$ and p, q are different prime divisors of $|G|$. Then any Hall $\{p, q\}$-subgroup of G is either nilpotent or Schmidt group.*

Proof. By Lemma 5, G is solvable, so G has a Hall $\{p, q\}$-subgroup K. Assume that K is non-nilpotent. Then K contains a Schmidt subgroup S. Since $G \in \mathfrak{H}$, it implies that S must be a Hall subgroup of G. Therefore $S = K$. □

Lemma 7. *Let $n \geq 2$ be a positive integer and p be a prime number. Denote by π the set of prime numbers q such that q divides $p^n - 1$, but q does not divide $p^{n_1} - 1$ for all $1 \leq n_1 < n$. Then $GL(n, p)$ has a cyclic Hall π-subgroup.*

Proof. The group $G = GL(n, p)$ has order

$$p^{n(n-1)/2}(p^n - 1)(p^{n-1} - 1) \ldots (p^2 - 1)(p - 1).$$

By Theorem II.7.3 [4], G contains a cyclic subgroup T of order $p^n - 1$. Denote by T_π a Hall π-subgroup of T. Since q does not divide $p^{n_1} - 1$ for all $q \in \pi$ and all $1 \leq n_1 < n$, it follows that T_π is a Hall π-subgroup of G. □

3. Main results

Theorem. *Let G be a finite non-nilpotent group in which every Schmidt subgroup is a Hall subgroup of G. Then the following statements hold:*

1) *if P is a non-normal Sylow p-subgroup of G, then P is cyclic and every maximal subgroup of P is contained in $Z(G)$;*
2) *if P is a normal Sylow p-subgroup of G and G is not p-decomposable, then P is either minimal normal in G or non-abelian, $Z(P) = P' = \Phi(P)$, and $P/\Phi(P)$ is minimal normal in $G/\Phi(P)$;*
3) *if P_1 is a normal p-subgroup of G, P_1 is not a Sylow p-subgroup of G, and G is not p-decomposable, then P_1 is contained in $Z(G)$;*
4) *if $Z(G) = 1$, then G has a normal abelian Hall subgroup A in which every Sylow subgroup is minimal normal in G, G/A is cyclic and $|G/A|$ is squarefree.*

Proof. 1. Let $G \in \mathfrak{H}$ and $p \in \pi(G)$. Assume that G has a non-normal Sylow p-subgroup P. By Lemma 5, G is solvable, hence G contains a Hall $\{p, q\}$-subgroup for any $q \in \pi(G) \setminus \{p\}$ by Theorem 5.3.13 [10]. Since P is non-normal in G, it follows that G contains a not p-closed Hall $\{p, q\}$-subgroup K for some $q \in \pi(G) \setminus \{p\}$. By Lemma 4, K has a q-closed Schmidt subgroup S. Under the condition of $G \in \mathfrak{H}$, S is the same as K. By the properties of Schmidt groups (see Lemma 1(1)), every Sylow p-subgroup of K is cyclic. Since K is a Hall subgroup of G, we see that a Sylow p-subgroup of K is a Sylow subgroup of G. Thus P is cyclic.

Let P_1 be a maximal subgroup of P. If $P_1 = 1$, then $P_1 \subseteq Z(G)$. Assume that $P_1 \neq 1$. It is clear that G has a Hall $\{p, q\}$-subgroup PQ for any prime $q \in \pi(G) \setminus \{p\}$, where Q is some Sylow q-subgroup of G. If PQ is nilpotent, then $Q \subseteq C_G(P_1)$. If PQ is non-nilpotent, then PQ is a Schmidt group by Lemma 6. If PQ is p-closed, then P has a prime order by Lemma 1(3), a contradiction. Hence PQ is q-closed and $P_1 \subseteq Z(PQ)$ by Lemma 1(1), i.e. $Q \subseteq C_G(P_1)$. Thus $C_G(P_1)$ contains a Sylow q-subgroup for every $q \in \pi(G) \setminus \{p\}$. Since $P \subseteq C_G(P_1)$, we have $C_G(P_1) = G$ and $P_1 \subseteq Z(G)$.

2. Let Sylow p-subgroup P be a normal subgroup of G. Suppose that P is not a minimal normal subgroup of G. In particular, $|P| > p$. By Schur–Zassenhaus theorem, G has a Hall p'-subgroup H. By the hypothesis of the theorem, G is not p-decomposable. Hence H has a Sylow subgroup Q such that $[P]Q$ is non-nilpotent. By Lemma 6, $[P]Q$ is a Schmidt subgroup. By our assumption, P is not minimal normal in G, it follows that P is not minimal normal in $[P]Q$. By the properties of Schmidt groups (see Lemma 1(3)), P is non-abelian and $Z(P) = P' = \Phi(P)$. Since $[P/\Phi(P)](Q\Phi(P)/\Phi(P))$ is a Schmidt group, $P/\Phi(P)$ is its minimal normal subgroup. We see that $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. The statement 2 is proved.

3. We denote by G_p a Sylow p-subgroup of G. Assume that $Z(G)$ does not contain P_1. Then $|P_1| \geq p$, $|G_p| \geq p^2$, and G_p is normal in G by claim 1 of the theorem. Let G_q be a Sylow q-subgroup of G, $q \in \pi(G) \setminus \{p\}$. By Lemma 6, the product G_pG_q either nilpotent or a Schmidt group. Suppose G_pG_q is nilpotent for all $q \in \pi(G) \setminus \{p\}$. In this case, $G = G_p \times G_{p'}$, a contradiction. Thus our assumption is false and there exists a prime $r \in \pi(G) \setminus \{p\}$ such that G_pG_r is non-nilpotent. It follows that G_pG_r is a p-closed Schmidt group and P_1 is its normal p-subgroup. By the properties of Schmidt groups (see Lemma 1(5)), $P_1 \subseteq Z(G_pG_r)$. Thus, $P_1 \subseteq Z(G_p)$ and $G_r \subseteq C_G(P_1)$ for all $r \in \pi(G) \setminus \{p\}$ such that G_pG_r is not nilpotent. If G_pG_r is nilpotent, then $G_q \subseteq C_G(P_1)$. Therefore $P_1 \subseteq Z(G)$.

4. We denote by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{E}$ the classes of all abelian, all nilpotent, and all finite groups respectively. We define $\mathfrak{N} \circ \mathfrak{A} = \{G \in \mathfrak{E} \mid G^{\mathfrak{A}} \in \mathfrak{N}\}$ and call $\mathfrak{N} \circ \mathfrak{A}$ the product of classes $\mathfrak{N}$ and $\mathfrak{A}$, where $G^{\mathfrak{A}}$ denotes $\mathfrak{A}$-residual of G, i.e. the smallest normal subgroup of G quotient by which belogs to $\mathfrak{A}$. For the other definition and terminology, the reader is referred to Doerk, Hawkes (1992), Huppert (1967)

and Shemetkov (1978). It is clear that $G^{\mathfrak{A}} = G'$ is the derived subgroup of G. Hence $\mathfrak{N} \circ \mathfrak{A}$ consists of all groups G whose the derived groups are nilpotent. The class $\mathfrak{N} \circ \mathfrak{A}$ is a saturated formation. Now, by induction on $|G|$, we prove that $\mathfrak{H} \subseteq \mathfrak{N} \circ \mathfrak{A}$. Suppose the assertion is false. Let G be a counterexample of minimal order and $G \in \mathfrak{H} \setminus \mathfrak{N} \circ \mathfrak{A}$. By Lemma 5, G is solvable and, by Lemma 3, $G/N \in \mathfrak{H}$ for every normal subgroup $N \neq 1$ of G. By induction, $G/N \in \mathfrak{N} \circ \mathfrak{A}$. Since $\mathfrak{N} \circ \mathfrak{A}$ is a saturated formation, it follows that G is primitive (see ([8], p. 143). By Theorem 4.42 [8], $F = F(G) = C_G(F) \simeq E_{p^n}$ is a minimal normal subgroup of G and, by the above claim 3 of the theorem, F is a Sylow subgroup of G.

If $n = 1$, then G/F is isomorphic to a subgroup of the automorphism group of F, where $|F| = p$. Thus $G \in \mathfrak{N} \circ \mathfrak{A}$. Next, we assume that $n \geq 2$. Since $[F]G_q$ is a Hall non-nilpotent subgroup of G, we have, by Lemma 6, that $[F]G_q$ is a Schmidt subgroup for every $q \in \pi = \pi(G/F)$. By Lemma 1(2), q divides $p^n - 1$, but q does not divide $p^{n_1} - 1$ for all $1 \leq n_1 < n$. The quotient group G/F is isomorphic to a subgroup K of $GL(n,p)$, K has a cyclic Hall π-subgroup T by Lemma 7. By Theorem 5.3.2 [10], G/F is contained in some subgroup T^x, $x \in GL(n,p)$. Thus G/F is cyclic and $G \in \mathfrak{N} \circ \mathfrak{A}$.

Let $G \in \mathfrak{H}$ and $Z(G) = 1$. Then G is not p-decomposable for any $p \in \pi(G)$. The assertion (3) implies that every minimal normal subgroup of G is a Sylow subgroup of G. So $F(G) = A$ is an abelian Hall subgroup of G in which every Sylow subgroup is minimal normal in G. Let B be a complement to A in G. The assertion 1 implies that $|B|$ is squarefree. Since $G \in \mathfrak{N} \circ \mathfrak{A}$, it follows that B is abelian. Therefore B is cyclic. □

Corollary. *Let G be a finite non-nilpotent group in which every Schmidt subgroup is a Hall subgroup of G. Then G contains a normal nilpotent Hall subgroup H such that G/H is cyclic. In particular, $G/\Phi(G)$ is metabelian.*

Proof. If $Z(G) = 1$, then the claim of the corollary is the same as assertion 4 of the theorem. Let $Z(G) \neq 1$. Denote by N a subgroup of prime order p, $N \subseteq Z(G)$. By induction, we have $\overline{G} = [A/N](B/N)$, where A/N is a nilpotent Hall subgroup of G/N and B/N is cyclic. Since $N \subseteq Z(G)$, we see that A and B are nilpotent, (see [8], Lemma 3.15). If A is a Hall subgroup of G, then, by Schur–Zassenhaus theorem, $B = N \times B_1$ and $G = [A]B_1$, where A is a nilpotent Hall subgroup of G and B_1 is a cyclic subgroup. In this case, the corollary is proved. Now we assume that A is not a Hall subgroup of G. Then $A = N \times A_1$, where A_1 is a normal nilpotent Hall subgroup of G and $G = [A_1]B$. Denote by B_1 the product of all Sylow subgroups P_i of B such that P_i are normal in G for all i. Respectively, denote by B_2 the product of all Sylow subgroups Q_j of B such that Q_j are non-normal in G for all j. It is clear that $G = [A \times B_1]B_2$, where $A \times B_1$ is a normal Hall subgroup of G and all Sylow subgroups of B_2 are cyclic by the assertion 1 of the theorem. Since B_2 is nilpotent, it follows that B_2 is cyclic. Therefore in any case, G contains a nilpotent Hall subgroup H such that G/H is cyclic. Since $\Phi(H) \subseteq \Phi(G)$, $H/\Phi(H)$ is abelian, we see that $G/\Phi(G)$ is metabelian. □

Remark 2. For any natural number $n \geq 3$ there exists a nilpotent group A such that the derived length of A is equal to n. Let p and q are distinct primes and $p, q \notin \pi(A)$. By Theorem 1.3 [7], there exists an $S_{\langle p,q \rangle}$-subgroup B. All Schmidt subgroups of $G = A \times B$ are Hall subgroups of G and the derived length of G is equal to n. Now, if G is a non-nilpotent group and $G \in \mathfrak{H}$, then its derived length is not bounded above.

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