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A class of Finsler metrics projectively related to a Randers metric

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Abstract. In this paper, we prove that the (α,β)-metrics in the form $F = (\alpha+\beta)^p/\alpha^{p-1}$ $(p \neq 1, 2)$ are projectively related to a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ on a manifold of dimension n $(n \geq 3)$ if and only if F is Berwald metric and $\bar{F}$ is Douglas metric and the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related.

1. Introduction

In Finsler geometry, it is an important topic to study projectively related Finsler metrics on a manifold. Two Finsler metrics are said to be projectively related if they have the same geodesics as point sets. It is well-known that two Finsler metrics F and $\bar{F}$ are projectively related if and only if their geodesic coefficients have the following relation

$$G^i = \bar{G}^i + P(x,y)y^i, \tag{1}$$

where $P(x,y)$ is a scalar function on $TM \setminus \{0\}$ with $P(x,\lambda y) = \lambda P(x,y)$, $\forall \lambda > 0$.

In Finsler geometry, there is a special class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is an 1-form with $\|\beta\|_\alpha < b_0$ and $\phi(s)$ is a C^∞ positive function on $(-b_0, b_0)$. In particular, when $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is Randers metric

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with $\|\beta\|_\alpha < 1$. Randers metric was introduced by the physicist G. Randers in 1941 from the standpoint of general relativity. The name was given by R. S. Ingarden, who used it to study the theory of the electron microscopein ([6]). Randers metrics form the simplest class of (α, β)-metrics, but they have some special properties which other (α, β)-metrics don't possess([1], [3]). In [14], the projectively related Randers metrics are studied. It shows that two Randers metrics are projectively related if and only if they have the same Douglas tensors and the corresponding Riemannian metrics are projectively related ([14]). Later on, N. Cui and Y. Shen prove that (α, β)-metrics in the form $F = (\alpha + \beta)^2/\alpha$ are projectively related to a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ if and only if both F and $\bar{F}$ are Douglas metrics and the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related ([5]).

The (α, β)-metrics in the form $F = (\alpha+\beta)^p/\alpha^{p-1}$ form a rich class of Finsler metrics. Obviously, when $p = 0$, $F = \alpha$ is just a Riemannian metric. Hence, we always assume that $p \neq 0$ in this paper. When $p = 1$, F becomes Randers metric $F = \alpha + \beta$. When $p = 2$, F is just the metric studied in [5]. If we substitute β with $-\beta$ and take $p = -1$, the resulting metric is just Matsumoto metric $F = \alpha^2/(\alpha - \beta)$. Matsumoto metric was introduced by M. Matsumoto as a realization of P. Finsler's idea "a slope measure of a mountain with respect to a time measure" ([10]). The purpose of this paper is to study (α, β)-metrics in the form $F = (\alpha + \beta)^p/\alpha^{p-1}$ which are projectively related to a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$. Firstly, we can prove the following

Theorem 1.1. *Let $F = (\alpha + \beta)^p/\alpha^{p-1} (p \neq 1)$ be an (α, β)-metric and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a Randers metric on a manifold M of dimension n ($n \geq 3$), where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero 1-forms. Then they have the same Douglas tensors if and only if F and $\bar{F}$ are Douglas metrics.*

Further, we have the following

Theorem 1.2. *Let $F = (\alpha + \beta)^p/\alpha^{p-1} (p \neq 1, 2)$ be an (α, β)-metric and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a Randers metric on a manifold M of dimension n ($n \geq 3$), where α and $\bar{\alpha}$ are two Riemannian metrics and β and $\bar{\beta}$ are two nonzero 1-forms. Then F is projectively related to $\bar{F}$ if and only if F is a Berwald metric and $\bar{F}$ is a Douglas metric and the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related.*

2. Preliminaries

For a given Finsler $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs as follows ([4]),

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

G^i are called the *geodesic coefficients* of F.

A Finsler metric F is called a *Berwald metric* if its geodesic coefficients

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$$

are quadratic in $y \in T_xM$ for any $x \in M$. It is easy to see that Riemannian metrics are special Berwald metrics.

Let

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big), \tag{2}$$

where G^i are the geodesic coefficients of F. The tensor $\mathbf{D} := D_j{}^i{}_{kl}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l$ is called the *Douglas tensor* of F. Douglas tensor is non-Riemannian. A Finsler metric is called *Douglas metric* if the Douglas tensor vanishes.

By (1), one can check easily that the Douglas tensor is a projectively invariant. A fundamental fact is that all Berwald metrics must be Douglas metrics.

By the definition, an (α, β)-metric is a Finsler metric expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemann metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_0$. It is proved that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying ([4])

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively, given by

$$G^i = \frac{g^{il}}{4}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}, \quad G^i_\alpha = \frac{a^{il}}{4}\Big\{[\alpha^2]_{x^m y^l}y^m - [\alpha^2]_{x^l}\Big\},$$

where $(g_{ij}) := \left(\frac{1}{2}[F^2]_{y^iy^j}\right)$ and $(a^{ij}) := (a_{ij})^{-1}$. Denote

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s^i{}_j := a^{il}s_{lj}, \quad s_i := b^j s_{ji},$$

where "|" denotes the horizontal covariant derivative with respect to α. Put $s_0 := s_i y^i$, $r_{00} := r_{ij}y^iy^j$. We have the following

Lemma 2.1. ([13]) *The geodesic coefficients of G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{3}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$

$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]},$$

$$\Psi := \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}.$$

In the following, we will compute the Douglas tensor of (α, β)-metrics. Let

$$\hat{G}^i := G^i_\alpha + \alpha Q s^i{}_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i.$$

Then (3) becomes

$$G^i = \hat{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i and $\hat{G}^i$ are projective equivalent sprays according to (1). Then they have the same Douglas tensor.

Denote

$$T^i := \alpha Q s^i{}_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{4}$$

Then $\hat{G}^i = G^i_\alpha + T^i$. We have

$$\begin{aligned} D_j{}^i{}_{kl} = \hat{D}_j{}^i{}_{kl} &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\Big) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\Big). \end{aligned} \tag{5}$$

Note that

$$\begin{aligned} \frac{\partial T^m}{\partial y^m} &= Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &\quad + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{6}$$

Thus, if two (α, β)-metrics F and $\bar{F}$ have the same Douglas tensors, i.e., $D_j{}^i{}_{kl} = \bar{D}_j{}^i{}_{kl}$, then by (5), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big[T^i - \bar{T}^i - \frac{1}{n+1}\Big(\frac{\partial T^m}{\partial y^m} - \frac{\partial \bar{T}^m}{\partial y^m}\Big)y^i\Big] = 0.$$

Then there are scalar functions $H^i{}_{jk} := H^i{}_{jk}(x)$ on M such that

$$T^i - \bar{T}^i - \frac{1}{n+1}\Big(\frac{\partial T^m}{\partial y^m} - \frac{\partial \bar{T}^m}{\partial y^m}\Big)y^i = H^i{}_{00}, \tag{7}$$

where $H^i{}_{00} := H^i{}_{jk} y^j y^k$.

3. The proof of Theorem 1.1

In this section, we consider the (α, β)-metrics in the following form:

$$F = \frac{(\alpha+\beta)^p}{\alpha^{p-1}} := \alpha\phi(s), \quad s := \frac{\beta}{\alpha},$$

where $\phi(s) = (1+s)^p$. Let $b_0 = b_0(p) > 0$ be the largest number such that

$$(1+s)^p > 0, \ (1+s)[1-(p-1)s] + p(p-1)(b^2 - s^2) > 0, \quad |s| \leq b < b_0. \tag{8}$$

Then $F = (\alpha+\beta)^p/\alpha^{p-1}$ is a Finsler metric if and only if β satisfies $b := \|\beta_x\|_\alpha < b_0$. It is easy to see that $b_0 = b_0(p) \leq 1$ for $p \neq 0$. Particularly, we have known that $b_0 = 1$ as $p = 1, 2$ and $b_0 = \frac{1}{2}$ as $p = -1$. In general, for fixed p, we always can determine b_0 such that (8) holds. For example, when $p > 1$ and $b := \|\beta_x\|_\alpha < min\{1, 1/(p-1)\}$, (8) holds.

By Lemma 2.1, the geodesic coefficients of F are given by (3) with

$$\begin{aligned}
Q &= \frac{p}{s(1-p)+1}, \\
\Theta &= \frac{1}{2}\frac{(1-2s(p-1))\,p}{s^2(1-p^2)+s(2-p)+1+b^2p(p-1)}, \\
\Psi &= \frac{1}{2}\frac{p\,(p-1)}{s^2(1-p^2)+s(2-p)+1+b^2p(p-1)}.
\end{aligned} \tag{9}$$

For a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, the geodesic coefficients of $\bar{F}$ are given by (3) with

$$\bar{Q} = 1, \quad \bar{\Theta} = \frac{1}{2(1+s)}, \quad \bar{\Psi} = 0. \tag{10}$$

To avoid clutter, we always assume

$$\lambda := \frac{1}{n+1}.$$

Proof of Theorem 1.1. The sufficiency is obvious. We just need to prove the necessity. If F and $\bar{F}$ have the same Douglas tensor, then (7) holds. By (4) and (6) and by Maple program, plugging (9) and (10) into (7) yields

$$\frac{A^i\alpha^6+B^i\alpha^5+C^i\alpha^4+D^i\alpha^3+E^i\alpha^2+F^i\alpha+H^i}{I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N}=\bar{\alpha}\bar{s}^i{}_0+H^i{}_{00}, \tag{11}$$

where

$$\begin{aligned} F^i &= -\lambda y^i\beta^3 r_{00}(p+4)(p-1)^2 p,\\ H^i &= 2\lambda y^i\beta^4 r_{00}(p+1)(p-1)^3 p,\\ M &= 2\beta^4(p+1)(p-5)(p-1)^2,\\ N &= 2\beta^5(p+1)^2(p-1)^3 \end{aligned} \tag{12}$$

and A^i, C^i, E^i, J, L denote the polynomials of odd degree in y and B^i, D^i, I, K denote the polynomials of even degree in y, which contain the terms $s^i{}_0$, s_0 and r_{00}.

Further, (11) is equivalent to

$$\begin{aligned} &A^i\alpha^6+B^i\alpha^5+C^i\alpha^4+D^i\alpha^3+E^i\alpha^2+F^i\alpha+H^i\\ &\qquad=(I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N)(H^i{}_{00}+\bar{\alpha}\bar{s}^i{}_0). \end{aligned} \tag{13}$$

Replacing y^i in (13) by $-y^i$ yields

$$\begin{aligned} &-A^i\alpha^6+B^i\alpha^5-C^i\alpha^4+D^i\alpha^3-E^i\alpha^2+F^i\alpha-H^i\\ &\qquad=(I\alpha^5-J\alpha^4+K\alpha^3-L\alpha^2+M\alpha-N)(H^i{}_{00}-\bar{\alpha}\bar{s}^i{}_0). \end{aligned} \tag{14}$$

(13)–(14) yields

$$\alpha(B^i\alpha^4+D^i\alpha^2+F^i)=\alpha H^i{}_{00}(I\alpha^4+K\alpha^2+M)+\bar{\alpha}\bar{s}^i{}_0(J\alpha^4+L\alpha^2+N). \tag{15}$$

(13)–(14) yields

$$\begin{aligned} &A^i\alpha^6+C^i\alpha^4+E^i\alpha^2+H^i\\ &\qquad=H^i{}_{00}(J\alpha^4+L\alpha^2+N)+\alpha\bar{\alpha}\bar{s}^i{}_0(I\alpha^4+K\alpha^2+M). \end{aligned} \tag{16}$$

Now we are ready to prove that $\bar{s}^i{}_j=0$.

Case 1: $p=2$. This case is discussed in ([5]), so we omit it.

Case 2: $p = -1$. In this case, plugging $p = -1$ into (13) yields

$$\bar{A}^i\alpha^5 + \bar{B}^i\alpha^4 + \bar{C}^i\alpha^3 + \bar{D}^i\alpha^2 + \bar{E}^i\alpha + \bar{F}^i = (\bar{I}\alpha^4 + \bar{J}\alpha^3 + \bar{K}\alpha^2 + \bar{L}\alpha)(\bar{\alpha}\bar{s}^i{}_0 + H^i{}_{00}), \tag{17}$$

where

$$\bar{F}^i = -6\lambda y^i\beta^3 r_{00} \tag{18}$$

and $\bar{A}^i$, $\bar{B}^i$, $\bar{C}^i$, $\bar{D}^i$, $\bar{E}^i$, $\bar{I}$, $\bar{J}$, $\bar{K}$, $\bar{L}$ denote the polynomials in y. From (17) and (18), there exists a scalar function $k := k(x)$ such that $r_{00} = k\alpha^2$ because α^2 is a irriducible polynomial of (y^i) and α^2 and β^2 are relatively prime polynomials of (y^i). Then $r_0 = k\beta$. Plugging them into (15) and (16) yields

$$\tilde{A}^i\alpha^5 + \tilde{C}^i\alpha^3 + \tilde{E}^i\alpha = \bar{\alpha}\bar{s}^i{}_0(\tilde{J}\alpha^2 + \tilde{L}) + H^i{}_{00}(\tilde{I}\alpha^3 + \tilde{K}\alpha), \tag{19}$$

$$\tilde{B}^i\alpha^4 + \tilde{D}^i\alpha^2 = \bar{\alpha}\bar{s}^i{}_0(\tilde{I}\alpha^3 + \tilde{K}\alpha) + H^i{}_{00}(\tilde{J}\alpha^2 + \tilde{L}), \tag{20}$$

where

$$\tilde{I} = (1+2b^2)^2, \quad \tilde{J} = 4\beta(1+2b^2)(2+b^2),$$
$$\tilde{K} = 3\beta^2(7+8b^2), \quad \tilde{L} = 18\beta^3 \tag{21}$$

and $\tilde{A}^i$, $\tilde{B}^i$, $\tilde{C}^i$, $\tilde{D}^i$, $\tilde{E}^i$ denote the polynomials in y.

(1) If $\bar{\alpha} \neq \mu(x)\alpha$, note that $\bar{s}^i{}_0\tilde{L} = 18\beta^3\bar{s}^i{}_0$, by (19), $\beta^6(\bar{s}^i{}_0)^2$ can be divided by α^2. Then there exists a scalar function $\tau^i = \tau^i(x)$ for each i such that $(\bar{s}^i{}_0)^2 = \tau^i\alpha^2$ which is equivalent to

$$\bar{s}^i{}_j\bar{s}^i{}_k = \tau^i a_{jk}.$$

When $n \geq 2$, if $\tau^i \neq 0$, then

$$1 \geq \mathrm{rank}(\bar{s}^i{}_j\bar{s}^i{}_k) = \mathrm{rank}(\tau^i a_{jk}) \geq 2,$$

which is impossible. Hence $\tau^i = 0$. Thus we get $\bar{s}^i{}_j = 0$.

(2) If $\bar{\alpha} = \mu(x)\alpha$, we have

$$y_i\bar{s}^i{}_0 = a_{ij}y^j\bar{s}^i{}_0 = \frac{1}{\mu(x)^2}\bar{a}_{ij}y^j\bar{s}^i{}_0 = \frac{1}{\mu(x)^2}\bar{y}_i\bar{s}^i{}_0 = 0.$$

On the other hand, (20) implies that $\left(H^i{}_{00}\tilde{L}\right)^2$ can be divided by α^2. Thus for each i, there exists a scalar function $\theta^i := \theta^i(x)$ such that $H^i{}_{00} = \theta^i\alpha^2$. Contracting (19) and (20) with y_i yields

$$A\alpha^4 + C\alpha^2 = \theta(\tilde{J}\alpha^2 + \tilde{L}), \qquad B\alpha^2 + D = \theta(\tilde{I}\alpha^2 + \tilde{K}), \tag{22}$$

where

$$
\begin{aligned}
A &= 3\lambda k b^2,\\
B &= (k + 2kb^2 + 2\lambda k b^2 - 2\lambda k)\beta + (2\lambda b^2 - 2\lambda)s_0,\\
C &= -\beta[(-4kb^2 - 5k + 13\lambda k + 8\lambda k b^2)\beta - (2 - 4b^2 + 10\lambda + 8\lambda b^2)s_0],\\
D &= -6\beta^2\,(3\lambda - 1)\,(k\beta + s_0)
\end{aligned}
$$

and

$$\theta := a_{ij}\theta^j y^i.$$

From (22) we know that $\theta\tilde{L}$ and $(D-\theta\tilde{K})$ can be divided by α^2, which imply that $\theta = 0$ and $k\beta + s_0 = 0 (n \geq 3)$. Then we can get that $k = 0$ and $s_0 = 0$. Hence, $r_{ij} = 0$ and $s_i = 0$. Plugging them into (17) yields

$$-\alpha^2 s^i{}_0 = (\alpha + 2\beta)(\mu\alpha\bar{s}^i{}_0 + H^i{}_{00}). \tag{23}$$

From (23), we obtain the following

$$-\alpha^2 s^i{}_0 = \mu\alpha^2\bar{s}^i{}_0 + 2\beta H^i{}_{00}, \qquad 0 = H^i{}_{00} + 2\mu\beta\bar{s}^i{}_0. \tag{24}$$

From (24) one gets $\alpha^2 s^i{}_0 + \mu\alpha^2\bar{s}^i{}_0 = 4\mu\beta^2\bar{s}^i{}_0$, which implies that $\bar{s}^i{}_0 = 0$.

Case 3: $p \neq 2, -1$. From (15) and by (12), it is obvious that $\left(\bar{\alpha}\beta\bar{s}^i{}_0\right)^2$ can be divided by α^2. If $\bar{\alpha} \neq \mu(x)\alpha$, then $\beta^2(\bar{s}^i{}_0)^2$ can be divided by α^2, which implies that $\bar{s}^i{}_0 = 0$.

If $\bar{\alpha} = \mu(x)\alpha$, then (15) and (16) are reduced to

$$
\begin{aligned}
B^i\alpha^4 + D^i\alpha^2 + F^i &= H^i{}_{00}(I\alpha^4 + K\alpha^2 + M)\\
&\quad + \mu(x)\bar{s}^i{}_0(J\alpha^4 + L\alpha^2 + N), \qquad (25)\\
A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + H^i &= H^i{}_{00}(J\alpha^4 + L\alpha^2 + N)\\
&\quad + \mu(x)\alpha^2\bar{s}^i{}_0(I\alpha^4 + K\alpha^2 + M). \qquad (26)
\end{aligned}
$$

The above two equations imply that $X^i := F^i - H^i{}_{00}M - \mu(x)\bar{s}^i{}_0 N$ and $Y^i := H^i - H^i{}_{00}N$ can be divided by α^2. Hence $2(p+1)(p-1)\beta X^i + (p+4)Y^i$ can also be divided by α^2. By (12) we have

$$
\begin{aligned}
&2(p+1)(p-1)\beta X^i + (p+4)Y^i\\
&\quad = -6H^i{}_{00}\beta^5(p-2)(p+1)^2(p-1)^3 - 4\mu(x)\bar{s}^i{}_0\beta^6(p+1)^3(p-1)^4. \qquad (27)
\end{aligned}
$$

Contracting (27) with $y_i := a_{ij}y^j$, we conclude that $H^0{}_{00} := H^i{}_{00}y_i$ can be divided by α^2, that is, there exists a 1-form $\eta := \eta(x)_i y^i$ such that $H^0{}_{00} = \eta\alpha^2$.

Contracting (13) with y_i yields

$$\bar{A}\alpha^5+\bar{B}\alpha^4+\bar{C}\alpha^3+\bar{D}\alpha^2+\bar{E}\alpha+\bar{F} = \eta(I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N), \quad (28)$$

where

$$\bar{E}=-\beta^3p(p-1)^2[\lambda(p+4)-3]r_{00}, \quad \bar{F}=\beta^4p(p+1)(p-1)^3(2\lambda-1)r_{00} \quad (29)$$

and $\bar{A}$, $\bar{C}$ denote the polynomials of odd degree in y and $\bar{B}$, $\bar{D}$ denote polynomials of even degree in y. Replacing y^i in (28) by $-y^i$ yields

$$-\bar{A}\alpha^5+\bar{B}\alpha^4-\bar{C}\alpha^3+\bar{D}\alpha^2-\bar{E}\alpha+\bar{F} = -\eta(I\alpha^5-J\alpha^4+K\alpha^3-L\alpha^2+M\alpha-N). \quad (30)$$

(28)–(30) yields

$$\bar{A}\alpha^4+\bar{C}\alpha^2+\bar{E}=\eta(I\alpha^4+K\alpha^2+M). \quad (31)$$

(28) + (30) yields

$$\bar{B}\alpha^4+\bar{D}\alpha^2+\bar{F}=\eta(J\alpha^4+L\alpha^2+N). \quad (32)$$

The above two equations imply that $X:=\bar{E}-\eta M$ and $Y:=\bar{F}-\eta N$ can be divided by α^2, so $(p+1)(p-1)(2\lambda-1)\beta X+[\lambda(p+4)-3]Y$ can also be divided by α^2. By (12) and (29) one gets

$$(p+1)(p-1)(2\lambda-1)\beta X+[\lambda(p+4)-3]Y = -2\eta\beta^5(p+1)^2(p-1)^3(p-2)(3\lambda-1). \quad (33)$$

Then $\eta=0$ because of $n\geq 3$. From (32) and by (29), we have $r_{00}=\tau\alpha^2$, where $\tau:=\tau(x)$ is a scalar function on M. Then $r_0=\tau\beta$. Plugging them into (28) yields

$$c_4\alpha^4+c_3\alpha^3+c_2\alpha^2+c_1\alpha+c_0=0, \quad (34)$$

where

$$\begin{aligned} c_0 &= -\beta^3(p-1)(p+1)\big\{\tau(p-1)(4\lambda-1)\beta+2[(3\lambda-1)p-\lambda]s_0\big\},\\ c_1 &= \beta^2\big[\tau(\lambda p^2+9\lambda p-10\lambda-3p+3)\beta+(10\lambda p-4p-2p^2\\ &\quad -6\lambda-2\lambda p^2)s_0\big] \end{aligned} \quad (35)$$

and c_2, c_3, c_4 denote polynomials in y. From (34), we obtain

$$c_4\alpha^4 + c_2\alpha^2 + c_0 = 0, \tag{36}$$

$$c_3\alpha^2 + c_1 = 0. \tag{37}$$

Combining (35), (36) and (37) yields

$$\begin{aligned} &\tau(p-1)(4\lambda-1)\beta + 2\big[(3\lambda-1)p-\lambda\big]s_0 = 0, \\ &\tau(p-1)\big[\lambda(p+10)-3\big]\beta + 2\big[\lambda(5p-p^2-3)-p(2-p)\big]s_0 = 0. \end{aligned} \tag{38}$$

Differentiating (38) with respect to y^i and contracting it by b^i yields

$$\tau(4\lambda-1) = 0, \tag{39}$$

$$\tau[\lambda(p+10)-3] = 0. \tag{40}$$

We claim $\tau = 0$. If $\tau \neq 0$, then (39) implies $\lambda = \frac{1}{4}$. Plugging it into (40) yields $\frac{\tau}{4}(p-2) = 0$, which is impossible. Hence $\tau = 0$. From (38), we have $s_0 = 0$. Then (13) becomes

$$\alpha^2 p s^i{}_0 = [\alpha + (1-p)\beta](\mu\alpha\bar{s}^i{}_0 + H^i{}_{00}).$$

From above equation, we have

$$\alpha^2 p s^i{}_0 = \mu\alpha^2\bar{s}^i{}_0 + (1-p)\beta H^i{}_{00}, \tag{41}$$

$$0 = (1-p)\mu\beta\bar{s}^i{}_0 + H^i{}_{00}. \tag{42}$$

Then (41)$-(1-p)\beta\times$(42) yields $(ps^i{}_0 - \mu\bar{s}^i{}_0)\alpha^2 = (p-1)\mu\beta\bar{s}^i{}_0$ which implies that $\beta\bar{s}^i{}_0$ can be divided by α^2. Then $\bar{s}^i{}_0 = 0$.

It is well known that Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed, i.e. $\bar{s}_{ij} = 0$. Then we have proved that $\bar{F}$ is a Douglas metric. By the assumption, F is also a Douglas metric. □

4. The Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemma

Lemma 4.1 ([9]). *Suppose that $Q/s \neq$ constant for an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ on a manifold M of dimension n $(n > 2)$. If F is a Douglas metric and $b := \|\beta_x\|_\alpha \neq 0$, then β is closed.*

Proof of Theorem 1.2. Firstly we prove the sufficiency. It is known that any regular (α,β)-metric F is Berwald metric if and only if β is parallel with respect to α, i.e., $b_{i|j}=0$ ([1], [7], [11]). Hence, if F is Berwald metric, its geodesic coefficients $G^i=G^i_\alpha$ by (3). Because $\bar{F}$ is Douglas metric, its geodesic coefficients $\bar{G}^i=\bar{G}^i_{\bar{\alpha}}+\frac{\bar{r}_{00}}{2\bar{F}}y^i$. Note that the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related, we have $G^i_\alpha=\bar{G}^i_{\bar{\alpha}}+\bar{P}(x,y)y^i$, where $\bar{P}(x,y)$ is a scalar function on $TM\setminus\{0\}$. Hence we have $G^i=\bar{G}^i+P(x,y)y^i$, where $P(x,y):=\bar{P}(x,y)-\frac{\bar{r}_{00}}{2\bar{F}}$. Thus F is projectively related to $\bar{F}$.

Next, we are going to prove the necessity. When $p=-1$, $F=\alpha^2/(\alpha+\beta)$. We can prove that F is a Douglas metric if and only if β is parallel with respect to α (see [9], [12]). Because F is projectively related to $\bar{F}$, F and $\bar{F}$ have the same Douglas tensors. By Theorem 1.1, we know that both of F and $\bar{F}$ are Douglas metrics. Then $b_{i|j}=0$ and $\bar{\beta}$ is closed. Further, we know that $G^i=G^i_\alpha$ and $\bar{G}^i=\bar{G}^i_{\bar{\alpha}}+\big[\bar{r}_{00}/(2\bar{F})\big]y^i$. By the assumption again, there is a scalar function $P:=P(x,y)$ on $TM\setminus\{0\}$ such that $G^i=\bar{G}^i+Py^i$. Then we obtain $G^i_\alpha=\bar{G}^i_{\bar{\alpha}}+(P+\frac{\bar{r}_{00}}{2\bar{F}})y^i$. Thus α is projectively related to $\bar{\alpha}$.

When $p\neq-1$, it is easy to prove that $\phi(s)=(1+s)^p$ satisfies $Q/s\neq$ constant. By Theorem 1.1 and Lemma 4.1, we obtain $s_{ij}=\bar{s}_{ij}=0$. Then (11) becomes

$$\frac{A^i_4\alpha^4+A^i_3\alpha^3+A^i_2\alpha^2+A^i_1\alpha+A^i_0}{I_4\alpha^4+I_3\alpha^3+I_2\alpha^2+I_1\alpha+I_0}={H^i}_{00}, \tag{43}$$

where

$$\begin{aligned} A^i_0&=2\lambda y^i\beta^3p(p+1)(p-1)^2r_{00},\\ A^i_1&=\lambda y^i\beta^2p(p-2)(p-1)r_{00},\\ I_0&=2\beta^4(p+1)^2(p-1)^2,\\ I_1&=4\beta^3(p+1)(p-1)(p-2) \end{aligned} \tag{44}$$

and A^i_4, A^i_3, A^i_2, I_4, I_3, I_2 denote polynomials in y. Then (15) and (16) become

$$A^i_4\alpha^4+A^i_2\alpha^2+A^i_0={H^i}_{00}(I_4\alpha^4+I_2\alpha^2+I_0), \tag{45}$$

$$A^i_3\alpha^2+A^i_1={H^i}_{00}(I_3\alpha^2+I_1). \tag{46}$$

Thus there exist a scalar function $\bar{\tau}:=\bar{\tau}(x)$ and a 1-form $\bar{\eta}:=\bar{\eta}_iy^i$ such that $r_{00}=\bar{\tau}\alpha^2$ and ${H^0}_{00}=\bar{\eta}\alpha^2$. Contracting (43) with y_i yields

$$B_5\alpha^5+B_4\alpha^4+B_3\alpha^3+B_2\alpha^2=\bar{\eta}(I_4\alpha^4+I_3\alpha^3+I_2\alpha^2+I_1\alpha+I_0), \tag{47}$$

where

$$B_2 = \bar{\tau}\beta^3 (4\lambda - 1)\, p\,(p-1)^2 (p+1)\,, \quad B_3 = \bar{\tau}\beta^2 (3\lambda - 1)\, p\,(p-2)(p-1) \tag{48}$$

and B_4, B_5 denote polynomials in y. Then we have

$$B_5\alpha^4 + B_3\alpha^2 = \bar{\eta}(I_3\alpha^2 + I_1), \tag{49}$$

$$B_4\alpha^4 + B_2\alpha^2 = \bar{\eta}(I_4\alpha^4 + I_2\alpha^2 + I_0). \tag{50}$$

From (49), we can see that $\bar{\eta}I_1$ can be divided by α^2, which implies that $\bar{\eta} = 0$. Thus, by (49) again, B_3 can be divided by α^2. By (48), we have $\bar{\tau} = 0$. Then $r_{ij}=0$. Thus F is Berwald metric. In this case, we still have $G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + (P + \frac{\bar{r}_{00}}{2F})y^i$, that is, α is projectively related to $\bar{\alpha}$. □

From Theorem 1.2, we immediately obtain the following corollary

Corollary 4.2. *Let $F = (\alpha+\beta)^p/\alpha^{p-1}$ $(p \neq 1, 2)$ be an (α,β)-metric on a manifold M of dimension n $(n \geq 3)$, where α is a Riemannian metric and β is a nonzero 1-form. Then F is projectively flat if and only if*

(1) *β is parallel with respect to α;*

(2) *α is locally projectively flat, i.e., α is of constant sectional curvature.*

Proof. If F is projectively flat, we can write $G^i = P(x,y)y^i$, where $P(x,y)$ is a scalar function on $TM \setminus \{0\}$ with $P(x,\lambda y) = \lambda P(x,y)$, $\forall\lambda > 0$. On the other hand, we can always chose a Riemann metric $\bar{\alpha}$ and an 1-form $\bar{\beta}$ such that $\bar{\alpha}$ is projectively flat and $\bar{\beta}$ is closed. Further we can construct a projectively flat Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ and its geodesic coefficients can be expressed as $\bar{G}^i = \bar{P}(x,y)y^i$, where $\bar{P}(x,y)$ is a scalar function on $TM \setminus \{0\}$. Thus $G^i = \bar{G}^i + (P - \bar{P})y^i$, i.e., F is projectively related to $\bar{F}$. Thus, by Theorem 1.2, we know that F is a Berwald metric and α is projectively related to $\bar{\alpha}$. It is obvious that α is projectively flat.

Conversely, because β is parallel with respect to α, we have $G^i = G^i_\alpha$ by Lemma 2.1. Since α is locally projectively flat, F is projectively flat. □

Corollary 4.2 is just the Theorem 1 in [2].

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