Publ. Math. Debrecen
45 / 1-2 (1994), 53–69

New classes of solutions of an alternative Cauchy equation

By GIAN LUIGI FORTI (Milano) and LUIGI PAGANONI (Milano)

Abstract. In this paper we consider the alternative Cauchy equation

$$g(x+y) \neq g(x)g(y) \text{ implies } f(x+y) = f(x)f(y)$$

where f, g are unknown functions from $\mathbb{R}$ into a group $(S, \cdot\,)$. Assuming a slightly different hypothesis than in [1] we describe new classes of solutions.

1. Introduction

In previous papers we studied the alternative Cauchy equation

$$g(x+y) \neq g(x)g(y) \text{ implies } f(x+y) = f(x)f(y), \tag{1}$$

where f, g are unknown functions from $\mathbb{R}^n$ ([1]) or $I := (0,1)$ ([2]) into a group $(S, \cdot\,)$ (for general references about the problem see [1]). In both cases we described the solutions of (1) under a suitable topological hypothesis concerning the function g.

In the present paper we study equation (1) on $\mathbb{R}$ assuming a slightly different hypothesis on g and we describe all solutions when the group S has no elements of order 2. In the general case we describe some classes of solutions and present open problems.

2. Notations and preliminary results

In this section we present the notations, some previous results and we state the problem treated in the present paper.

Mathematics Subject Classification: 39B30, 39B50, 39B70.
Work supported by M.U.R.S.T. Research funds (40%).

Denote by $\mathbb{Z}$ and $\mathbb{N}_0$ the classes of the integers and the non-negative integers respectively, and by $p_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, the maps given by:

$$p_1(x, y) = x, \quad p_2(x, y) = y, \quad p_3(x, y) = x + y.$$

Given an open interval $E \subset \mathbb{R}$ and a function $\varphi : E \to S$, we define

$$\Omega_\varphi := \{(x, y) \in (E \times E) \cap p_3^{-1}(E) : \varphi(x + y) \neq \varphi(x)\varphi(y)\}$$

and

$$A_\varphi := \{(x, y) \in (E \times E) \cap p_3^{-1}(E) : \varphi(x + y) = \varphi(x)\varphi(y)\}.$$

A_φ° and Ω_φ° denote the interior of A_φ and Ω_φ respectively.

A function $\varphi : E \to S$ is said *locally affine* in $x \in E$ if there exists $a \in \mathrm{Hom}(\mathbb{R}, S)$ such that $\varphi(x + u) = \varphi(x)a(u)$ for all u in an open interval $U \ni 0$ (Note that the homomorphism a may depend on the point x). A function $\varphi : E \to S$ is said *locally affine* in an interval $V \subset E$ if it is locally affine in each point of V.

We shall use the following simple properties:

Lemma 1([2]). i) *If $(x_0, y_0) \in A_\varphi^\circ$ then φ is locally affine in x_0, y_0, $x_0 + y_0$.*

ii) *If $E \subset \mathbb{R}$ is an open interval and φ is locally affine in each point of E, then there exist $a \in \mathrm{Hom}(\mathbb{R}, S)$ and $\alpha \in S$ such that*

$$\varphi(x) = \alpha a(x), \quad x \in E.$$

iii) *Let J, K, L be open intervals and*

$$\varphi(x) = \begin{cases} \alpha a(x), & x \in J, \\ \beta b(x), & x \in K, \\ \gamma c(x), & x \in L \end{cases} \qquad a, b, c \in \mathrm{Hom}(\mathbb{R}, S).$$

If there exists $(x_0, y_0) \in A_\varphi^\circ$ with $x_0 \in J$, $y_0 \in K$, $x_0 + y_0 \in L$, then

$$\gamma = \alpha\beta \quad \text{and} \quad b(x) = c(x) = \beta^{-1} a(x)\beta.$$

For any function $\varphi : \mathbb{R} \to S$ define

(2) $$H_\varphi := \{t \in \mathbb{R} : \forall x \in \mathbb{R},\ \varphi(t + x) = \varphi(t)\varphi(x) = \varphi(x)\varphi(t)\} = \mathbb{R} \setminus \Big(p_1(\Omega_\varphi) \cup p_2(\Omega_\varphi)\Big).$$

Lemma 2 ([1]). *The set H_φ, if not empty, is a subgroup of $\mathbb{R}$.*

Note that if $\Omega_\varphi^\circ \neq \emptyset$ then by (2) either $H_\varphi = \emptyset$ or H_φ is a discrete subgroup of $\mathbb{R}$, i.e. $H_\varphi = h\mathbb{Z}$ for some $h \geq 0$.

Lemma 3 ([1], [2]). *Let $\varphi : \mathbb{R} \to S$ be any function with $H_\varphi \neq \emptyset$. For every $t \in H_\varphi$ and for every $m, n \in \mathbb{Z}$*

$$(x, y) \in \Omega_\varphi \iff (x + nt, y + mt) \in \Omega_\varphi.$$

In [1] among other results we described the solutions (f, g) of (1) on $\mathbb{R}^n$, under the assumption

$$p_i(\Omega_g) = p_i(\Omega_g^\circ), \quad i = 1, 2. \tag{3}$$

In our case (i.e. $n = 1$), if $\Omega_\varphi^\circ \neq \emptyset$, we have three possibilities for H_g:

$$H_g = \emptyset, \quad H_g = \{0\}, \quad H_g = h\mathbb{Z}, \quad h > 0.$$

Without loss of generality we may always assume, and we do, that in the last case the solution (f, g) is "normalized", i.e. $h = 1$. Then (3) is equivalent, by Lemma 3, to

$$p_i(\Omega_g \cap Q) = p_i(\Omega_g^\circ \cap Q), \quad i = 1, 2, \tag{4}$$

where Q is the open square $I \times I$.

In [2] we solved the following local form of equation (1):

$$\tilde{g}(x + y) \neq \tilde{g}(x)\tilde{g}(y) \quad \text{implies} \quad \tilde{f}(x + y) = \tilde{f}(x)\tilde{f}(y), \quad (x, y) \in T \tag{5}$$

with $\tilde{f}, \tilde{g} : I \to S$ and $T := \{(x, y) \in I^2 : x, y, x + y \in I\}$, under the corresponding assumption (3) for $\tilde{g}$; since $\Omega_{\tilde{g}} \subset T$ this hypothesis can be written in the form

$$p_i(\Omega_{\tilde{g}} \cap T) = p_i(\Omega_{\tilde{g}}^\circ \cap T), \quad i = 1, 2. \tag{3'}$$

Take now any solution of (1) satisfying (3) and having $H_g = \mathbb{Z}$ (and so satisfying (4)). If moreover $p_i(\Omega_g \cap T) = p_i(\Omega_g^\circ \cap T)$, $i = 1, 2$, obviously its restriction $(\tilde{f}, \tilde{g})$ to I belongs to the class of solutions of (5) described in [2] (see Theorem 1 below).

It is so natural to ask whether there exist solutions of (5) satisfying (3′) which can be extended to solutions (f, g) of (1) on the whole $\mathbb{R}$, having $H_g = \mathbb{Z}$ but not satisfying (4). The problem can be reformulated as follows:

Describe, if there are, the solutions (f, g) of equation (1) such that:

$$H_g = \mathbb{Z} \quad \text{and so} \quad g(1 + x) = g(1)g(x) = g(x)g(1),\ x \in \mathbb{R}, \tag{6}$$

$$p_i(\Omega_g \cap T) = p_i(\Omega_g^\circ \cap T),\ i = 1, 2, \tag{7}$$

$$p_i(\Omega_g \cap Q) \neq p_i(\Omega_g^\circ \cap Q) \text{ for at least one index } i = 1, 2. \tag{8}$$

We are obviously interested in "non-trivial" solutions, i.e. solutions such that f is not a homomorphism of $\mathbb{R}$ into S.

Remark 1. Instead of condition (7) concerning the triangle T one can equivalently assume

$$p_i(\Omega_g \cap T') = p_i(\Omega_g^\circ \cap T'), \quad i = 1, 2, \tag{7'}$$

where $T' := \{(x, y) \in \mathbb{R}^2 : x, y \in I,\ x + y \in (1, 2)\}$. Actually, by Lemma 3, (7′) is equivalent to

$$p_i(\Omega_g \cap (-T)) = p_i(\Omega_g^\circ \cap (-T)), \quad i = 1, 2, \tag{7''}$$

and (f, g) is a solution of (1) under (7″) if and only if $(\bar{f}, \bar{g})$ given by

$$\bar{f}(x) = f(-x), \quad \bar{g}(x) = g(-x)$$

is a solution of (1) satisfying (7).

In order to solve our problem we use the following result proved in [2].

Theorem 1. *Let $(\tilde{f}, \tilde{g})$ be a solution of (5) satisfying (3′) and define*

$$W = I \setminus (p_1(\Omega_{\tilde{g}}) \cup p_2(\Omega_{\tilde{g}})).$$

If $W = \emptyset$ $[W = I]$ then $\tilde{f}[\tilde{g}]$ is the restriction to I of a homomorphism of $\mathbb{R}$ into S.

If $\emptyset \neq W \neq I$ then W has a minimum $\tau (> 0)$ and either $\tilde{f}$ is the restriction of a homomorphism or the pair $(\tilde{f}, \tilde{g})$ has one of the following forms:

$$\begin{cases} \tilde{f}(x) = \alpha^{i+1} a(x) \\ \tilde{g}(x) = \gamma^i c(x) \end{cases} \quad \text{if } x \in [i\tau, (i+1)\tau) \cap I,\ \alpha, \gamma \neq e,\ i \in \mathbb{N}_0, \tag{9}$$

$$\begin{cases} \tilde{f}(x) = \alpha^{i} a(x) \\ \tilde{g}(x) = \gamma^{i+1} c(x) \end{cases} \quad \text{if } x \in (i\tau, (i+1)\tau] \cap I,\ \alpha, \gamma \neq e,\ i \in \mathbb{N}_0, \tag{10}$$

$$\begin{cases} \tilde{f}(x) = a(x) \quad \text{if} \quad x \in I \setminus E, \qquad \tilde{f}(x) \neq a(x) \ \text{ if } x \in E \\ \text{where } \emptyset \neq E \subset \tau\mathbb{N}_0 \cap I \\ \text{and } \tilde{g} \text{ satisfies the conditions} \\ \tilde{g}(x+\tau) = \tilde{g}(x)\tilde{g}(\tau) = \tilde{g}(\tau)\tilde{g}(x), \ x \in (0, 1-\tau) \\ \tilde{g}(\tau) = \tilde{g}(x)\tilde{g}(\tau - x), \qquad\qquad x \in (0, \tau), \end{cases} \tag{11}$$

$$(12)\quad \begin{cases} \tilde f(x) = a(x) \text{ if } x \in I \setminus \{\xi\}, & \tilde f(\xi) \neq a(\xi) \\ \text{with } \xi \in W \setminus \tau\mathbb{N}_0,\ \max\{\tau, 1-\tau\} < \xi < 1 & \\ \text{and } \tilde g \text{ satisfies the conditions} & \\ \tilde g(x+\tau) = \tilde g(x)\tilde g(\tau) = \tilde g(\tau)\tilde g(x), & x \in (0, 1-\tau) \\ \tilde g(x+\xi) = \tilde g(x)\tilde g(\xi) = \tilde g(\xi)\tilde g(x), & x \in (0, 1-\xi) \\ \tilde g(\xi) = \tilde g(x)\tilde g(\xi - x), & x \in (0, \xi), \end{cases}$$

where $a, c \in \mathrm{Hom}(\mathbb{R}, S)$ *and in cases* (9), (10) a *and* c *commute with* α *and* γ *respectively.*

3. The case $W = \emptyset$ or $W = I$

Let (f, g) be a non-trivial solution of equation (1) satisfying (6)–(8). Obviously if $\tilde f = f_{|I}$ and $\tilde g = g_{|I}$ then $(\tilde f, \tilde g)$ is a solution of (5) satisfying (3′) and so it has one of the forms described in Theorem 1. We refer to the pair $(\tilde f, \tilde g)$ as *the solution on* T *associated to* (f, g). In this section we show that f on $\mathbb{R} \setminus \mathbb{Z}$ cannot equal a homomorphism and that for the associate solution the cases listed in Theorem 1 relative to $W = \emptyset$ and $W = I$ cannot appear.

Lemma 4. *Let* (f, g) *be a non-trivial solution of equation* (1) *satisfying* (6)–(8). *Then there exists* $t_0 \in \mathbb{R}$ *such that*

$$g(t_0)g(-t_0) \neq g(0)\ (= e) \quad \text{i.e.} \quad (t_0, -t_0) \notin A_g.$$

Proof. Assume

$$(13)\qquad g(t)g(-t) = g(0)\ (= e), \quad t \in \mathbb{R}.$$

By (13) we have

$$g(-y-x) = g(x+y)^{-1}, \quad g(-y)g(-x) = g(y)^{-1}g(x)^{-1},$$

thus $(x, y) \in \Omega_g$ if and only if $(-y, -x) \in \Omega_g$ or, equivalently, $(1-y, 1-x) \in \Omega_g$. This means that the set Ω_g is symmetric with respect to the diagonal $y = -x + 1$.

So we have

$$p_i(\Omega_g \cap T) = 1 - p_{3-i}(\Omega_g \cap T'),\ p_i(\Omega_g^\circ \cap T) = 1 - p_{3-i}(\Omega_g^\circ \cap T'), \quad i = 1, 2$$

and by (7) $p_i(\Omega_g^\circ \cap T') = p_i(\Omega_g \cap T')$, $i = 1, 2$. Since by (13) and Lemma 3 $\{(t, 1-t), t \in \mathbb{R}\} \subset A_g$, it follows $p_i(\Omega_g \cap Q) = p_i(\Omega_g^\circ \cap Q)$, $i = 1, 2$, contrary to the assumption (8).

Proposition 1. *Let* (f,g) *be a non-trivial solution of equation* (1) *satisfying* (6)–(8)*. The function* f *cannot be of the form*

$$f(x)=a(x),\quad x\in\mathbb{R}\setminus\mathbb{Z} \tag{14}$$

for any $a\in\operatorname{Hom}(\mathbb{R},S)$.

Proof. Assume f of the form (14) and let $n_0\in\mathbb{Z}$ such that $f(n_0)\neq a(n_0)$. Then by (6), $\{(x,n_0-x),x\in\mathbb{R}\}\subset A_g$ and, by Lemma 3, the same holds for the set $\{(x,-x),x\in\mathbb{R}\}$, contrary to Lemma 4.

Proposition 2. *Let* (f,g) *be a non-trivial solution of equation* (1) *satisfying* (6)–(8) *and let* $(\tilde f,\tilde g)$ *be its associate solution on* T*. Then the set*

$$W=I\setminus\Big(p_1(\Omega_{\tilde g})\cup p_2(\Omega_{\tilde g})\Big)=I\setminus\Big(p_1(\Omega_g\cap T)\cup p_2(\Omega_g\cap T)\Big)$$

is a proper non-empty subset of I.

Proof. Assume $W=I$. Then, by Theorem 1, there is $a\in\operatorname{Hom}(\mathbb{R},S)$ such that

$$\tilde g(x)=c(x),\quad x\in I.$$

By (6)

$$g(x)=g(1)g(x-1)=g(1)\tilde g(x-1)=g(1)c(-1)c(x),\quad x\in(1,2).$$

If $g(1)c(-1)=e$, then $Q\subset A_g$; if $g(1)c(-1)\neq e$, then $\Omega_g\cap Q=Q\setminus T$. In both cases we have $p_i(\Omega_g\cap Q)=p_i(\Omega_g^\circ\cap Q)$, $i=1,2$, contrary to (8).

Let now $W=\emptyset$. By Theorem 1 we have

$$\tilde f(x)=a(x),\ x\in(0,1)\quad\text{for some } a\in\operatorname{Hom}(\mathbb{R},S).$$

For each $k\in\mathbb{Z}\setminus\{0\}$, let $T_k:=T+(k,k)$; by (7) and Lemma 3 we have

$$p_i(\Omega_g^\circ\cap T)+k=p_i(\Omega_g^\circ\cap T_k)=p_i(\Omega_g\cap T_k)=p_i(\Omega_g\cap T)+k$$

and so

$$p_1(\Omega_g^\circ\cap T_k)\cup p_2(\Omega_g^\circ\cap T_k)=(k,k+1).$$

This relation, by Lemma 1 and the property $A_f^\circ\supset\Omega_g^\circ$, implies f locally affine in $(k,k+1)$, i.e.

$$f(x)=\alpha_k a_k(x),\ x\in(k,k+1)\ \text{for some } a_k\in\operatorname{Hom}(\mathbb{R},S)\text{ and }\alpha_k\in S.$$

Let $(x,y)\in\Omega_g^\circ$, $x\in(k,k+1)$, $y\in(0,1)$, $x+y\in(k,k+1)$: we have

$$\alpha_k a_k(x)a_k(y)=\alpha_k a_k(x+y)=f(x+y)=f(x)f(y)=\alpha_k a_k(x)a(y),$$

and so $a_k(y) = a(y)$ in an interval, this implies $a_k = a$ for all $k \in \mathbb{Z} \setminus \{0\}$. By Proposition 1 we cannot have $\alpha_k = e$ for all $k \in \mathbb{Z} \setminus \{0\}$, so there exists k for which $\alpha_k \neq e$ and let $\bar{k}$ be the smallest, in absolute value, of these integers k. If we take the triangle

$$T' = \{(x, y) \in \mathbb{R}^2; x, y \in (0, 1), \quad x + y \in (1, 2)\},$$

then it is immediately verified that

$$T' + (\bar{k} - 1, 0) \subset A_g \text{ when } \bar{k} > 0 \text{ and } T' + (\bar{k}, 0) \subset A_g \text{ when } \bar{k} < 0$$

and so, by Lemma 3, we get $T' \subset A_g$. Lemma 1–i) implies that $\tilde{g}$ is locally affine in (0,1) and so, by Lemma 1–ii), $\tilde{g}(x) = \beta c(x)$, $x \in (0, 1)$ for some $c \in \mathrm{Hom}(\mathbb{R}, S)$ and $\beta \in S$. We have $\beta \neq e$ otherwise $T \subset A_{\tilde{g}}$ and this implies $W = (0, 1)$. So $T \subset \Omega_{\tilde{g}}$; from this we get

$$(0, 1) \supset p_i(\Omega_g \cap Q) \supset p_i(\Omega_g^\circ \cap Q) \supset p_i(\Omega_g^\circ \cap T) = p_i(\Omega_{\tilde{g}}^\circ) = (0, 1);$$

thus $p_i(\Omega_g \cap Q) = p_i(\Omega_g^\circ \cap Q)$, contrary to condition (8).

4. On the representations (11) and (12)

From the results of Section 3, we obtain that, if (f, g) is a non-trivial solution of (1) satisfying (6)–(8), then f cannot equal a homomorphism on $\mathbb{R} \setminus \mathbb{Z}$ and moreover for its associate solution $(\tilde{f}, \tilde{g})$ on T the set W satisfies $\emptyset \neq W \neq I$ and has a minimum $\tau > 0$. In the present section first we show that do not exist non-trivial solutions (f, g) of our problem with $\tilde{f} = a$ on I, $a \in \mathrm{Hom}(\mathbb{R}, S)$. Furthermore we prove that, if S has no elements of order 2, then the associate solution of a non-trivial one must have one of the forms (9) or (10).

Define

$$\begin{aligned}
I_n :=& (n, n+1), \quad n \in \mathbb{Z} \quad (I_0 = I) \\
J_k :=& \{x \in I : k\tau < x < (k+1)\tau\}, \quad k \in \mathbb{N}_0 \\
T^1_{i,j} :=& \{(x, y) \in T : x \in J_i,\ y \in J_j,\ x + y \in J_{i+j}\} \\
T^2_{i,j} :=& \{(x, y) \in T : x \in J_i,\ y \in J_j,\ x + y \in J_{i+j+1}\} \\
\nu :=& \begin{cases} \max\{k \in \mathbb{N}_0 : (k+1)\tau \leq 1\}, & \text{if } \tau \leq 1/2 \\ 1, & \text{if } \tau > 1/2. \end{cases}
\end{aligned}$$

Let (f, g) be a non-trivial solution of (1) satisfying (6)–(8). If $\tilde{f}$ does not equal a homomorphism on I, then, by the proof of Theorem 1 in [2], we have :

$(\tilde{f}, \tilde{g})$ of the form (9) if $\Omega_{\tilde{g}} \cap T^1_{0,0} = \emptyset$,

$(\tilde{f}, \tilde{g})$ of the form (10) if $\Omega_{\tilde{g}} \cap T^2_{0,0} = \emptyset$,

$(\tilde{f}, \tilde{g})$ of the form (11) or (12) if $\Omega_{\tilde{g}} \cap T^1_{0,0} \neq \emptyset$ and $\Omega_{\tilde{g}} \cap T^2_{0,0} \neq \emptyset$.

Proposition 3. *Let* (f, g) *be a non-trivial solution of equation* (1) *satisfying* (6)–(8) *and let* $(\tilde{f}, \tilde{g})$ *be its associate solution on* T. *Assume*

$$\text{either} \quad \Omega_{\tilde{g}} \cap T^1_{0,0} = \emptyset \quad \text{or} \quad \Omega_{\tilde{g}} \cap T^2_{0,0} = \emptyset.$$

Then $\tilde{f}$ *cannot be a homomorphism on* I *and* $(\tilde{f}, \tilde{g})$ *has the form* (9) *or* (10) *respectively.*

Proof. By the proof of Theorem 1 in [2], the hypothesis on $\Omega_{\tilde{g}}$ assures that $\tilde{g}$ has the form given in (9) or (10), independently on the form of $\tilde{f}$. Assume $\tilde{f} = a \in \mathrm{Hom}(\mathbb{R}, S)$. By Lemma 1, on all intervals

$$(n + i\tau, n + (i+1)\tau) \cap I_n, \quad n \in \mathbb{Z},\ i \in \mathbb{N}_0$$

the function f is locally affine. Let $F := \{x \in \mathbb{R} \setminus \mathbb{Z} : f(x) \neq a(x)\}$; by Proposition 1, $F \neq \emptyset$. First we assume that $F \cap \mathbb{R}^+ =: F^+ \neq \emptyset$ and we define $x_0 = \inf F^+$ ($x_0 \geq 1$). Note that x_0 is a point of the form $n_0 + i_0\tau$, with $n_0 \in \mathbb{Z} \setminus \{0\}$, $i_0 \geq 0$.

i) x_0 is a limit point of F^+.
In this case f is locally affine on $(x_0, x_0 + \delta)$ with $\delta = \min(\tau, 1 - i_0\tau)$, and so

$$f(x) = \alpha b(x),\ x \in (x_0, x_0 + \delta), \quad f(x) = a(x),\ x \in (0, x_0) \setminus \mathbb{Z}$$

where either $b \neq a$ or $\alpha \neq e$. Thus

$$\{(x, y) : x \in (0, x_0),\ y \in (0, x_0),\ x + y \in (x_0, x_0 + \delta)\} \subset A_g.$$

By Lemma 1 it follows

$$g(x) = \gamma c(x), \quad c \in \mathrm{Hom}(\mathbb{R}, S),\ x \in (0, x_0) \supset I,$$

a contradiction

ii) x_0 is not a limit point of F^+.
We have $x_0 = n_0 + i_0\tau$, $i_0 > 0$ and $\{(x, n_0 + i_0\tau - x) : x \in (n_0, n_0 + i_0\tau)\} \subset A_g$. It follows $\{(x, i_0\tau - x) : x \in (0, i_0\tau)\} \subset A_g$: a contradiction.

In the case $F \cap \mathbb{R}^+ = \emptyset$, i.e. $F \subset \mathbb{R}^-$, we define $x_0 = \sup F$.

i) x_0 is a limit point of F.
Then $\{(x, y) : x \in (x_0 - \delta, x_0),\ y \in I,\ x + y > x_0,\ x + y \notin \mathbb{Z}\} \subset A_g$.

By Lemma 1 it follows that $\tilde{g}$ is locally affine on I, contrary to (9) and (10).

ii) x_0 is not a limit point of F.
We have $x_0 = n_0 + i_0\tau$, $i_0 > 0$ and $\{(x, x_0 - x) : x \in (0, \tau)\} \subset A_g$. By Lemma 3 it follows $\{(x, i_0\tau - x) : x \in (0, \tau)\} \subset A_g$, again contrary to (9) and (10).

Lemma 5. *Let (f, g) be a non-trivial solution of equation* (1) *satisfying* (6)–(8) *and let $(\tilde{f}, \tilde{g})$ be its associate solution on T. Assume*

$$\Omega_{\tilde{g}} \cap T^1_{0,0} \neq \emptyset \quad \textit{and} \quad \Omega_{\tilde{g}} \cap T^2_{0,0} \neq \emptyset.$$

Then there exists $a \in \mathrm{Hom}(\mathbb{R}, S)$ such that f has one of the following forms

$$(15) \qquad f = a \text{ on } I_n \setminus E_n, \; n \in \mathbb{Z} \text{ with } E_n \subset n + (\tau\mathbb{N}_0 \cap I)$$

$$(16) \qquad f = a \text{ on } I_n \setminus E_n, \; n \in \mathbb{Z} \text{ with } E_n \subset \{n + \xi\}, \; \xi \in W \setminus \tau\mathbb{N}_0$$

where $E_n \neq \emptyset$ for at least one $n \in \mathbb{Z}$.

Proof. By the meaning of W and τ and by Lemmas 1 and 3, f is locally affine in all intervals

$$(17)_n \qquad (n + i\tau, n + (i+1)\tau), \quad n \in \mathbb{Z}, \; 0 \leq i \leq \nu - 1.$$

In all cases, $\tilde{f}$ equals a homomorphism a on the intervals $(17)_0$; moreover

$$f(x) = \alpha_{i,n} a_{i,n}(x), \quad x \in (n + i\tau, n + (i+1)\tau), \quad n \in \mathbb{Z}, \; 0 \leq i \leq \nu - 1,$$

with $a_{i,n} \in \mathrm{Hom}(\mathbb{R}, S)$ and $\alpha_{i,n} \in S$. Now, by Lemma 1–iii) $a_{i,n} = a$; moreover by Lemma 3 and the properties of W (see [2]), it is $\Omega_g \cap (T^1_{i,0} + (n, 0)) \neq \emptyset$ and so we obtain $\alpha_{i,n} = e$. Thus

$$(18) \qquad f(x) = a(x) \quad \text{on the intervals } (17)_n.$$

We remark that the proof of Theorem 1 concerning the cases (11) and (12) shows, by an iterative procedure, that $\tilde{f} = a$ on I except for the points of a finite set E, where

$$\begin{cases} E \subset \tau\mathbb{N}_0 \cap I & \text{in case (11)} \\ E = \{\xi\}, \; \xi \in W \setminus \tau\mathbb{N}_0 & \text{in case (12).} \end{cases}$$

This procedure depends only on the properties of $\Omega_{\tilde{g}}$ and works as follows:

if $\tilde{f} = a$ on $(0, A_n) \setminus F_n$ (F_n a finite set), then we have $\tilde{f} = a$ on $(0, A_{n+1}) \setminus F_{n+1}$ where $A_{n+1} > A_n$, $F_{n+1} \supset F_n$, F_{n+1} finite.

By Lemma 3 and (18) we can apply the same procedure to f on all intervals $(n, n+1)$ and we get (15) and (16) in the cases (11) and (12) respectively.

Since by Proposition 1 f is not equal to a on $\mathbb{R} \setminus \mathbb{Z}$, there exists an interval I_n where f is not identically equal to a and so it must have one of the forms (15) or (16) with $E_n \neq \emptyset$.

Lemma 6. *Assume* $(S, \cdot)$ *has no elements of order 2. Let* (f, g) *be a non-trivial solution of equation* (1) *satisfying* (6)–(8). *Then there exists no interval* I_n *where* f *has the form* (15) *with* $E_n \neq \emptyset$.

Proof. Assume on the contrary there exists $\bar{n}$ such that f has the form (15) on $I_{\bar{n}}$ with $E_{\bar{n}} \neq \emptyset$, i.e.

$$f(x) = a(x), \quad x \in I_{\bar{n}} \setminus E_{\bar{n}}, \quad E_{\bar{n}} \subset \bar{n} + (\tau\mathbb{N}_0 \cap I), \quad E_{\bar{n}} \neq \emptyset.$$

It is always possible to find a pair of consecutive intervals I_m, I_{m+1}, $m \in \mathbb{Z}$, such that $E_m \neq \emptyset$ and either $E_{m+1} = \emptyset$ or $\min(E_m - m) \leq \min(E_{m+1} - (m+1))$. Let $k\tau = \min(E_m - m)$, and so $f(m + k\tau) \neq a(m + k\tau)$. We show that

$$\{(k\tau, y) : y \in I\} \subset A_g, \quad \{(x, k\tau) : x \in I\} \subset A_g.$$

By Lemma 3 this implies $k\tau \in H_g$, contrary to the assumption (6).

- The points $(m + k\tau, y)$ with $y \in (0, 1 - k\tau)$ are in A_g by the definition of τ.
- The points $(m + k\tau, y)$ with $y \in (1 - k\tau, 1)$, $y \neq i\tau \in E$ are in A_g since $m + k\tau + y \notin E_{m+1}$.

By Lemma 3 it follows

$$\{(k\tau, y) : y \in (0, 1 - k\tau)\} \cup \{(k\tau, y) : y \in (1 - k\tau, 1) \setminus E\} \subset A_g.$$

It remains to show that $\{(k\tau, i\tau),\, i\tau \in E,\, (i + k)\tau > 1\} \cup \{(k\tau, 1 - k\tau)\} \subset A_g$.

First we prove that either all points $(i\tau, r\tau)$, $(i\tau, 1 - i\tau)$, $(1 - i\tau, i\tau)$, with $(i + r)\tau > 1$ are in A_g or none of them is in A_g, i.e. all are in A_f.

- $(i\tau, 1 - i\tau) \in A_g$ if and only if $(1 - i\tau, i\tau) \in A_g$:
  $g(1) = g(i\tau)g(1 - i\tau)$, or equivalently $g(1 - i\tau) = g(\tau)^{-i}g(1) = g(1)g(\tau)^{-i}$, if and only if $g(1) = g(1 - i\tau)g(\tau)^i = g(1 - i\tau)g(i\tau)$.
- $(i\tau, 1 - i\tau) \in A_g$ if and only if $(i\tau, r\tau) \in A_g$, $(i + r)\tau > 1$:
  by the last equation in (11),

  $$g(r\tau) = g((r+i)\tau - 1)g(1 - i\tau) \text{ i.e. } g(1 - i\tau) = g(r\tau)g((r+i)\tau - 1)^{-1};$$

  by the definition of H_g,

  $$g((r+i)\tau) = g(1)g((r+i)\tau - 1), \text{ i.e. } g((r+i)\tau - 1)^{-1} = g((r+i)\tau)^{-1}g(1).$$

It follows $g(i\tau)g(1-i\tau)=g(i\tau)g(r\tau)g(r+i)\tau)^{-1}g(1)$ and this relation implies

$$g(1)=g(i\tau)g(1-i\tau) \quad \text{if and only if} \quad g((r+i)\tau)=g(i\tau)g(r\tau).$$

Assume $1=(\nu+1)\tau$ and consider the point $(k\tau,1-k\tau)=(k\tau,(\nu+1-k)\tau)$. For all $y\in(0,\tau)$ we have:

$$g(k\tau)g(1-k\tau+y)=\begin{cases} g(1+y)=g(1)g(y)\\ g(k\tau)g[(\nu+1-k)\tau+y]=g(k\tau)g[(\nu+1-k)\tau]g(y).\end{cases}$$

It follows $g(1)g(y)=g(k\tau)g(1-k\tau)g(y)$, i.e. $(k\tau,1-k\tau)\in A_g$. So

$$\{(k\tau,i\tau),\ i\tau\in E,\ (i+k)\tau>1\}\cup\{(k\tau,1-k\tau)\}\subset A_g.$$

Let now $1\notin\tau\mathbb{N}_0$. If there is no $i\tau\in E$ with $(k+i)\tau>1$, then

$$L:=\{(k\tau,y):y\in(1-k\tau,1)\}\subset A_g.$$

Since there exists at least a point $(k\tau,r\tau)\in L$, then $(k\tau,1-k\tau)\in A_g$ as well.

Conversely if there exists $r\tau\in E$ with $(k+r)\tau>1$, then at least one of the numbers $r\tau,k\tau$ is greater than $1/2$. To conclude the proof it is then enough to show that $i\tau\in E$, $i\tau>1/2$ implies $(i\tau,i\tau)\in A_g$. If not, we have $(i\tau,i\tau)$, $(i\tau,1-i\tau)$, $(1-i\tau,i\tau)\in A_f$. Put $f(i\tau)=\gamma a(i\tau)$, $\gamma\neq e$. We have:

$$\begin{aligned}\gamma a(1)=&\gamma a(1-i\tau)a(i\tau)=\gamma a(i\tau)a(1-i\tau)=f(i\tau)f(1-i\tau)=f(1)=\\ =&f(1-i\tau)f(i\tau)=a(1-i\tau)\gamma a(i\tau),\end{aligned}$$

and this implies

$$\gamma a(1-i\tau)=a(1-i\tau)\gamma. \tag{19}$$

By Lemma 3, $(i\tau,2-i\tau)$, $(2-i\tau,i\tau)\in A_f$ as well; so

$$\begin{aligned}a(1)\gamma a(1)&=a(1)f(1)=a(1)a(1-i\tau)f(i\tau)=f(2-i\tau)f(i\tau)=f(2)=\\ &=f(i\tau)f(2-i\tau)=f(i\tau)a(1-i\tau)a(1)=f(1)a(1)=\gamma a(1)^2;\end{aligned}$$

this implies

$$\gamma a(1)=a(1)\gamma. \tag{20}$$

From (19) and (20) we get

$$\gamma a(i\tau)=a(i\tau)\gamma. \tag{21}$$

Since $(i\tau, i\tau) \in A_f$,

$$a(i\tau)^2 = a(2i\tau) = f(2i\tau) = f(i\tau)^2 = \gamma a(i\tau)\gamma a(i\tau) \overset{(21)}{=} \gamma^2 a(i\tau)^2,$$

i.e. $\gamma^2 = e$; a contradiction since S has no elements of order 2.

Lemma 7. *Assume $(S,\cdot)$ has no elements of order 2. Let (f,g) be a non-trivial solution of equation* (1) *satisfying* (6)–(8). *Then there exists no interval I_n where f has the form* (16) *with $E_n \neq \emptyset$.*

Proof. Assume on the contrary there exists $\bar{n}$ such that f has the form (16) on $I_{\bar{n}}$ with $E_{\bar{n}} \neq \emptyset$, i.e. $E_{\bar{n}} = \{\bar{n}+\xi\}$, $\xi \in W \setminus \tau\mathbb{N}_0$. We shall prove that the segments

$$\{(\xi, y), y \in I\} \quad \text{and} \quad \{(x,\xi), x \in I\}$$

are in A_g; this implies $\xi \in H_g$, contrary to the assumption (6).

By Proposition 3 it must be $\Omega_{\tilde{g}} \cap T^1_{0,0} \neq \emptyset$ and $\Omega_{\tilde{g}} \cap T^2_{0,0} \neq \emptyset$. Therefore, by Lemma 5,

$$f(x) = a(x), \quad x \in I_{n+1} \setminus E_{n+1},\ E_{n+1} \subset \{n+1+\xi\}.$$

It follows immediately that $\big\{(\bar{n}+\xi, y) : y \in I \setminus (\{1-\xi\} \cup \{\xi\})\big\} \subset A_g$ and so, by Lemma 3, $\big\{(\xi, y) : y \in I \setminus (\{1-\xi\} \cup \{\xi\})\big\} \subset A_g$. Now we prove that either the three points (ξ,ξ), $(1-\xi,\xi)$ and $(\xi,1-\xi)$ are all in A_g or none of them is in A_g. By the last equation in (12), $g(\xi) = g(2\xi-1)g(1-\xi) = g(1-\xi)g(2\xi-1)$, and so

$$g(\xi)^2 = g(2\xi-1)g(1-\xi)g(\xi) = g(\xi)g(1-\xi)g(2\xi-1).$$

By the definition of H_g we have

$$g(2\xi) = g(2\xi-1)g(1) = g(1)g(2\xi-1).$$

These relations immediately imply

$$g(1) = g(1-\xi)g(\xi) \iff g(2\xi) = g(\xi)^2 \iff g(1) = g(\xi)g(1-\xi).$$

If $f(\xi) = a(\xi)$, we have $(\bar{n}+\xi,\xi) \in A_g$ and so (ξ,ξ), $(\xi,1-\xi)$, $(1-\xi,\xi) \in A_g$. Assume $f(\xi) \neq a(\xi)$ and let $f(\xi) = \gamma a(\xi)$, $\gamma \neq e$. Suppose that one of the points (ξ,ξ), $(1-\xi,\xi)$, $(\xi,1-\xi)$ is not in A_g, then all three are in A_f and moreover $(2-\xi,\xi)$, $(\xi,2-\xi) \in A_f$. This implies

$$a(1-\xi)\gamma a(\xi) = f(1-\xi)f(\xi) = f(1) = f(\xi)f(1-\xi) = \gamma a(\xi)a(1-\xi) = \\ = \gamma a(1-\xi)a(\xi) = \gamma a(1),$$

so

$$a(1-\xi)\gamma = \gamma a(1-\xi), \tag{22}$$

and

$$a(1)\gamma a(1) = a(1)f(1) = a(1)a(1-\xi)\gamma a(\xi) = f(2-\xi)f(\xi) = f(2) = \\ = f(\xi)f(2-\xi) = \gamma a(\xi)a(1)a(1-\xi) = f(1)a(1) = \gamma a(1)^2;$$

thus we have

$$\gamma a(1) = a(1)\gamma. \tag{23}$$

From (22) and (23) we obtain $\gamma a(\xi) = a(\xi)\gamma$. Then

$$a(2\xi) = f(2\xi) = f(\xi)^2 = \gamma a(\xi)\gamma a(\xi) = \gamma^2 a(2\xi),$$

i.e. $\gamma^2 = e$: a contradiction since S has no elements of order 2.

We have so proved that $\{(\xi, y) : y \in I\} \subset A_g$; in a similar way we may obtain $\{(x, \xi) : x \in I\} \subset A_g$.

We summarize the results of this section in the following

Proposition 4. *Assume S is a group without elements of order 2. Let (f, g) be a non-trivial solution of equation (1) satisfying (6)–(8). Then its associate solution on T, $(\tilde{f}, \tilde{g})$, has one of the forms (9) or (10).*

5. New solutions

A) S has no elements of order 2.

From the results of the previous sections we know that, when S has no elements of order 2, the only possible non-trivial solutions of our problem must have on I one of the forms (9) and (10) of Theorem 1.

Now we prove that actually such solutions exist.

Theorem 2. *Let S be a group without elements of order 2. The functional equation (1) under the conditions (6)–(8) has non-trivial solutions if and only if $(\nu+1)\tau = 1$ and, in this case, the pair (f, g) has one of the following forms:*

$$\begin{cases} f(x) = \alpha^{n(\nu+2)+i+1}a(x) & , \quad x \in [n+i\tau, n+(i+1)\tau) \\ g(x) = \gamma^{n\nu+i}c(x) & , \quad x \in [n+i\tau, n+(i+1)\tau) \end{cases} \tag{24}$$

$$\begin{cases} f(x) = \alpha^{n\nu+i}a(x) & , \quad x \in (n+i\tau, n+(i+1)\tau] \cap I_n \\ g(x) = \gamma^{n(\nu+2)+i+1}c(x) & , \quad x \in (n+i\tau, n+(i+1)\tau] \cap I_n \\ f(n) = \alpha^{n\nu-1}a(n) & , \quad g(n) = \gamma^{n(\nu+2)}c(n) \end{cases} \tag{25}$$

where $i = 0, \dots, \nu$, $n \in \mathbb{Z}$, $\alpha \neq e$, $\gamma \neq e$, $a, c \in \mathrm{Hom}(\mathbb{R}, S)$ *and* a, c *commute with* α, γ *respectively.*

Proof. Let (f, g) be a non-trivial solution of (1). By the results of the previous sections we have $\emptyset \neq W \neq I$ and the pair $(\tilde{f}, \tilde{g})$ has one of the forms (9) or (10) of Theorem 1. Assume $(\tilde{f}, \tilde{g})$ has the form (9); then $T^1_{0,0} \subset A_g$ and $T^2_{0,0} \subset \Omega_g$. By the definition of H_g, we have

$$g(n+x) = g(1)^n g(x), \quad x \in I,\ n \in \mathbb{Z}, \quad \text{and} \quad g(0) = e.$$

Denote $\rho = g(1)c(-1)$; we immediately have

$$g(n+x) = \rho^n \gamma^i c(n+x), \quad x \in [i\tau, (i+1)\tau) \cap [0,1), \quad i = 0, 1, \dots\ .$$

Consider now the function f in the interval I_1. It is locally affine in all subintervals $(1 + i\tau, 1 + (i+1)\tau) \cap I_1$, $i = 0, 1, \dots$ and so

$$f(x) = \beta_i a_i(x), \quad x \in (1 + i\tau, 1 + (i+1)\tau) \cap I_1,$$

for some $a_i \in \mathrm{Hom}(\mathbb{R}, S)$ and $\beta_i \in S$. By Lemma 1–iii) we have $a_i = a$ for all i. Let us denote $\beta_0 = \eta\alpha$. by the properties of Ω_g (and so of A_f) and by Lemma 3 we get $\beta_i = \eta\alpha^{i+1}$.

Now we show that $(\nu+1)\tau = 1$. Assume the contrary and let $1 = \sigma\tau + \rho$ where $0 < \rho < \tau$ and

$$\sigma = \begin{cases} \nu + 1 & \text{if} \quad \tau < 1/2 \\ 1 & \text{if} \quad \tau > 1/2 \end{cases} \tag{26}$$

Consider the three sets

$$U_1 = \{(x, y) : (\sigma - 1)\tau < x < \sigma\tau,\ 0 < y < \tau,\ x + y > 1\}$$
$$U_2 = \{(x, y) : \sigma\tau < x < 1,\ 0 < y < \tau,\ x + y > 1\}$$
$$U_3 = \{(x, y) : \sigma\tau < x < 1,\ \tau < y < 2\tau,\ x + y < 1 + \tau\}$$

and let $(x_i, y_i) \in U_i$. We have the following possibilities:

a) $(x_1, y_1) \in A_g$: thus $\gamma^{\sigma-1} = \rho$. Since S has no elements of order 2 and $\gamma \neq e$, both points (x_2, y_2) and (x_3, y_3) are in A_f. This implies $\alpha^{\sigma+1} = \eta = \alpha^{\sigma+2}$ i.e. $\alpha = e$; a contradiction.

b) $(x_1, y_1) \in A_f$: thus $\alpha^\sigma = \eta$. So (x_2, y_2), $(x_3, y_3) \in A_g$ and this implies $\gamma^{\sigma+1} = \rho = \gamma^\sigma$ i.e. $\gamma = e$; contradiction.

We prove that $(\tau, \nu\tau) \notin A_g$. On the contrary we get $\gamma^{\nu+1} = \rho$ and it follows

$$\{(\tau, y) : \nu\tau \leq y < 1\} \cup \{(x, \tau) : \nu \leq x < 1\} \subset A_g.$$

Thus $\tau = \frac{1}{\nu+1} \in H_g$: a contradiction. So $(\tau, \nu\tau)$ must belong to $A_f \cap \Omega_g$ and this implies $f(1) = \alpha^{\nu+3} a(1)$. From this we have

$$\{(x, 1-x) : x \in [0,1],\ x \notin \mathbb{N}\tau\} \subset A_g$$

and this implies $\rho = \gamma^{\nu}$. So g has the form described in (24). Take now (x, y) with $\nu\tau < x < 1$, $\tau < y < 2\tau$, $1 < x + y < 1 + \tau$; since $(x, y) \notin A_g$ we must have $(x, y) \in A_f$ and this implies $\eta = \alpha^{\nu+2}$.

By induction we easily obtain

$$f(x) = \alpha^{n(\nu+2)+i+1} a(x),\ x \in [n + i\tau, n + (i+1)\tau),\ i = 0, 1, \ldots, \nu; n \in \mathbb{Z}.$$

In the case $(\tilde{f}, \tilde{g})$ of the form (10), in the same way we obtain the solutions given by (25).

A simple check shows that (24) and (25) are solutions of (1).

B) S has elements of order 2.

We examine the role of the assumption that S has no elements of order 2. This hypothesis appeared in Section 4 and it has been used in Lemma 7 in order to assure that $(\xi, \xi) \in A_g$, in Lemma 6 to prove that $(i\tau, i\tau) \in A_g$ for all $i\tau \in E$ and so to exclude the case $1 \notin \tau\mathbb{N}$. Again it has been used in Theorem 2 to prove that $(\nu + 1)\tau = 1$. In this last case, that is when $\emptyset \neq W \neq I$ and the associate solution has one of the forms (9) or (10) of Theorem 1, it is possible to describe the solutions of our problem in the case S has elements of order 2.

Assume $(\tilde{f}, \tilde{g})$ has the form (9) and $\sigma\tau < 1$, where σ is given by (26). Consider the sets U_1, U_2, U_3 defined in the proof of Theorem 2 and let $(x_i, y_i) \in U_i$, $i = 1, 2, 3$. We have two cases:

I) $(x_1, y_1) \in A_g$: then $\gamma^{\sigma-1} = \rho$. Since $\gamma \neq e$, it is $(x_2, y_2) \in A_f$ and this implies $\alpha^{\sigma+1} = \eta$. Moreover $(x_2, y_2) \in A_f$ implies $(x_3, y_3) \in A_g$, i.e. $\gamma^2 = e$. Looking to the diagonal $\{(x, 1-x) : x \in I\}$ we immediately realize that $\alpha^{\sigma+2} a(1) = f(1)$.

II) $(x_1, y_1) \in A_f$: then $\alpha^{\sigma} = \eta$. Since $\alpha \neq e$, it is $(x_2, y_2) \in A_g$ and this implies $\gamma^{\sigma} = \rho$. Moreover $(x_2, y_2) \in A_g$ implies $(x_3, y_3) \in A_f$, i.e. $\alpha^2 = e$. Looking to the diagonal $\{(x, 1-x) : x \in I\}$ we immediately realize that $\alpha^{\sigma+3} a(1) = f(1) = \alpha^{\sigma+1} a(1)$.

In the same way we argue when $(\tilde{f}, \tilde{g})$ has the form (10).

Summarizing we have the following.

Theorem 3. *If S has elements of order 2, the functional equations (1) under the conditions (6)–(8), besides the solutions described in Theorem 2, has the following ones, with $\sigma\tau < 1$:*

$$\begin{cases} f(x) = \alpha^{n(\sigma+1)+i+1} a(x), & x \in [n+i\tau, n+(i+1)\tau) \cap [n, n+1) \\ g(x) = \gamma^{n(\sigma-1)+i} c(x), & x \in [n+i\tau, n+(i+1)\tau) \cap [n, n+1) \end{cases}$$

where $\gamma^2 = e$, $\gamma \neq e$, $\alpha \neq e$;

$$\begin{cases} f(x) = \alpha^{n\sigma+i+1} a(x), & x \in [n+i\tau, n+(i+1)\tau) \cap [n, n+1) \\ g(x) = \gamma^{n\sigma+i} c(x), & x \in [n+i\tau, n+(i+1)\tau) \cap [n, n+1) \end{cases}$$

where $\alpha^2 = e$, $\gamma \neq e$, $\alpha \neq e$;

$$\begin{cases} f(x) = \alpha^{n\sigma+i} a(x), & x \in (n+i\tau, n+(i+1)\tau] \cap (n, n+1] \\ g(x) = \gamma^{n\sigma+i+1} c(x), & x \in (n+i\tau, n+(i+1)\tau] \cap (n, n+1] \\ f(n) = \alpha^{n\sigma-1} a(n), & g(n) = \gamma^{n\sigma} c(n) \end{cases}$$

where $\gamma^2 = e$, $\gamma \neq e$, $\alpha \neq e$;

$$\begin{cases} f(x) = \alpha^{n(\sigma-1)+i} a(x), & x \in (n+i\tau, n+(i+1)\tau] \cap (n, n+1] \\ g(x) = \gamma^{n(\sigma+1)+i+1} c(x), & x \in (n+i\tau, n+(i+1)\tau] \cap (n, n+1] \\ f(n) = \alpha^{n(\sigma-1)+1} a(n), & g(n) = \gamma^{n(\sigma+1)} c(n) \end{cases}$$

where $\alpha^2 = e$, $\gamma \neq e$, $\alpha \neq e$.

In all cases $a, c \in \mathrm{Hom}(\mathbb{R}, S)$ and commute with α, γ respectively.

6. Open problems and final remarks

About the results of Section 4, the following example shows that the condition on S is essential. Let $S = \mathbb{R}/2\mathbb{Z}$ and $\tau \in (1/2, 1)$ and define $g : \mathbb{R} \to \mathbb{R}/2\mathbb{Z}$ as a periodic function of period 1 given on [0,1] by

$$\begin{cases} g(0) = g(\tau) = g(1) = 0 \\ g(x) = 1 \quad \text{elsewhere.} \end{cases}$$

So $H_g = \mathbb{Z}$ and $A_g \cap Q$ is the set

$$\begin{aligned} \{(x, \tau - x) : x \in (0, \tau)\} &\cup \{(x, 1 + \tau - x) : x \in (\tau, 1)\} \\ &\cup \{(x, 1 - x) : x \in I \setminus \{1 - \tau, \tau\}\} \\ &\cup \{(\tau, y) : y \in I \setminus \{1 - \tau, \tau\}\} \\ &\cup \{(x, \tau) : x \in I \setminus \{1 - \tau, \tau\}\}. \end{aligned}$$

Thus g satisfies conditions (7) and (8). If we take $f : \mathbb{R} \to \mathbb{R}/2\mathbb{Z}$ periodic of period 1 and given on [0,1] by

$$\begin{cases} f(0) = f(\tau) = 1 \\ f(x) = 0 \quad \text{elsewhere,} \end{cases}$$

the pair (f, g) is a non-trivial solution of our problem and the restriction $(\tilde{f}, \tilde{g})$ on I has the form (11) of Theorem 1 with $1 \notin \tau\mathbb{N}_0$.

It seems that the existence of such special solutions depends not only on the fact that S has elements of order 2, but also on what group S is. So it remains open the problem of describing all solutions in this latter case.

As proved in Lemma 4, all non-trivial solutions of equation (1) satisfying (6)–(8) does not satisfy (13), i.e. g is not odd. Note that starting from an arbitrary solution (f, g) of (1) on $\mathbb{R}$ (without any other condition) it is always possible to construct another one (f_1, g_1) in the following way:

$$f_1(x) = \begin{cases} f(x), & x \geq 0 \\ [f(-x)]^{-1}, & x < 0 \end{cases} \qquad g_1(x) = \begin{cases} g(x), & x \geq 0 \\ [g(-x)]^{-1}, & x < 0 \end{cases}.$$

Observe that since either $f(0) = e$ or $g(0) = e$, then either f_1 or g_1 is odd. So in this way it is possible to obtain new classes of solutions (f, g) where at least one of the functions f_1 and g_1 is odd.

References

[1] G. L. Forti and L. Paganoni, On an alternative Cauchy equation in two unknown functions Some classes of solutions, *Aequationes Math.* **42** (1991), 271–295.

[2] G. L. Forti and L. Paganoni, Local solutions of an alternative Cauchy equation, *Publ. Math. Debrecen* (*to appear*).

G. L. FORTI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI MILANO
VIA C. SALDINI 50
I–20133 MILANO
ITALY

L. PAGANONI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI MILANO
VIA C. SALDINI 50
I–20133 MILANO
ITALY

(Received February 15, 1993)